ISSN 1066-369X, Russian Mathematics, 2023, Vol. 67, No. 10, pp. 14–29. © Allerton Press, Inc., 2023.
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Inverse Coefficient Problems for a Time-Fractional Wave Equation with the Generalized Riemann–Liouville Time Derivative

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Received March 29, 2023; revised May 9, 2023; accepted May 29, 2023

Abstract—This paper considers the inverse problem of determining the time-dependent coefficient in the fractional wave equation with Hilfer derivative. In this case, the direct problem is initial-boundary value problem for this equation with Cauchy type initial and nonlocal boundary conditions. As overdetermination condition nonlocal integral condition with respect to direct problem solution is given. By the Fourier method, this problem is reduced to equivalent integral equations. Then, using the Mittag–Leffler function and the generalized singular Gronwall inequality, we get apriori estimate for solution via unknown coefficient which we will need to study of the inverse problem. The inverse problem is reduced to the equivalent integral of equation of Volterra type. The principle of contracted mapping is used to solve this equation. Local existence and global uniqueness results are proved.

Keywords: fractional derivative, Riemann–Liouville fractional integral, inverse problem, integral equation, Fourier series, Banach fixed point theorem

DOI: 10.3103/S1066369X23100092

1. INTRODUCTION

Nowadays, a great attention has been focused on the study of initial and initial boundary value problems for fractional differential equations. Because fractional differential equations have been widely used in engineering, physics, chemistry, biology, and other fields. The researcher can find many applications in the work [1–4] and references therein.

The identification of the right hand side and the order of time fractional derivative equation in applied fractional modeling plays an important role. In [5–8], an inverse problem for determining these unknowns of time fractional derivative in a subdiffusion equation with an arbitrary second order elliptic differential operator is considered. It is proved that the additional information about the solution at a fixed time instant at a monitoring location, as the observation data, identified uniquely the order of the fractional derivative.

In [9–12], the unique solvability of the nonlocal direct problems and inverse source problems for the various fractional differential equations with Caputo and Riemann–Liouville integral-differential operators were investigated.

Inverse problems for classical integro-differential wave propagation equations have been extensively studied. Nonlinear inverse coefficient problems with various types of overdetermination conditions are often found in the literature (e.g., [13–18] and references therein). In [19–24], inverse problems of determining unknown coefficients in Cauchy problem for fractional diffusion-wave equation were investigated. Local existence and uniqueness in whole are proved and estimates of conditional stability are obtained.

2. FORMULATION OF THE PROBLEM

In this paper, we investigate the local existence and global uniqueness of an inverse problem of determining time-dependent coefficient in the generalized time fractional wave equation with initial, nonlocal boundary and overdetermination integral conditions.

In the next section, we provide some necessary preliminaries are given.

Let $T > 0$, $l > 0$ be fixed numbers and $\Omega_{lT} := \{(x,t) : 0 < x < l, 0 < t \le T\}$. Consider the time-fractional diffusion equation

$$D^{\alpha,\beta}_{0+,t}u(x,t) - u_{xx} + q(t)u(x,t) = p(t)f(x,t), \quad x \in (0,l), \ t \in (0,T], \tag{1}$$

the initial conditions of Cauchy type

$$\begin{gathered} I^{(2-\alpha)(1-\beta)}_{0+,t}u(x,t)\Big|_{t=0} = \varphi_1(x), \\ \frac{\partial}{\partial t}\left(I^{(2-\alpha)(1-\beta)}_{0+,t}u\right)(x,t)\Big|_{t=0} = \varphi_2(x), \ \ x \in [0,l], \end{gathered} \tag{2}$$

the boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 \le t \le T, \tag{3}$$

and the nonlocal additional condition

$$\int_0^l w_i(x)u(x,t)dx = h_i(t), \ \ i = 1,2, \ \ t \in [0,T]. \tag{4}$$

Here the generalized Riemann–Liouville (Hilfer) fractional differential operator $D^{\alpha,\beta}_{0+,t}$ of the order $1 < \alpha < 2$ and type $0 \le \beta \le 1$ is defined as follows ([1], pp. 112–118; [2], pp. 62–65):

$$D^{\alpha,\beta}_{0+,t}u(\cdot,t) = \left(I^{\beta(2-\alpha)}_{0+,t}\frac{\partial^2}{\partial t^2}\left(I^{(1-\beta)(2-\alpha)}_{0+,t}u\right)\right)(\cdot,t),$$

$$I^{\gamma}_{0+,t}u(x,t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\gamma}}d\tau, \ \ \gamma \in (0,1)$$

is the Riemann–Liouville fractional integral of the function $u(x,t)$ with respect to t ([3]; [4], pp. 69–72), $\Gamma(\cdot)$ is the Euler's gamma function. The functions $f(x,t)$, $w_i(x)$, $\varphi_i(x)$, $h_i(t)$, $i = 1,2$ are known functions.

In ([1], pp. 112–118; [3], pp. 28–37), by R. Hilfer was introduced a generalized form of the Riemann–Liouville fractional derivative of order α and a type $\beta \in [0,1]$, which coincides with the Riemann–Liouville fractional derivative at $\beta = 0$ and with Gerasimov–Caputo fractional derivative at $\beta = 1$, and the case $\beta \in (0,1)$ interpolates these both fractional derivatives.

Assume that throughout this article, given functions $\varphi_1, \varphi_2, f, w$, and h satisfy the following assumptions:

(A1) $\varphi_i \in C^3[0,l]$, $\varphi_i^{(4)} \in L_2[0,l]$, $\varphi_i(0) = \varphi_i(l) = 0$, $\varphi_i''(0) = \varphi_i''(l) = 0$, $i = 1,2$;

(A2) $f(x,\cdot) \in C[0,T]$ *and for* $t \in [0,T]$, $f(\cdot,t) \in C^3[0,l]$, $f(\cdot,t)^{(4)} \in L_2[0,l]$, $f(0,t) = f(l,t) = 0$, $f_{xx}(0,t) = f_{xx}(l,t) = 0$;

(A3) $w(x) \in C^2[0,l]$ *and* $w(0) = w(l) = 0$ *and* $w''(0) = w''(l) = 0$;

(A4) $\left(D^{\alpha,\beta}_{0+,t}h\right)(t) \in C[0,T]$, $|h(t)| \ge h_0 > 0, h_0$ *is a given number*,

$$\int_0^l w_i(x)\varphi_1(x)dx = I^{(2-\alpha)(1-\beta)}_{0+,t}h_i(t)\Big|_{t=0+},$$

$$\int_0^l w_i(x)\varphi_2(x)dx = \frac{\partial}{\partial t}\left(I^{(2-\alpha)(1-\beta)}_{0+,t}h_i(t)\right)(t)\Big|_{t=0+}, \quad i = 1,2.$$

3. PRELIMINARIES

In this section, we present some useful definitions and results of fractional calculus.

Two parameter Mittag–Leffler (M–L) function. The two parameter M–L function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta,z\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\alpha)$ denotes the real part of the complex number α. The Mittag–Leffler function has been studied by many authors who have proposed and studied various generalizations and applications.

From the above, there exist some positive constants M_i, $i=1,2,3$, such that

$$M=\max\left\{\max_{0\le t\le T}\left|E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2t^\alpha\right)\right|,\max_{0\le t\le T}\left|E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2t^\alpha\right)\right|,\right.$$

$$\left.\max_{0\le s<t\le T}\left|E_{\alpha,\alpha}(-\lambda_n^2(t-s)^\alpha)\right|\right\}.$$

Proposition 1. *Let* $0<\alpha<2$ *and* $\beta\in\mathbb{R}$ *be arbitrary. We suppose that* κ *is such that* $\pi\alpha/2<\kappa<\min\{\pi,\pi\alpha\}$. *Then there exists a constant* $C=C(\alpha,\beta,\kappa)>0$ *such that*

$$\left|E_{\alpha,\beta}(z)\right|\le\frac{C}{1+|z|},\quad \kappa\le|\arg(z)|\le\pi.$$

For the proof, we refer to ([4], pp. 40–45), *for example*.

We consider the weighted spaces of continuous functions ([4], pp. 4–5, 162–163).

$$C_\gamma[a,b]:=\left\{g:(a,b]\to R:(t-a)^\gamma g(t)\in C[a,b],0\le\gamma<1\right\},$$

$$C_\gamma^{\alpha,\beta}(\Omega)=\left\{g(t):D_{0+,t}^{\alpha,\beta}g(t)\in C_\gamma(0,T];1<\alpha\le 2,0\le\beta\le 1\right\},$$

$$C_\gamma^{2,\alpha,\beta}(\Omega)=\left\{u(x,t):u(\cdot,t)\in C^2(0,1);t\in[0,T]\text{ and}\right.$$

$$\left.D_{0+,t}^{\alpha,\beta}u(x,\cdot)\in C_\gamma(0,T];x\in[0,1],1<\alpha\le 2,0\le\beta\le 1\right\},$$

$$C_\gamma^0[a,b]=C_\gamma[a,b],$$

with the norms

$$\|f\|_{C_\gamma}=\left\|(t-a)^\gamma f(t)\right\|_C,\quad \|f\|_{C_\gamma^n}=\sum_{k=0}^{n-1}\left\|f^{(k)}\right\|_C+\left\|f^{(n)}\right\|_{C_\gamma}.$$

Lemma 1 ([25], p. 188). *Suppose* $b\ge 0$, $\alpha>0$ *and* $a(t)$ *nonnegative function locally integrable on* $0\le t<T$ (*some* $T\le+\infty$) *and suppose* $u(t)$ *is nonnegative and locally integrable on* $0\le t<T$ *with*

$$u(t)\le a(t)+b\int_0^t(t-s)^{\alpha-1}u(s)ds$$

then

$$u(t)\le a(t)+b\Gamma(\alpha)\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(b\Gamma(\alpha)(t-s)^\alpha\right)a(s)ds.$$

Lemma 2 ([25], p. 189)**.** *Suppose* $b \geq 0, \alpha > 0, \gamma > 0, \alpha + \gamma > 1$ *and* $a(t)$ *nonnegative function locally integrable on* $0 \leq t < T$ *and suppose* $t^{\gamma-1}u(t)$ *is nonnegative and locally integrable on* $0 \leq t < T$ *with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\alpha-1} s^{\gamma-1} u(s)ds,$$

then

$$u(t) \leq a(t)E_{\alpha,\gamma}\left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}} t \right),$$

where

$$E_{\alpha,\gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}, \quad c_0 = 1, \quad \frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$$

for $m \geq 0$. *As* $t \to +\infty$ $E_{\alpha,\gamma}(t) = O\left(t^{\frac{1}{2}\frac{\alpha+\gamma-1}{\alpha-\gamma}} \exp\left(\frac{\alpha+\gamma-1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}} \right) \right)$.

The proof of these assertions come from the definition of Caputo fractional derivative and differentiation of the two-parameter M–L function.

Proposition 2 ([26], pp. 40–45). *For* $0 < \alpha < 1$, $t > 0$, *we have* $0 < E_{\alpha,1}(-t) < 1$. *Moreover*, $E_{\alpha,1}(-t)$ *is completely monotonic*, *that is*

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t) \geq 0, \quad \forall n \in \mathbb{N}.$$

Proposition 3 ([26], pp. 42–45). *For* $0 < \alpha < 1, \eta > 0$, *we have* $0 \leq E_{\alpha,\alpha}(-\eta) \leq \dfrac{1}{\Gamma(\alpha)}$. *Moreover*, $E_{\alpha,\alpha}(-\eta)$ *is a monotonic decreasing function with* $\eta > 0$.

4. INVESTIGATION OF DIRECT PROBLEM (1)–(3)

By applying the Fourier method, the solution $u(x;t)$ of the direct problem (1)–(3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{5}$$

where

$$X_n(x) = \sqrt{\frac{2}{l}} \sin(\lambda_n x), \quad \lambda_n = \frac{\pi n}{l}, \quad n = 1, 2, 3, \ldots. \tag{6}$$

The coefficients $u_n(t)$ for $n \geq 1$ are to be found by making use of the orthogonality of the eigenfunctions $X_n(x)$. The scalar product in $L_2[0,l]$ is defined by $(f,g) = \int_0^l f(x)g(x)dx$. Let us note the expansion coefficients of $\varphi_i(x)$, $i = 1,2$ and $f(x,t)$ in the eigenfunctions of (6) for $n \geq 1$ are definded respectively by

$$\left(f(x,t), X_n(x)\right) = f_n(t), \quad \left(\varphi_i(x), X_n(x)\right) = \varphi_{n,i}, \quad i = 1,2, \quad n = 1,2,\ldots.$$

We obtain in view of (1) and with $\left(u(x,t), X_n(x)\right) = \int_0^l u(x,t)X_n(x)dx = u_n(t)$, and we may write

$$\left(D_{0+,t}^{\alpha,\beta} u_n\right)(t) + \lambda_n^2 u_n(t) = p(t) f_n(t) - q(t) u_n(t), \tag{7}$$

where

$$f_n(t) = \sqrt{\frac{2}{l}} \int_0^l f(x,t) \sin(\lambda_n x) dx.$$

The initial condition (2) give:

$$I_{0+,t}^{(2-\alpha)(1-\beta)}u_n(t)\Big|_{t=0} = \varphi_{n,1}, \quad \frac{d}{dt}\left(I_{0+,t}^{(2-\alpha)(1-\beta)}u_n\right)(t)\Big|_{t=0} = \varphi_{n,2}. \tag{8}$$

In view of ([27], pp. 61–114), we have that the initial problem (7)–(8) is equivalent in the space $C_\gamma^{\alpha,\beta}[0,T]$ to the Volterra integral equation of the second kind

$$\begin{aligned} u_n(t) &= t^{(\beta-1)(2-\alpha)}E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2 t^\alpha\right)\varphi_{n,1} \\ &+ t^{1+(\beta-1)(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2 t^\alpha\right)\varphi_{n,2} \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)p(\tau)f_n(\tau)d\tau \\ &- \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)q(\tau)u_n(\tau)d\tau. \end{aligned} \tag{9}$$

We prove the following assertions for $u_n(t)$:

Lemma 3. *For fixed* $n \in N$ *we have the estimates*

$$\begin{aligned} t^\gamma |u_n| \le M \Bigg(& t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,1}| \\ &+ t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,2}| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg) \\ &\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}t\right), \quad t\in[0,T], \end{aligned}$$

$$\begin{aligned} t^\gamma \left|\left(D_{0+,t}^{\alpha,\beta}u_n\right)(t)\right| &\le \|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]} \\ &+ M\left(\lambda_n^2 + \|q\|_{C[0,T]}\right)\Bigg(t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,2}| \\ &+ \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}t\right), \quad t\in[0,T], \end{aligned}$$

where $1 > \gamma > (1-\beta)(2-\alpha)$.

Proof. Multiplying Eq. (9) by t^γ, we have

$$\begin{aligned} t^\gamma|u_n| \le M\Bigg(& t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,2}| \\ &+ \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg) + \frac{\|q\|_{C[0,T]}t^\gamma}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u_n(\tau)|d\tau. \end{aligned} \tag{10}$$

Further, via Lemma 2, we get

$$t^{\gamma}\left|u_{2n-1}\right| \le M\left(t^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right| + t^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right) \times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right), \quad t\in[0,T].$$

Taking into account the first equation in (7) and the first estimate of Lemma 3, we have the second part of the Lemma 3:

$$t^{\gamma}\left|\left(D_{0+,t}^{\alpha,\beta}u_n\right)(t)\right| \le \|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]} + M\left(\lambda_n^2+\|q\|_{C[0,T]}\right)\left(t^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right| + t^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right).$$

Lemma 3 is proven.

□

Formally, from (5) by term-by-term differentiation we compose the series

$$D_{0+,t}^{\alpha,\beta}u(x,t) = \sum_{n=1}^{\infty} D_{0+,t}^{\alpha,\beta}u_n(t)\sin(\lambda_n x), \tag{11}$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} \lambda_n^2 u_n(t)\sin(\lambda_n x). \tag{12}$$

In view of Lemma 3, the series (5), (11), and (12) for any $(x,t)\in\overline{\Omega}_{lT}$,

$$M\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right| + T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right),$$

$$\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left[\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]} + M\left(\lambda_n^2+\|q\|_{C[0,T]}\right)\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right| + T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)\right],$$

$$M\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\lambda_n^2\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right|\right.$$
$$\left.+\,T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right|+\frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right),$$

where $\overline{\Omega}_{lT} := \{(x,t) : 0 \le x \le l, 0 \le t \le T\}$.

We hold the following auxiliary lemma.

Lemma 4. *If the conditions* (A1)–(A2) *then there are equalities*

$$\varphi_{n,i} = \frac{1}{\lambda_n^3}\varphi_{n,i}^{(3)}, \quad i = 1,2, \quad f_n(t) = \frac{1}{\lambda_n^3}f_n^{(3)}(t), \tag{13}$$

where

$$\varphi_{n,i}^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(3)}(x)_i\cos(\lambda_n x)dx, \quad i = 1,2,$$
$$f_n^{(3)}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xxx}^{(3)}(x,t)\cos(\lambda_n x)dx,$$

with the following estimates:

$$\sum_{n=1}^{\infty}\left|\varphi_{n,i}^{(3)}\right|^2 \le \left\|\varphi_i^{(3)}\right\|_{L_2[0,l]}, \quad \sum_{n=1}^{\infty}\left|f_n^{(3)}(t)\right|^2 \le \left\|f^{(3)}(t)\right\|_{L_2[0,l]\times C[0,T]}. \tag{14}$$

If the functions $\varphi_i(x)$, $i = 1,2$ and $f(x,t)$ satisfy the conditions of Lemma 3.2, then due to representations (13) and (14) series (5), (11), and (12) converge uniformly in the rectangle $\overline{\Omega}_{lT}$, therefore, function $u(x,t)$ satisfies relations (1)–(3).

Using the above results, we obtain the following assertion.

Lemma 5. *Let* $p(t), q(t) \in C[0,T]$, (A1)–(A2) *are satisfied*, *then there exists a unique solution of the direct problem* (1)–(3) $u(x,t) \in C_{\gamma}^{2,\alpha,\beta}(\overline{\Omega}_{lT})$.

Let us derive an estimate for the norm of the difference between the solution of the original integral Eq. (9) and the solution of this equation with perturbed functions $\tilde{p}$, $\tilde{q}$, $\tilde{\varphi}_{n,i}$, $i = 1,2$, $\tilde{f}_n$. Let $\tilde{u}_n(t)$ be solution of the integral Eq. (9) corresponding to the functions $\tilde{p}$, $\tilde{q}$, $\tilde{\varphi}_{n,i}$, $i = 1,2$, $\tilde{f}_n$; i.e.,

$$\begin{aligned}\tilde{u}_n(t) &= t^{(\beta-1)(2-\alpha)}E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right)\tilde{\varphi}_{n,1}\\ &\quad + t^{1+(\beta-1)(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right)\tilde{\varphi}_{n,2}\\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\tilde{p}(\tau)\tilde{f}_n(\tau)d\tau\\ &\quad - \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\tilde{q}(\tau)\tilde{u}_n(\tau)d\tau.\end{aligned} \tag{15}$$

Composing the difference $u-\tilde{u}$ with the help of Eqs. (9), (15) and introducing the notations $u-\tilde{u} = \bar{u}_n$, $p-\tilde{p} = \bar{p}$, $q-\tilde{q} = \bar{q}$, $f_n - \tilde{f}_n = \bar{f}_n$, we obtain the integral equation

$$\begin{aligned}\bar{u}_n(t) &= t^{(\beta-1)(2-\alpha)}E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right)\bar{\varphi}_{n,1}\\ &\quad + t^{1+(\beta-1)(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right)\bar{\varphi}_{n,2}\\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\bar{p}(\tau)f_n(\tau)d\tau\end{aligned}$$

$$+\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)\tilde{p}(\tau)\overline{f}_n(\tau)d\tau$$

$$-\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)\overline{q}(\tau)u_n(\tau)d\tau$$

$$-\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)\tilde{q}(\tau)\overline{u}_n(\tau)d\tau$$

from which we derive the following linear integral inequality for $\|\overline{u}_n(t)\|_{C_\gamma^{2,\alpha,\beta}}$:

$$\|\overline{u}_n(t)\|_{C_\gamma^{2,\alpha,\beta}} \le M\left(t^{\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,1}\right| + t^{1+\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,2}\right|\right.$$
$$\left. + \frac{\left(\|\overline{p}\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]} + \|\tilde{p}\|_{C[0,T]}\|\overline{f}_n\|_{C_\gamma[0,T]}\right)t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$
$$+\frac{\|\overline{q}\|_{C[0,T]}t^\alpha}{\Gamma(\alpha+1)}M\left(t^{\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,1}\right| + t^{1+\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,2}\right|\right.$$
$$\left.+\frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}t\right)$$
$$+\frac{\|\tilde{q}\|_{C[0,T]}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|\overline{u}_n(\tau)\right|d\tau.$$

Using Lemma 1 from last inequality, we arrive at the estimate:

$$\|\overline{u}_n(t)\|_{C_\gamma^{2,\alpha,\beta}} \le \left\{M\left(t^{\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,1}\right| + t^{1+\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,2}\right|\right.\right.$$
$$\left. + \frac{\left(\|\overline{p}\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]} + \|\tilde{p}\|_{C[0,T]}\|\overline{f}_n\|_{C_\gamma[0,T]}\right)t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$
$$+\frac{\|\overline{q}\|_{C[0,T]}t^\alpha}{\Gamma(\alpha+1)}M\left(t^{\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,1}\right| + t^{1+\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,2}\right|\right. \tag{16}$$
$$\left.\left.+\frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}t^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}t\right)\right\}$$
$$\times E_{\alpha,\gamma}\left(\left(\|\tilde{q}\|_{C[0,T]}t^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}t\right).$$

Indeed, the expression (16) is stability estimate for the solution to the problem (1)–(3). The uniqueness of this solution follows from (16).

5. INVESTIGATION OF THE INVERSE PROBLEM (1)–(4)

In this section it is studied the inverse problem as the problem of determining of functions $q(t)$ from relations (1)–(4), using the contraction mapping principle.

Let us multiply (1) by $\omega_i(x)$, $(i = 1,2)$ and integrate over x from 0 to l:

$$\int_0^l \omega_i(x) D_{0+,t}^{\alpha,\beta} u(x,t)dx - \int_0^l \omega_i(x) u_{xx} dx + q(t)\int_0^l \omega_i(x) u(x,t)dx$$
$$= p(t)\int_0^l \omega_i(x) f(x,t)dx, \quad i := 1,2, \quad x \in (0,l), \quad t \in (0,T].$$

After integrating by parts, in view of conditions (2)–(4), we obtain the equality

$$D_{0+,t}^{\alpha,\beta} h_i(t) - \omega_i(l) u_x(l,t) + \omega_i(0) u_x(0,t) - \int_0^l \omega_i''(x) u(x,t)dx + q(t)h_i(t)$$
$$= p(t)\int_0^l \omega_i(x) f(x,t)dx, \quad i := 1,2. \tag{17}$$

Suppose that the condition $\omega_i(0) = \omega_i(l) = 0$ are satisfied for (17). Solving the system (17) with respect to the unknown functions $p(t)$ and $q(t)$, we obtain the following integral equations with respect to the unknowns

$$p(t) = \frac{1}{\Delta(t)} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} (-1)^j h_j(t) \left[D_{0+,t}^{\alpha,\beta} h_i(t) - \sum_{n=1}^{\infty} u_n(t;p;q) \lambda_n^2 \omega_{in}^{(2)} \right], \tag{18}$$

$$q(t) = \frac{1}{\Delta(t)} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} (-1)^j \left[\sum_{n=1}^{\infty} f_n(t)\omega_{jn} \left(D_{0+,t}^{\alpha,\beta} h_i(t) - \sum_{n=1}^{\infty} u_n(t;p;q) \lambda_n^2 \omega_{in}^{(2)} \right) \right], \tag{19}$$

where

$$\Delta(t) = h_1(t)\sum_{n=1}^{\infty} f_n(t)\omega_{2n} - h_2(t)\sum_{n=1}^{\infty} f_n(t)\omega_{1n};$$

$$\omega_{in} = \sqrt{\frac{2}{l}}\int_0^l \omega_i(x)\sin(\lambda_n x)dx; \quad \omega_{in}^{(2)} = \frac{1}{\lambda_n^2}\sqrt{\frac{2}{l}}\int_0^l \omega_i''(x)\sin(\lambda_n x)dx.$$

Equations (18), (19) form a complete system of integral equations for the unknown functions $p(t), q(t)$. We introduce this system in the form of the operator equation

$$g(t) = \Lambda[g](t), \tag{20}$$

where $g = (g_1, g_2) := (p(t); q(t))$ is vector function. $\Lambda = (\Lambda_1, \Lambda_2)$, defining it by the right hand sides of (18), (19)

$$\Lambda_1[g](t) = g_{01}(t) - \frac{1}{\Delta(t)} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} (-1)^j h_j(t) \sum_{n=1}^{\infty} u_n(t; g_1; g_2) \lambda_n^2 \omega_{in}^{(2)},$$

$$\Lambda_2[g](t) = g_{02}(t) - \frac{1}{\Delta(t)} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} (-1)^j \left[\sum_{n=1}^{\infty} f_n(t)\omega_{jn} \sum_{n=1}^{\infty} u_n(t; g_1; g_2) \lambda_n^2 \omega_{in}^{(2)} \right].$$

Let $g_0 := (g_{01}, g_{02})$, where

$$g_{01}(t) = \frac{1}{\Delta(t)} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} (-1)^j h_j(t) D_{0+,t}^{\alpha,\beta} h_i(t);$$

$$g_{02}(t) = \frac{1}{\Delta(t)} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} (-1)^j \sum_{n=1}^{\infty} f_n(t)\omega_{jn} D_{0+,t}^{\alpha,\beta} h_i(t).$$

Fix a number $\rho > 0$ and consider the ball

$$B^T(g_0,\rho) := \left\{g : \|g - g_0\|_{C[0,T]} \le \rho\right\}.$$

Theorem. *Let* (A1)–(A4) *are satisfied. Then there exists a number* $T^* \in (0,T)$, *such that there exists a unique solution* $p(t),\ q(t) \in C[0,T^*]$ *of the inverse problem* (1)–(4).

Proof. Let us first prove that for an enough small $T > 0$ the operator Λ maps the ball $B^T(g_0,\rho)$ implies that $\Lambda[g](t) \in B^T(g_0,\rho)$. Indeed, for any continuous function $g(t)$, the function $\Lambda[g](t)$ calculated using formula (20) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\|\Lambda_1[g](t) - g_{01}(t)\| \le \frac{2\omega_0 h_0}{\Delta_0}\left|\sum_{n=1}^{\infty} u_n(T; g_1; g_2)\right|$$

$$\le \frac{2\omega_0 h_0}{\Delta_0}\sum_{n=1}^{\infty} M\left(T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,1}|\right.$$

$$\left.+\, T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,2}| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right),$$

$$\|\Lambda_2[g](t) - g_{02}(t)\| \le \frac{2\omega_0 f_0}{\Delta_0}\left|\sum_{n=1}^{\infty} u_n(T; g_1; g_2)\right|$$

$$\le \frac{2\omega_0 f_0}{\Delta_0}\sum_{n=1}^{\infty} M\left(T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,1}|\right.$$

$$\left.+\, T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,2}| + \frac{\|p\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right),$$

where $\omega_0 = \max_{x\in[0,l]}|\omega(x)|$.

Here we have used the estimate (9). In view of Lemmas 3.1 and 3.2 last series is convergent series. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $g(t)$ belongs to the ball $B^T(g_0,\rho)$ implies the inequality

$$\|g\| \le \rho + \|g_0\|. \tag{21}$$

Therefore, we only strengthen the inequality if we replace $\|g\|$ in this inequality with the expression $\rho + \|g_0\|$. Performing these replacements, we obtain the estimate

$$\|\Lambda_1[g](t) - g_{01}(t)\| \le \frac{2\omega_0 h_0}{\Delta_0}\sum_{n=1}^{\infty} M\left(T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,1}|\right.$$

$$\left.+\, T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{n,2}| + \frac{(\rho+\|g_0\|)\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left((\rho+\|g_0\|)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right),$$

$$\left\|\Lambda_2[g](t)-g_{02}(t)\right\| \le \frac{2\omega_0 f_0}{\Delta_0}\sum_{n=1}^{\infty} M\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right|\right.$$

$$\left.+\,T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right|+\frac{(\rho+\|g_0\|)\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left((\rho+\|g_0\|)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

These together with (18)–(20) imply the estimates

$$\|\Lambda[g](t)-g_0(t)\| = \max\{\|\Lambda_1[g](t)-g_{01}(t)\|,\|\Lambda_2[g](t)-g_{02}(t)\|\}$$

$$\le \max\left\{\frac{2\omega_0 h_0}{\Delta_0},\frac{2\omega_0 f_0}{\Delta_0}\right\}\sum_{n=1}^{\infty} M\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right|\right.$$

$$\left.+\,T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right|+\frac{(\rho+\|g_0\|)\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left((\rho+\|g_0\|)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

Let T_1 be a positive root of the equation

$$\max\left\{\frac{2\omega_0 h_0}{\Delta_0},\frac{2\omega_0 f_0}{\Delta_0}\right\}\sum_{n=1}^{\infty} M\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,1}\right|\right.$$

$$\left.+\,T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\varphi_{n,2}\right|+\frac{(\rho+\|g_0\|)\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)$$

$$\times E_{\alpha,\gamma}\left(\left((\rho+\|g_0\|)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)=\rho.$$

Then for $T\in(0,T_1)$ we have $\Lambda[g](t)\in B^T(g_0,\rho)$.

Now consider two functions $g(t)$ and $\tilde{g}(t)$ belonging to the ball $B^T(g_0,\rho)$ and estimate the distance between their images $\Lambda[g](t)$ and $\Lambda[\tilde{g}](t)$ in the space $C[0,T]$. The function $\tilde{u}_n(t)$ corresponding to $\tilde{g}(t)$ satisfies the integral Eq. (15) with the functions $\varphi_{n,i}=\tilde{\varphi}_{n,i}$, $i=1,2$ and $f_n=\tilde{f}_n$. Composing the difference $\Lambda[g](t)-\Lambda[\tilde{g}](t)$ with the help of Eq. (9), (15) and then estimating its norm, we obtain

$$\left\|\Lambda_1[g](t)-\Lambda_1[\tilde{g}](t)\right\| \le \frac{2\omega_0 h_0}{\Delta_0}\sum_{n=1}^{\infty}\left\|u_n(T;g_1;g_2)-\tilde{u}_n(T;\tilde{g}_1;\tilde{g}_2)\right\|$$

$$\le \frac{2\omega_0 h_0}{\Delta_0}\sum_{n=1}^{\infty}\left\{M\left(T^{\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,1}\right|+T^{1+\gamma+(\beta-1)(2-\alpha)}\left|\overline{\varphi}_{n,2}\right|\right.\right.$$

$$+\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Big(\|g_1-\tilde{g}_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}$$

$$\left.\left.+\,\|\tilde{g}_1\|_{C[0,T]}\|f_n-\tilde{f}_n\|_{C_\gamma[0,T]}\Big)\right)\right.$$

$$+\frac{\|g_2-\tilde{g}_2\|_{C[0,T]}T^{\alpha}}{\Gamma(\alpha+1)}M\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}|\overline{\varphi}_{n,1}|$$

$$+T^{1+\gamma+(\beta-1)(2-\alpha)}|\overline{\varphi}_{n,2}|+\frac{\|g_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)$$

$$\times E_{\alpha,\gamma}\left(\left(\|g_2\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg\}\times E_{\alpha,\gamma}\left(\left(\|\tilde{g}_2\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right),$$

$$\|\Lambda_2[g](t)-\Lambda_2[\tilde{g}](t)\|\le\frac{2\omega_0 f_0}{\Delta_0}\sum_{n=1}^{\infty}\|u_n(T;g_1;g_2)-\tilde{u}_n(T;\tilde{g}_1;\tilde{g}_2)\|$$

$$\le\frac{2\omega_0 f_0}{\Delta_0}\sum_{n=1}^{\infty}\Bigg\{M\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}|\overline{\varphi}_{n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}|\overline{\varphi}_{n,2}|$$

$$+\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Big(\|g_1-\tilde{g}_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}$$

$$+\|\tilde{g}_1\|_{C[0,T]}\|f_n-\tilde{f}_n\|_{C_\gamma[0,T]}\Big)\Bigg)$$

$$+\frac{\|g_2-\tilde{g}_2\|_{C[0,T]}T^{\alpha}}{\Gamma(\alpha+1)}M\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}|\overline{\varphi}_{n,1}|$$

$$+T^{1+\gamma+(\beta-1)(2-\alpha)}|\overline{\varphi}_{n,2}|+\frac{\|g_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)$$

$$\times E_{\alpha,\gamma}\left(\left(\|g_2\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg\}\times E_{\alpha,\gamma}\left(\left(\|\tilde{g}_2\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

Using inequality (10) and the estimate (16) with $\varphi_{n,i}=\tilde{\varphi}_{n,i}$, $i=1,2$ and $f_n=\tilde{f}_n$, we continue the previous inequality in following form:

$$\|\Lambda_1[g](t)-\Lambda_1[\tilde{g}](t)\|$$

$$\le\frac{2\omega_0 h_0}{\Delta_0}\sum_{n=1}^{\infty}\Bigg\{M\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg(\|g_1-\tilde{g}_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}$$
$$+\frac{M\|g_2-\tilde{g}_2\|_{C[0,T]}T^{\alpha}}{\Gamma(\alpha+1)}\frac{\|g_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg) \tag{22}$$

$$\times E_{\alpha,\gamma}\left(\left(\|g_2\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg\}\times E_{\alpha,\gamma}\left(\left(\|\tilde{g}_2\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

$$\left\|\Lambda_2[g](t)-\Lambda_2[\tilde{g}](t)\right\|$$
$$\le \frac{2\omega_0 f_0}{\Delta_0}\sum_{n=1}^{\infty}\left\{M\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\left(\left\|g_1-\tilde{g}_1\right\|_{C[0,T]}\left\|f_n\right\|_{C_\gamma[0,T]}\right.\right.$$
$$\left.+\frac{M\left\|g_2-\tilde{g}_2\right\|_{C[0,T]}T^{\alpha}}{\Gamma(\alpha+1)}\frac{\left\|g_1\right\|_{C[0,T]}\left\|f_n\right\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right) \tag{23}$$
$$\left.\times E_{\alpha,\gamma}\left(\left(\left\|g_2\right\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\right\}\times E_{\alpha,\gamma}\left(\left(\left\|\tilde{g}_2\right\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).$$

The functions $g(t)$ and $\tilde{g}(t)$ belong to the ball $B^T(g_0,\rho)$, and hence for each of these functions one has inequality (21). Note that the function on the right-hand side in inequality (22), (23) at the factor $\|g\|-\|\tilde{g}\|$ is monotone increasing with $\|g\|,\|\tilde{g}\|$, and T. Consequently, replacing $\|g\|$ and $\|\tilde{g}\|$ in inequality (22), (23) with $\rho+\|g\|$ will only strengthen the inequality. This, we have

$$\left\|\Lambda_1[g](t)-\Lambda_1[\tilde{g}](t)\right\|$$
$$\le \frac{2\omega_0 h_0}{\Delta_0}\sum_{n=1}^{\infty}\left\{M\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\left(\left\|f_n\right\|_{C_\gamma[0,T]}+\frac{MT^{\alpha}}{\Gamma(\alpha+1)}\right.\right.$$
$$\left.\left.\times\frac{\left\|g_1\right\|_{C[0,T]}\left\|f_n\right\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\left\|g_2\right\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\right\}$$
$$\times E_{\alpha,\gamma}\left(\left(\left\|\tilde{g}_2\right\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\left\|g-\tilde{g}\right\|,$$

$$\left\|\Lambda_2[g](t)-\Lambda_2[\tilde{g}](t)\right\|$$
$$\le \frac{2\omega_0 f_0}{\Delta_0}\sum_{n=1}^{\infty}\left\{M\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\left(\left\|f_n\right\|_{C_\gamma[0,T]}+\frac{MT^{\alpha}}{\Gamma(\alpha+1)}\right.\right.$$
$$\left.\left.\times\frac{\left\|g_1\right\|_{C[0,T]}\left\|f_n\right\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\right)E_{\alpha,\gamma}\left(\left(\left\|g_2\right\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\right\}$$
$$\times E_{\alpha,\gamma}\left(\left(\left\|\tilde{g}_2\right\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\left\|g-\tilde{g}\right\|.$$

Hence,

$$\left\|\Lambda[g](t)-\Lambda[\tilde{g}](t)\right\|\le \max\left\{\frac{2\omega_0 h_0}{\Delta_0},\frac{2\omega_0 f_0}{\Delta_0}\right\}$$
$$\times\sum_{n=1}^{\infty}\left\{M\frac{T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\left(\left\|f_n\right\|_{C_\gamma[0,T]}+\frac{MT^{\alpha}}{\Gamma(\alpha+1)}\right.\right.$$

$$\times \frac{\|g_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg) E_{\alpha,\gamma}\left(\left(\|g_2\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg\}$$

$$\times E_{\alpha,\gamma}\left(\left(\|\tilde{g}_2\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\|g-\tilde{g}\|.$$

Let T_2 be a positive root of the equation

$$\|\Lambda[g](t)-\Lambda[\tilde{g}](t)\| \le \max\left\{\frac{2\omega_0 h_0}{\Delta_0},\frac{2\omega_0 f_0}{\Delta_0}\right\}$$

$$\times \sum_{n=1}^{\infty}\left\{M\frac{T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg(\|f_n\|_{C_\gamma[0,T]}+\frac{MT^\alpha}{\Gamma(\alpha+1)}\right.$$

$$\times \frac{\|g_1\|_{C[0,T]}\|f_n\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg) E_{\alpha,\gamma}\left(\left(\|g_2\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg\}$$

$$\times E_{\alpha,\gamma}\left(\left(\|\tilde{g}_2\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}T\right)=1.$$

Then for $T \in (0,T_2)$ the operator Λ contracts the distance between the elements $g(t)$, $\tilde{g}(t) \in B^T(g_0,\rho)$. Consequently, if we choose $T^* < \min(T_1,T_2)$ then the operator Λ is a contraction in the ball $B^T(g_0,\rho)$. However, in accordance with the Banach theorem ([28], pp. 87–97), the operator Λ has unique fixed point in the ball $B^T(g_0,\rho)$; i.e., there exists a unique solution of Eq. (21).

□

6. CONCLUSIONS

In this work, the solvability of a nonlinear inverse problem for a time-fractional wave equation with initial-boundary conditions and integral type overdetermination conditions was studied. Firstly we investigated solvability direct problem. The (1)–(3) problem replaced by an equivalent of Volterra integral equations of the second kind. Existence and uniqueness of direct problem solution was proven. The inverse problem was considered for determining pair functions $p(t)$, $q(t)$ included in Eq. (1) with additional conditions (4) of the solution of this system with the initial and boundary conditions (2), (3). Conditions for given functions are obtained, under which the inverse problem has unique solutions for a sufficiently small interval.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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