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# Global solvability of the determination convolutional kernel in a hyperbolic system of integro-differential equations

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**Abstract.** In this paper we consider first-order hyperbolic system with memory, the inverse problem of determining the convolutional kernel. The direct problem is the initial-boundary value problem for this system on a finite segment $[0, H]$. Under certain conditions of data matching, the inverse problem is reduced to solving a system of Volterra-type integral equations for unknown functions. Further, the principle of contracted mappings is applied to this system in the space of continuous functions with a weighted norm, and a theorem on the global unique solvability of the problem is proved for any fixed $H$.

**Keywords:** Hyperbolic system, convolutional kernel, weighted norm, integral equation, contraction mapping principle.

**Mathematics Subject Classification (2010):** 41A15.

# 1 Introduction

Hyperbolic systems of partial differential equations of the first order are cover many important mathematical models found in various questions of natural science. As a rule, second-order equations are derived from them under some additional assumptions. In this regard, it is desirable to study inverse problems directly in terms of the system itself. For hyperbolic systems, inverse problems of determining the coefficients and right-hand sides of equations began to be studied from the 70s of the last century in the works of L.P. Nizhnik [1], S.P. Belinsky [2], V.G. Romanova and L.I. Slinyucheva [3].

There are physical phenomena, where not only the present state of the system is taken into account, but also all the previous positions that the given system occupied, in other words, it depends on the entire previous history. An example of such a phenomenon is the propagation of elastic waves in viscous media, in which the deformation of a viscoelastic medium depends not only on the nature of the applied forces, but also on the previous deformations to which the medium was subjected. Such an environment is called a "memory"or "aftereffect"environment [4]. Other phenomena of this kind are the propagation of electromagnetic waves in dispersive media [5], the population of animals or plants of various species in mathematical biology [6]. Mathematically, such phenomena are described mainly by a hyperbolic system of integro-differential equations with partial derivatives of the first order with an integral term of the convolution type with respect to a time variable. The problems of determining the kernel of integrals in these systems play an important role in applied sciences and relate to inverse problems. By now, the problems of determining kernels from a single integro-differential equation of the second order have been widely studied (see, for example, [7] – [15] and ite referencess).

In this paper, we investigate the inverse problem of determining the kernel in a hyperbolic system of integro-differential equations of the first order, which has the form of a matrix of dimension $n \times n$ depending on the time variable $t$. Unique solution in the global sense is proved, i.e. the existence and uniqueness of the solution take place for any segment of the definition of unknown functions.

The article is organized as follows: in the first section, the problem statements are given, the second section is devoted to the study of the direct problem in the third section, the inverse problem is reduced to the study of an equivalent system of integral equations; in the fourth section, the main result of the work is proved - the theorem of the unique solvability of the inverse problem; at the end, a list of references.

## 2 Formulation of the problem

Let us investigate a system of $n$ equations in the case of one spatial variable

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \int_0^t B(\tau)u(x,t-\tau)d\tau + f(x,t),\ (x,t) \in D, \tag{2.1}$$

where $D := \{0 \le x \le H,\ t > 0\}$, $u(x,t)$ is a vector function with components $u_1,\ u_2 \dots u_n$. Here $A,\quad B$ are square matrices of dimension $n$, moreover

$$A = diag(\lambda_1, \lambda_2, \dots, \lambda_n); \quad B(t) = (b_{ij})\,(t),\ i,j = \overline{1,n},$$

$\lambda_i$ real, various constants and

$$\lambda_i > 0, \quad i = \overline{1,s};\ \lambda_i < 0, \quad i = \overline{s+1,n}; \quad 0 \le s \le n; \tag{2.2}$$

$$f(x,t) = (f_1(x,t),\ f_2(x,t), \dots, f_n(x,t))\,.$$

In what follows, the notation of vectors are understood as a column vector.

*Statement of the direct problem:* for given matrices $A,\quad B$ and a vector function $f$, it is required to determine a vector function $u(x,t)$ that satisfies (2.1) with the following initial and boundary conditions

$$u(x,0) = \varphi(x),\ 0 \le x \le H, \tag{2.3}$$

$$u_i(0,t) = g_i(t),\ i = \overline{1,s}; \quad u_i(H,t) = g_i(t),\ i = \overline{s+1,n}, \tag{2.4}$$

where $\varphi(x) = (\varphi_1,\ \varphi_2, ..., \varphi_n)\,(x)$ and $g(t) = (g_1,\ g_2, ..., g_n)\,(t), \quad i = \overline{1,n}$ given smooth vector functions.

Throughout this article, writing a vector of functions in a product with matrices is understood as a string, if it is multiplied on the left and as a row if multiplication is on the right.

Let now consider $n$ problems of the form (2.1)–(2.4), each with its own set of functions $f,\ \varphi,\ g_i$, but with the same matrices $A,\ B$

$$\left(E\frac{\partial}{\partial t}+A\frac{\partial}{\partial x}\right)u^l=\int_0^t B(\tau)u^l\,(x,t-\tau)\,d\tau+f^l(x,t),$$

$$u^l|_{t=0}=\varphi^l(x),\ 0\le x\le H, \tag{2.5}$$

$$u_i^l(0,t)=g_i^l(t),\ i=\overline{1,s},\quad u_i^l(H,t)=g_i^l(t),\ i=\overline{s+1,n},\ l=\overline{1,n},$$

where $E-$ is the identity matrix of dimension $n\times n$.

*Inverse problem:* find the matrix $B(t)$, $t>0$, if the following information is known about direct to the problem (2.5):

$$u_i^l(0,t)=h_i^l(t),\ i=\overline{s+1,n},\quad u_i^l(H,t)=h_i^l(t),\ i=\overline{1,s},\ l=\overline{1,n}. \tag{2.6}$$

**Remark 1.** In [16, pp. 164–171] and [17, pp. 76–86] the inverse problem of determining the matrix function $B(x)$ from the hyperbolic system of equations

$$\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}+B(x)u(x,t)=f(x,t)$$

according to (2.6).

**Remark 2.** Instead of system (2.1), a more general system of equations, hyperbolic in the sense of I.G. Petrovsky [18] can be considered. But, then there is a non-degenerate transformation of functions that such a system is reduced to the form (2.1) [19] .

# 3 Investigation of the direct problem

Consider an arbitrary point $(x,t)\in D$ on the plane of variables $\xi,\tau$ and draw through it the characteristic of the $i-$ th equation of system (2.1):

$$\xi=x+\lambda_i(\tau-t) \tag{3.1}$$

before intersection in area $\tau\le t$ with edge $D$. The intersection point is denoted by $(x_0^i,t_0^i)$. For $\lambda_i>0$ (i.e. $i=\overline{1,s}$), this point lies either on the segment $[0,H]$ of the $t=0$ axis, either on the straight line $x=0$, and for $\lambda_i<0$, (i.e. $i=\overline{s+1,n}$) either on the segment $[0,H]$ or on the line $x=H$. Integrating the $i-$ th component of equality (2.1) by characteristics (3.1) from the point $(x_0^i,\ t_0^i)$ to the point $(x,t)$, we obtain

$$u_i(x,t)=u_i\left(x_0^i,t_0^i\right)+$$

$$+\int\limits_{t_0^i}^{t}\left[\int\limits_0^{\tau}\sum_{j=1}^{n} b_{ij}(\alpha)u_j(\xi,\tau-\alpha)d\alpha + f_i(\xi,\tau)\right]_{\xi=x+\lambda_i(\tau-t)} d\tau,\ i=\overline{1,n}. \tag{3.2}$$

Let us first define the value $t_0^i$ in (3.2). It depends on the coordinates of the point $(x,t)$ and on what is a $i$. It is easy to see that $t_0^i(x,t)$ has the form

$$t_0^i(x,t)=\begin{cases} t-\frac{x}{\lambda_i}, & t\geq \frac{x}{\lambda_i},\\ 0, & 0<t<\frac{x}{\lambda_i}, \quad i=\overline{1,s};\end{cases}$$
$$t_0^i(x,t)=\begin{cases} t+\frac{H-x}{\lambda_i}, & t\geq \frac{x-H}{\lambda_i},\\ 0, & 0<t<\frac{x-H}{\lambda_i},\ i=\overline{s+1,n}.\end{cases} \tag{3.3}$$

Then, from the condition that the pair $(x_0^i,\ t_0^i)$ satisfies equation (3.1) it follows that

$$x_0^i(x,t)=\begin{cases} 0, & t\geq \frac{x}{\lambda_i},\\ x-\lambda_i t, & 0<t<\frac{x}{\lambda_i}, \quad i=\overline{1,s};\end{cases}$$
$$x_0^i(x,t)=\begin{cases} H, & t\geq \frac{x-H}{\lambda_i},\\ x-\lambda_i t, & 0<t<\frac{x-H}{\lambda_i},\ i=\overline{s+1,n}.\end{cases} \tag{3.4}$$

The constant terms of integral equations (3.2) are determined through the initial and boundary conditions (2.3) and (2.4) as follows:

$$u_i(x_0^i,t_0^i)=\begin{cases} g_i(t-\frac{x}{\lambda_i}), & t\geq \frac{x}{\lambda_i},\\ \varphi_i(x-\lambda_i t), & 0\leq t<\frac{x}{\lambda_i},\ i=\overline{1,s};\end{cases}$$
$$u_i(x_0^i,t_0^i)=\begin{cases} g_i(t+\frac{H-x}{\lambda_i}), & t\geq \frac{x-H}{\lambda_i},\\ \varphi_i(x-\lambda_i t), & 0\leq t<\frac{x-H}{\lambda_i},\ i=\overline{s+1,n}.\end{cases} \tag{3.5}$$

Let the functions $u_i(x_0^i,\ t_0^i)$ be continuous in the domain $D$. Note that for these conditions to be satisfied, the given functions $\varphi_i(x)$ and $g_i(t)$ should be satisfy the following matching conditions at the corner points of the domain $D$:

$$\varphi_i(0)=g_i(0),\ i=\overline{1,s};\ \ \varphi_i(H)=g_i(0),\ i=\overline{s+1,n}. \tag{3.6}$$

Here and below, the values of functions $g_i$ at $t=0$ and functions $\varphi_i$ at $x=0$ and $x=H$ are understood as the limit at these points when the argument tends from the other side of the point, where these functions are defined.

Suppose that the functions $b_{ij}(t)$ and $f_i(x,t)$ are continuous functions of their arguments in $D$. Then the system of equations (3.2) are closed system of integral equations of the Volterra type of the second kind with continuous kernels and free terms. As usual, such a system has a unique solution in any bounded subdomain $D_T = \{(x,t) : 0 \le x \le H, \quad 0 \le t \le T\}$ of the domain $D$, where $T > 0$ is some fixed number.

In order to study the properties of the first derivatives of the functions $u_i(x,t)$, we differentiate equation (3.2). We have

$$\frac{\partial}{\partial x}u_i(x,t) = \frac{\partial}{\partial x}u_i\left(x_0^i,\ t_0^i\right) - f_i\left(x_0^i,\ t_0^i\right)\frac{\partial t_0^i}{\partial x} + \\ + \int\limits_{t_0^i}^{t}\frac{\partial}{\partial x}f_i(x+\lambda_i(\tau-t),\ \tau)d\tau - \int\limits_0^{t_0^i}\sum_{j=1}^n b_{ij}(\tau)u_j\left(x_0^i,\ t_0^i-\tau\right)d\tau\frac{\partial t_0^i}{\partial x} + \\ + \int\limits_{t_0^i}^{t}\int\limits_0^{\tau}\sum_{j=1}^n b_{ij}(\alpha)\frac{\partial}{\partial x}u_j(x+\lambda_i(\tau-t),\ \tau-\alpha)d\alpha d\tau,\ i=\overline{1,n}. \quad (3.7)$$

For $i=\overline{1,s}$, (3.7) are equivalent to the following equations:

$$\frac{\partial}{\partial x}u_i(x,t) = -\frac{1}{\lambda_i}\frac{d}{dt}g_i\left(t-\frac{x}{\lambda_i}\right) + \frac{1}{\lambda_i}f_i\left(0,\ t-\frac{x}{\lambda_i}\right) + \\ + \int\limits_{t-\frac{x}{\lambda_i}}^{t}\frac{\partial}{\partial x}f_i(x+\lambda_i(\tau-t),\ \tau)d\tau + \frac{1}{\lambda_i}\int\limits_0^{t-\frac{x}{\lambda_i}}\sum_{j=1}^n b_{ij}(\tau)u_j\left(0,\ t-\frac{x}{\lambda_i}-\tau\right)d\tau + \\ + \int\limits_{t-\frac{x}{\lambda_i}}^{t}\int\limits_0^{\tau}\sum_{j=1}^n b_{ij}(\alpha)\frac{\partial}{\partial x}u_j(\xi,\tau-\alpha)d\alpha d\tau, \quad t \ge \frac{x}{\lambda_i};$$

$$\frac{\partial}{\partial x}u_i(x,t) = -\lambda_i\frac{d}{dx}\varphi_i(x-\lambda_i t) + \int\limits_0^{t}\frac{\partial}{\partial x}f_i(x+\lambda_i(\tau-t),\ \tau)d\tau + \\ + \int\limits_0^{t}\int\limits_0^{\tau}\sum_{j=1}^n b_{ij}(\alpha)\frac{\partial}{\partial x}u_j(\xi,\tau-\alpha)d\alpha d\tau, \quad 0 \le t < \frac{x}{\lambda_i}.$$

These equalities show that, possible discontinuities of the first kind of functions $\frac{\partial u_i}{\partial x}$ for $i=\overline{1,s}$ are only characteristics $x=\lambda_i t$.

Requiring the fulfillment of the conditions of agreement

$$f_i(0,0) - \lambda_i \Big[\frac{d}{dx}\varphi_i\Big]_{x=0} = \Big[\frac{d}{dt}g_i(t)\Big]_{t=0}, \quad i=\overline{1,s}, \tag{3.8}$$

we have that the functions $\partial u_i/\partial x$, defined as solutions of equations (3.7) for $i=\overline{1,s}$ to be continuous in the domain $D_T$.

Proceeding in the same way for $i=\overline{s+1,n}$, we obtain that if the matching conditions are follow

$$f_i(H,0) - \lambda_i \frac{d}{dx}\varphi_i^l\Big|_{x=H} = \frac{d}{dt}g_i(t)\Big|_{t=0}, \quad i=\overline{s+1,n}, \tag{3.9}$$

then the functions $\frac{\partial u_i}{\partial x}$ for $i=\overline{s+1,n}$ will also be continuous in the domain $D_T$.

Thus, under conditions (3.8), (3.9) equalities (3.7) are a system of integral equations of Volterra type with continuous free terms (the first four terms on the right-hand side of (3.7)) and kernels. According to the theory it is known that such a system has a unique continuous solution.

Further, we denote the vector function $v(x,t) := (\partial/\partial t)u(x,t)$. Let $u(x,t)$ be a solution of problem (2.1), (2.3), (2.4).

To obtain a problem for a function $v(x,t)$ similar to (2.1), (2.3), (2.4), we differentiate equation (2.1) and boundary conditions (2.4) with respect to the variable $t$, and using be condition at $t=0$ is found equation (2.1) and initial condition (2.3). In this case, we get

$$\left(E\frac{\partial}{\partial t} + A\frac{\partial}{\partial x}\right)v = B(t)\varphi(x) + \int_0^t B(\tau)v(x,t-\tau)d\tau + \frac{\partial}{\partial t}f(x,t), \tag{3.10}$$

$$v\big|_{t=0} = f(x,0) - A\frac{d}{dx}\varphi(x), \ 0\le x\le H, \tag{3.11}$$

$$v_i(0,t) = \frac{d}{dt}g_i(t), \ i=\overline{1,s}; \ \ v_i(H,t) = \frac{d}{dt}g_i(t), \ i=\overline{s+1,n}. \tag{3.12}$$

Integration along the corresponding characteristics again leads to problem (3.10)-(3.12) to the integral equations

$$v_i(x,t) = v_i\left(x_0^i,\ t_0^i\right) + \int_{t_0^i}^t \frac{\partial}{\partial\tau}f_i(x+\lambda_i(\tau-t),\ \tau)d\tau + \int_{t_0^i}^t \sum_{j=1}^n b_{ij}(\tau)\varphi_j(x+\lambda_i(\tau-t))d\tau +$$

$$+\int_{t_0^i}^t \int_0^\tau \sum_{j=1}^n b_{ij}(\alpha)v_j(x+\lambda_i(\tau-t),\ \tau-\alpha)d\alpha d\tau, \tag{3.13}$$

where $v_i\left(x_0^i,\ t_0^i\right)$ are determined by the formulas:

$$v_i(x_0^i, t_0^i) = \begin{cases} \frac{d}{dt} g_i(t - \frac{x}{\lambda_i}), & t \geq \frac{x}{\lambda_i}, \\ f_i(x - \lambda_i t, 0) - \lambda_i \frac{d}{dt}\varphi_i(x - \lambda_i t), & 0 \leq t < \frac{x}{\lambda_i}, \quad i = \overline{1, s}; \end{cases}$$

$$v_i(x_0^i, t_0^i) = \begin{cases} \frac{d}{dt} g_i(t + \frac{H-x}{\lambda_i}), & t \geq \frac{x-H}{\lambda_i}, \\ f_i(x - \lambda_i t, 0) - \lambda_i \frac{d}{dt}\varphi_i(x - \lambda_i t), & 0 \leq t < \frac{x-H}{\lambda_i}, \quad i = \overline{s+1, n}. \end{cases}$$

It is easy to see that the conditions for matching the initial (3.11) and boundary (3.12) data at the corner points of the domain $D$ coincide with relations (3.8) and (3.9). Hence, it is clear that if the same equalities (3.8) and (3.9) are satisfied, then integral equations (3.13) have a unique continuous solutions $v_i(x,t)$, or the same $(\partial/\partial t)u_i(x,t)$.

So, we have proved the following theorem:

**Theorem 3.1.** *Let $\varphi(x) \in C^1\left[0,\ H\right]$, $B(t) \in C^1\left[0,\ \infty\right)$, $g(t) \in C^1\left[0,\ \infty\right)$, $f(x,t) \in C^1\left(D\right)$, and the matching conditions (3.6), (3.8), (3.9) are hold. Then, there is a unique classical solution to the problem (2.1)-(2.3) in the domain $D$.*

# 4 Investigation of the inverse problem. Derivation of an equivalent system of integral equations

Consider an arbitrary point $(x,0) \in D$ and draw characteristic (3.1) through it until the lateral boundaries cross the domain $D$. Integrating the $i-$component of equations (3.10) and using differentiated data (3.2) with respect to the variable $t$, we find

$$v_i^l(x,0) - \frac{d}{dt}h_i^l(t_i(x)) = \int\limits_0^{t_i(x)} \sum_{j=1}^n b_{ij}(\tau)\varphi_j^l(x + \lambda_i\tau)d\tau +$$

$$+ \int\limits_0^{t_i(x)} \int\limits_0^{\tau} \sum_{j=1}^n b_{ij}(\alpha)v^l(x + \lambda_i\tau, \tau - \alpha)d\alpha d\tau + \int\limits_0^{t_i(x)} \frac{\partial}{\partial t} f^l(x + \lambda_i\tau, \tau)d\tau, \quad (4.1)$$

where

$$t_i(x) = \frac{1}{|\lambda_i|} \begin{cases} H - x, & i = \overline{1, s}; \\ x, & i = \overline{s+1, n}. \end{cases}$$

Taking into account (3.11), we rewrite (4.1) in the form

$$\int_0^{t_i(x)} \sum_{j=1}^n b_{ij}(\tau)\varphi_j^l(x+\lambda_i\tau)d\tau + \int_0^{t_i(x)} \int_0^{\tau} \sum_{j=1}^n b_{ij}(\alpha)v^l(x+\lambda_i\tau,\tau-\alpha)d\alpha d\tau =$$

$$= f_i^l(x,0) - \lambda_i \frac{d}{dx}\varphi_i^l(x) - \frac{d}{dt}h_i^l(t_i(x)) - \int_0^{t_i(x)} \frac{\partial}{\partial t} f^l(x+\lambda_i\tau,\tau)d\tau.$$

Let us differentiate this equation with respect to the variables $x$. Then we have

$$-\frac{1}{\lambda_i}\sum_{j=1}^n b_{ij}(t_i(x))\varphi_j^l(x+\lambda_i t_i(x)) + \int_0^{t_i(x)} \sum_{j=1}^n b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(x+\lambda_i\tau)d\tau -$$

$$-\frac{1}{\lambda_i}\int_0^{t_i(x)} \sum_{j=1}^n b_{ij}(\tau)v_j^l(x+\lambda_i t_i(x), t_i(x)-\tau)d\tau +$$

$$+\int_0^{t_i(x)} \int_0^{\tau} \sum_{j=1}^n b_{ij}(\alpha)\frac{\partial}{\partial x}v_j^l(x+\lambda_i\tau,\tau-\alpha)d\alpha d\tau = \frac{\partial}{\partial x} f_i^l(x,0) -$$

$$-\lambda_i\frac{d^2}{dx^2}\varphi_i^l(x) - \frac{1}{\lambda_i}\frac{d^2}{dt^2}h_i^l(t_i(x)) + \frac{1}{\lambda_i}\frac{\partial}{\partial t}f^l(x+\lambda_i t_i(x), t_i(x)) +$$

$$+\int_0^{t_i(x)} \frac{\partial^2}{\partial t\partial x} f_i^l(x+\lambda_i\tau,\tau)d\tau,\ i,l=\overline{1,n}. \quad (4.2)$$

Let us write equation (4.2) in the following form:

$$\sum_{j=1}^n b_{ij}\left(\frac{H-x}{\lambda_i}\right)\varphi_j^l(H) = F_i^l(x) - \int_0^{\frac{H-x}{\lambda_i}} \sum_{j=1}^n b_{ij}(\tau)G_j^l\left(\frac{H-x}{\lambda_i}-\tau\right)d\tau +$$

$$+\lambda_i \int_0^{\frac{H-x}{\lambda_i}} \sum_{j=1}^n b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(x+\lambda_i\tau)d\tau +$$

$$+\lambda_i \int_0^{\frac{H-x}{\lambda_i}} \int_0^{\tau} \sum_{j=1}^n b_{ij}(\alpha)\frac{\partial}{\partial x}v_j^l(x+\lambda_i\tau,\tau-\alpha)d\alpha d\tau,\ i=\overline{1,s}; \quad (4.3)$$

$$\sum_{j=1}^{n} b_{ij}\left(-\frac{x}{\lambda_i}\right)\varphi_j^l(0) = F_i^l(x) - \int_0^{-\frac{x}{\lambda_i}} \sum_{j=1}^{n} b_{ij}(\tau) G_j^l\left(-\frac{x}{\lambda_i} - \tau\right) d\tau +$$

$$+\lambda_i \int_0^{-\frac{x}{\lambda_i}} \sum_{j=1}^{n} b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(x+\lambda_i\tau)d\tau +$$

$$+\lambda_i \int_0^{-\frac{x}{\lambda_i}} \int_0^{\tau} \sum_{j=1}^{n} b_{ij}(\alpha)\frac{\partial}{\partial x}v_j^l(x+\lambda_i\tau, \tau-\alpha)d\alpha d\tau,\ i=\overline{s+1,n}, \quad (4.4)$$

where $l=\overline{1,n}$, $F_i(x)$ denote by:

$$F_i(x) := -\lambda_i \frac{\partial}{\partial x} f_i^l(x,0) + \lambda_i^2 \frac{d^2}{dx^2}\varphi_i^l(x) + \frac{d^2}{dt^2}h_i^l(t_i(x)) -$$

$$-\frac{\partial}{\partial t} f^l(x+\lambda_i t_i(x), t_i(x)) - \lambda_i \int_0^{t_i(x)} \frac{\partial^2}{\partial t \partial x} f_i^l(x+\lambda_i\tau, \tau)d\tau,$$

$$G_j^l(\cdot) := \begin{cases} \frac{d}{dt}h_j^l(\cdot), & j=\overline{1,s}, \\ \frac{d}{dt}g_j^l(\cdot), & j=\overline{s+1,n},\ l=\overline{1,n}. \end{cases}$$

In what follows, in equations (4.3) and (4.4), we replace $\frac{H-x}{\lambda_i}$ and $-\frac{x}{\lambda_i}$ by $t$. Then we get

$$\sum_{j=1}^{n} b_{ij}(t)\varphi_j^l(H) = F_i^l(H-\lambda_i t) - \int_0^{t} \sum_{j=1}^{n} b_{ij}(\tau) G_j^l(t-\tau)\, d\tau +$$

$$+\lambda_i \int_0^{t} \sum_{j=1}^{n} b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(H-\lambda_i(t-\tau))d\tau +$$

$$+\lambda_i \int_0^{t} \int_0^{\tau} \sum_{j=1}^{n} b_{ij}(\alpha)\frac{\partial}{\partial x}v_j^l(H-\lambda_i(t-\tau), \tau-\alpha)d\alpha d\tau,\ i=\overline{1,s}; \quad (4.5)$$

$$\sum_{j=1}^{n} b_{ij}(t)\varphi_j^l(0) = F_i^l(-\lambda_i t) - \int\limits_0^t \sum_{j=1}^{n} b_{ij}(\tau) G_j^l\,(t-\tau)\,d\tau +$$

$$+\lambda_i \int\limits_0^t \sum_{j=1}^{n} b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(-\lambda_i(t-\tau))d\tau +$$

$$+\lambda_i \int\limits_0^t \int\limits_0^\tau \sum_{j=1}^{n} b_{ij}(\alpha)\frac{\partial}{\partial x}v_j^l(-\lambda_i(t-\tau),\tau-\alpha)d\alpha d\tau,\ i=\overline{s+1,n},$$

where $t \in [0, \frac{H}{|\lambda_i|}]$, $i=\overline{1,n}$.

Let $\Phi(x)$ be the matrix function by columns $\varphi^l(x)$, $l=\overline{1,n}$:

$$\Phi(x) = \left(\varphi^1, \varphi^2, \ldots, \varphi^n\right)(x).$$

In what follows, we will assume that the conditions

$$\det\Phi(0)\neq 0, \quad \det\Phi(H)\neq 0. \tag{4.6}$$

Solving the system of (4.5) with respect to $b_{il}(t)$, we obtain

$$b_{il}(t) = \frac{1}{\det\Phi(\nu_i)}\sum_{j=1}^{n}\Bigg[F_i^j(\bar{t}_i(t)) -$$

$$-\int\limits_0^t \sum_{k=1}^{n} b_{ik}(\tau)\left(G_k^j\,(t-\tau) - \lambda_i\frac{d}{dx}\varphi_k^j(\bar{t}_i(t)+\lambda_i\tau)\right)d\tau\Bigg]\Psi_j^l(\nu_i) -$$

$$-\frac{\lambda_i}{\det\Phi(\nu_i)}\int_0^t \int\limits_0^\tau \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ik}(\alpha)\frac{\partial}{\partial x}v_k^j(\bar{t}_i(t)+\lambda_i\tau,\tau-\alpha)d\alpha d\tau\Psi_j^l(\nu_i), \tag{4.7}$$

$i=\overline{1,n}$, $l=\overline{1,n}$, where

$$\nu_i = \begin{cases} H, & i=\overline{1,s}, \\ 0, & i=\overline{s+1,n}, \end{cases} \qquad \bar{t}_i(t) = \begin{cases} H-\lambda_i t, & i=\overline{1,s}, \\ -\lambda_i t, & i=\overline{s+1,n}. \end{cases}$$

Equations (4.7) include unknown functions $(\partial/\partial x)v_j^l$, $j,l=\overline{1,n}$. For them, we obtain integral equations from (3.13) using differentiation with respect to the variable $x$, having previously rewritten them for each problem with numbers $l$ labeled with a superscript. Moreover, we have

$$\frac{\partial}{\partial x}v_i^l(x,t) = \frac{\partial}{\partial x}v_i^l(x_0^i,t_0^i) -$$

$$-\sum_{j=1}^{n} b_{ij}(t_0^i)\varphi_j^l(x_0^i)\frac{\partial}{\partial x}t_0^i - f_i^l(x_0^i,t_0^i)\frac{\partial}{\partial x}t_0^i + \int\limits_{t_0^i}^{t}\frac{\partial^2}{\partial\tau\partial x}f_i^l(x+\lambda_i(\tau-t),\tau)d\tau +$$

$$+\int\limits_{t_0^i}^{t}\sum_{j=1}^{n} b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(x+\lambda_i(\tau-t))d\tau + \int\limits_{0}^{t_0^i}\sum_{j=1}^{n} b_{ij}(\tau)G\left(t_0^i-\tau\right)\frac{\partial}{\partial x}t_0^i d\tau +$$

$$+\int\limits_{t_0^i}^{t}\int\limits_{0}^{\tau}\sum_{j=1}^{n} b_{ij}(\alpha)\frac{\partial}{\partial x}v_j^l(x+\lambda_i(\tau-t),\tau-\alpha)d\alpha d\tau, \quad i,l=\overline{1,n}. \quad (4.8)$$

In integral equations (4.8), the requirement of continuity of free terms in $D$ entails matching conditions, which would also include the values of the elements of matrix $B$ at the points $t=0$, $t=H$, which is illogical from the point of view of the inverse problem. Then integral equations (4.8) have piecewise continuous free terms and continuous kernels in a domain $D$. Consequently, functions $\partial v_i^l/\partial x$, $i=\overline{1,s}$, as solutions of these equations, are discontinuous at point $(0,0)$, functions $\partial v_i^l/\partial x$, $i=\overline{s+1,n}$, are discontinuous at point $(0,H)$. These breaks will spread according to characteristics into the area $D$. Thus, each component $\partial v_i^l/\partial x$ of function $\partial v^l/\partial x$ turns out to be discontinuous along characteristic (3.1) outgoing either from the point $(0,0)$ for $i=\overline{1,s}$ or from the point $(0,H)$ for $i=\overline{s+1,n}$.

We transfer the second term on the right-hand side of (4.8), which contains the elements of the matrix $B$, to the left-hand side and introduce into consideration the new functions

$$p_i^l(x,t) := \frac{\partial}{\partial x}v_i^l(x,t) + \sum_{j=1}^{n} b_{ij}(t_0^i)\varphi_j^l(x_0^i)\frac{\partial}{\partial x}t_0^i.$$

Then from (4.1) follows

$$p_i^l(x,t) = \frac{\partial}{\partial x}v_i^l(x_0^i,t_0^i) - f_i^l(x_0^i,t_0^i)\frac{\partial}{\partial x}t_0^i +$$

$$+\int_{t_0^i}^{t}\frac{\partial^2}{\partial\tau\partial x}f_i^l(x+\lambda_i(\tau-t),\tau)d\tau + \int\limits_{0}^{t_0^i}\sum_{j=1}^{n} b_{ij}(\tau)G\left(t_0^i-\tau\right)\frac{\partial}{\partial x}t_0^i d\tau +$$

$$+\int\limits_{t_0^i}^{t}\sum_{j=1}^{n} b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(x+\lambda_i(\tau-t))d\tau + \int\limits_{t_0^i}^{t}\int\limits_{0}^{\tau}\sum_{j=1}^{n} b_{ij}(\alpha)\Big[p_j^l(\xi,\tau-\alpha) -$$

$$-\sum_{k=1}^{n} b_{jk}\left(t_0^i(\xi,\tau-\alpha)\right)\varphi_k^l\left(x_0^i(\xi,\tau-\alpha)\right)\frac{\partial}{\partial x}t_0^i\Bigg]_{\xi=x+\lambda_i(\tau-t)} d\alpha d\tau, \qquad (4.9)$$

$i,l=\overline{1,n}$.
Let $i=\overline{1,s},\ l=\overline{1,n}$. Then equations (4.9) are equivalent to the following equations:

$$p_i^l(x,t)=-\frac{1}{\lambda_i}\frac{d^2}{dt^2}g_i^l\left(t-\frac{x}{\lambda_i}\right)+\frac{1}{\lambda_i}f_i^l\left(0,t-\frac{x}{\lambda_i}\right)+$$

$$\int_{t-\frac{x}{\lambda_i}}^{t}\frac{\partial^2}{\partial\tau\partial x}f_i^l(x+\lambda_i(\tau-t),\tau)d\tau+\int\limits_{t-\frac{x}{\lambda_i}}^{t}\sum_{j=1}^{n}b_{ij}(\tau)\left[\frac{d}{dx}\varphi_j^l(x+\lambda_i(\tau-t))+\right.$$

$$\left.+\frac{1}{\lambda_i}v_j^l\left(0,t-\frac{x}{\lambda_i}-\tau\right)\right]d\tau+\int\limits_{t-\frac{x}{\lambda_i}}^{t}\int\limits_{0}^{\tau}\sum_{j=1}^{n}b_{ij}(\alpha)\left[p_j^l(x+\lambda_i(\tau-t),\tau-\alpha)+\right.$$

$$\left.+\frac{1}{\lambda_i}\sum_{k=1}^{n}b_{jk}\left(\tau-\alpha-\frac{x}{\lambda_i}\right)\varphi_k^l\left(\tau-\alpha-\frac{x}{\lambda_i}\right)\right]d\alpha d\tau,\quad t\geq\frac{x}{\lambda_i}; \qquad (4.10)$$

$$p_i^l(x,t)=\frac{\partial}{\partial x}f_i^l(x-\lambda_i t,0)-\lambda_i\frac{d^2}{dx^2}\varphi_i^l(x-\lambda_i t)+$$

$$+\int_0^t\frac{\partial^2}{\partial\tau\partial x}f_i^l(x+\lambda_i(\tau-t),\tau)d\tau+\int\limits_0^t\sum_{j=1}^{n}b_{ij}(\tau)\frac{d}{dx}\varphi_j^l(x+\lambda_i(\tau-t))d\tau+$$

$$+\int\limits_0^t\int\limits_0^\tau\sum_{j=1}^{n}b_{ij}(\alpha)p_j^l(x+\lambda_i(\tau-t),\tau-\alpha)d\alpha d\tau,\quad 0\leq t<\frac{x}{\lambda_i}. \qquad (4.11)$$

It can be seen from equalities (4.10) and (4.11) that possible discontinuities of the first kind of functions $p_i^l(x,t)$ can occur only on characteristics $x=\lambda_i t$.

Requiring the fulfillment of the matching conditions

$$-\frac{1}{\lambda_i}\frac{d^2}{dt^2}g_i^l(0)+\frac{1}{\lambda_i}f_i^l(0,0)=\frac{\partial}{\partial x}f_i^l(0,0)-\lambda_i\frac{d^2}{dx^2}\varphi_i^l(0), \qquad (4.12)$$

we have that functions $p_i^l(x,t)$, defined as solutions of integral equations (4.9) for $i=\overline{1,s},\ l=\overline{1,n}$ will be continuous in the domain $D$. Proceeding in the same way for $i=\overline{s+1,n},\ l=\overline{1,n}$, we obtain that if the matching conditions of the form

$$-\frac{1}{\lambda_i}\frac{d^2}{dt^2}g_i^l(0)+\frac{1}{\lambda_i}f_i^l(H,0)=\frac{\partial}{\partial x}f_i^l(H,0)-\lambda_i\frac{d^2}{dx^2}\varphi_i^l(H), \qquad (4.13)$$

then the functions $p_i^l(x,t)$ for $i=\overline{s+1,n},\ l=\overline{1,n}$ will be continuous in the domain $D$.

# 5 Main result and its proof

The main result of this work is the following statement:

**Theorem 5.1.** *Let $\varphi(x) \in C^2[0,H]$, $g(t) \in C^2[0,\infty)$, $f(x,t) \in C^1(D)$, $h_i^l(t) \in C^2(0,\infty)$ and condition (4.6), matching conditions (3.6), (3.8), (3.9), equalities (4.12),(4.13) are hold. Then, for all $H>0$ on the segment $\left[0, \frac{H}{\mu}\right]$, $\mu = \min_{1\le i\le n}|\lambda_i|$ there exists a unique solution to the inverse problem (2.5),(2.6) in the class $B(t) \in C\left[0, \frac{H}{\mu}\right]$, and each component $b_{il}(t)$ is determined by $h_i^l(t)$ for $t \in \left[0, \frac{H}{|\lambda_i|}\right]$.*

Equations (3.13), written for each $\vartheta_i^l(x,t)$ with numbers $l=\overline{1,n}$, together with (4.7) and (4.9), form a closed system of integral equations of the Volterra type of the second kind with respect to unknowns $\vartheta_i^l(x,t)$, $b_{il}(t)$, $p_i^l(x,t)$ $i,l=\overline{1,n}$. Consider now a rectangular area

$$D(\mu) := \left\{(x,t) : 0 \le x \le H,\, 0 \le t \le \frac{H}{\mu}\right\}.$$

Equations (3.13),(4.7) and (4.8) show that the values of the functions $\vartheta_i^l(x,t)$, $b_{il}(t)$, $p_i^l(x,t)$ at $(x,t)\in D(\mu)$ are expressed in terms of integrals of some combinations of the same functions over the intervals lying in $D(\mu)$. Let us write them in the form of an operator equation. For this, we introduce vector functions $\psi(x,t) = \left(\psi_{il}^1,\ \psi_{il}^2,\ \psi_{il}^3,\quad i,l=\overline{1,n}\right)$, into consideration, specifying their components by the equalities:

$$\psi_{il}^1(x,t) = \vartheta_i^l(x,t), \quad \psi_{il}^2(x,t) = \psi_{il}^2(t) = b_{il}(t), \quad \psi_{il}^3(x,t) = p_i^l(x,t).$$

Then system of equations (3.13),(4.7) and (4.9) takes the operator-vector form

$$\psi = U\psi, \tag{5.1}$$

where the operator $U = \left(U_{il}^1,\quad U_{il}^2,\quad U_{il}^3,\quad i,l=\overline{1,n}\right)$ in accordance with the right-hand sides of equations (3.13),(4.7) and (4.9) is defined by the relations

$$U_{il}^1\psi = \psi_{il}^{10}(x,t)+$$

$$+\int_{t_0^i}^{t}\sum_{j=1}^{n}\psi_{ij}^2(\tau)\varphi_j^l(x+\lambda_i(\tau-t))d\tau+\int_{t_0^i}^{t}\int_0^{\tau}\sum_{j=1}^{n}\psi_{ij}^2(\alpha)\psi_{jl}^1(x+\lambda_i(\tau-t),\tau-\alpha)d\alpha d\tau, \tag{5.2}$$

$$U_{il}^2\psi = \psi_{il}^{20}(x,t) - \frac{1}{\det \Phi(\nu_i)} \int\limits_0^t \sum_{j=1}^n \sum_{k=1}^n \psi_{ik}^2(\tau) \Big[ G_k^j(t-\tau) -$$

$$- \lambda_i \frac{d}{dx}\varphi_k^j(\bar{t}_i(t) + \lambda_i\tau) \Big] d\tau \Psi_j^l(\nu_i) - \frac{\lambda_i}{\det \Phi(\nu_i)} \int_0^t \sum_{j=1}^n \sum_{k=1}^n \int_0^\tau \psi_{ik}^2(\alpha) \Big[ \psi_{kj}^3(\xi, \tau-\alpha) -$$

$$- \sum_{p=1}^n \psi_{kp}^2 \left(t_0^i(\xi,\tau-\alpha)\right) \varphi_p^j \left(x_0^i(\xi,\tau-\alpha)\right) \times \frac{\partial}{\partial x} t_0^i \Big]_{\xi=\bar{t}_i(t)+\lambda_i\tau} d\alpha d\tau \Psi_j^l(\nu_i), \quad (5.3)$$

$$U_{il}^3\psi = \psi_{il}^{30}(x,t) + \int\limits_{t_0^i}^t \sum_{j=1}^n \psi_{ij}^2(\tau) \frac{d}{dx}\varphi_j^l(x+\lambda_i(\tau-t))d\tau +$$

$$+ \int\limits_0^{t_0^i} \sum_{j=1}^n \psi_{ij}^2(\tau) G\left(t_0^i - \tau\right) \frac{\partial}{\partial x} t_0^i d\tau + \int\limits_{t_0^i}^t \int\limits_0^\tau \sum_{j=1}^n \psi_{ij}^2(\alpha) \Big[ \psi_{jl}^3(\xi,\tau-\alpha) -$$

$$- \sum_{k=1}^n \psi_{jk}^2 \left(t_0^i(\xi,\tau-\alpha)\right) \varphi_k^l \left(t_0^i(\xi,\tau-\alpha)\right) \frac{\partial}{\partial x} t_0^i \Big]_{\xi=x+\lambda_i(\tau-t)} d\alpha d\tau, \quad (5.4)$$

$i = \overline{1,n}$. In these formulas, we have introduced the notation

$$\psi_{il}^{10}(x,t) = v_i^l\left(x_0^i, t_0^i\right) + \int_{t_0^i}^t f_i^l(x+\lambda_i(\tau-t),\tau)d\tau,$$

$$\psi_{il}^{20}(t) = \frac{1}{\det \Phi(\nu_i)} \sum_{j=1}^n F_i^j(\bar{t}_i(t))\Psi_j^l(\nu_i),$$

$$\psi_{il}^{30}(t) = \frac{\partial}{\partial x} v_i^l(x_0^i, t_0^i) -$$

$$- f_i^l(x_0^i, t_0^i) \frac{\partial}{\partial x} t_0^i + \int_{t_0^i}^t \frac{\partial^2}{\partial\tau\partial x} f_i^l(x+\lambda_i(\tau-t),\tau)d\tau. \quad (5.5)$$

We define the Banach space $C_\sigma(D(\mu))$ of continuous functions on the set $D(\mu)$ generated by the family of weighted norms

$$\|\psi\|_\sigma = \max_{i,l,s} \sup_{(x,t)\in D(\mu)} \left|\psi_{il}^s(x,t)e^{-\sigma t}\right|,$$

$i,l = \overline{1,n}$; $s = 1,2,3$, $\sigma \geq 0$ is a number that will be chosen later. Obviously, for $\sigma = 0$ this space coincides with the space of continuous functions with the usual norm $\|\psi\|_\sigma$. Due to the inequality

$$e^{-\sigma \frac{H}{\mu}} \|\psi\|_\sigma \leq \|\psi\|_\sigma \leq \|\psi\|,$$

norms $\|\psi\|_\sigma$ and $\|\psi\|$ are equivalent for any fixed $H \in (0, \infty)$.

Next, consider the set of functions $S\left(\psi^0, \rho\right) \subset C_\sigma\left(D(\mu)\right)$, satisfying the inequality

$$\|\psi - \psi^0\|_\sigma \leq \rho, \tag{5.6}$$

where the vector is a function $\psi^0(x,t) = \left(\psi_{il}^{10}(x,t),\, \psi_{il}^{20}(t),\, \psi_{il}^{30}(x,t),\, i,l = \overline{1,n}\right)$, the components of which are defined by formulas (5.6). It is easy to see that for $\psi \in S\left(\psi^0, \rho\right)$ the estimate $\|\psi\|_\sigma \leq \|\psi^0\|_\sigma + \rho \leq \|\psi^0\| + \rho := \rho_0$ holds. So $\rho_0$ is a known number.

Let us introduce the following notation:

$$\varphi_0 := \max_{1\leq i,l\leq n} \left\|\varphi_i^l\right\|_{C^2[0,H]},\ g_0 := \max_{1\leq i,l\leq n} \left\|g_i^l\right\|_{C^2\left[0,\frac{H}{\mu}\right]},\ f_0 := \max_{1\leq i,l\leq n} \left\|f_i^l\right\|_{C^1[D(\mu)]},$$

$$h_0 := \max_{1\leq i,l\leq n} \left\|h_i^l\right\|_{C^2\left[0,\frac{H}{\mu}\right]},\ \Gamma_0 := \max\left\{g_0,\ f_0\right\},\ \Phi_0 := \min\left\{|\Phi(0)|,\ |\Phi(H)|\right\},$$

$$\lambda_0 = \max_{1\leq i\leq n} |\lambda_i|,\ \Psi_0 := \max\left\{\max_{1\leq i,l\leq n}\left|\Psi_j^l(0)\right|,\ \max_{1\leq i,l\leq n}\left|\Psi_j^l(H)\right|\right\}.$$

The operator $U$ takes the space $C_\sigma\left(D(\mu)\right)$ into itself. Let us show that for a suitable choice of $\sigma$ (note that $H > 0$ is an arbitrary fixed number), it is a contraction operator on the set $S\left(\psi^0, \rho\right)$. First, let us verify that the operator $U$ takes the set $S\left(\psi^0, \rho\right)$ into itself, that is, condition $\psi(x,t) \in S\left(\psi^0, \rho\right)$ implies that $U\psi \in S\left(\psi^0, \rho\right)$, if $\sigma$ satisfies some restrictions. In fact, for any $(x,t) \in D(\mu)$ and any $\psi \in S\left(\psi^0, \rho\right)$, the following inequalities hold:

$$\left|\left(U_{il}^1\psi - \psi_{il}^{10}\right)e^{-\sigma t}\right| = \left|\int\limits_{t_0^i}^{t} \sum_{j=1}^{n} \psi_{ij}^2(\tau)e^{-\sigma\tau}\varphi_j^l(x+\lambda_i(\tau-t))e^{-\sigma(t-\tau)}d\tau + \right.$$

$$\left. + \int\limits_{t_0^i}^{t} e^{-\sigma(t-\tau)} \int\limits_{0}^{\tau} \sum_{j=1}^{n} \psi_{ij}^2(\alpha)e^{-\sigma\alpha}\psi_{jl}^1(x+\lambda_i(\tau-t), \tau-\alpha)e^{-\sigma(\tau-\alpha)}d\alpha d\tau\right| \leq$$

$$\leq n\left[\varphi_0\|\psi\|_\sigma + \tau\|\psi\|_\sigma^2\right]\int_0^t e^{-\sigma(t-\tau)}d\tau \leq \frac{1}{\sigma}n\left(\varphi_0 + \frac{H}{\mu}\rho_0\right)\rho_0 =: \frac{1}{\sigma}\beta_1,$$

similarly, we obtain the following estimates

$$\left|\left(U_{il}^2\psi - \psi_{il}^{20}\right)e^{-\sigma t}\right| \leq \frac{1}{\sigma}\frac{n^2\Psi_0}{\Phi_0}\left[\Gamma_0 + \lambda_0\varphi_0 + \frac{H}{\mu}\left(1+n\varphi_0\right)\rho_0\right]\rho_0 =: \frac{1}{\sigma}\beta_2,$$

$$\left|\left(U_{il}^3\psi - \psi_{il}^{30}\right)e^{-\sigma t}\right| \leq \frac{1}{\sigma}n\left[\Gamma_0 + \varphi_0 + \frac{H}{\mu}\left(1 + \frac{n\varphi_0}{\mu}\right)\rho_0\right]\rho_0 =: \frac{1}{\sigma}\beta_3.$$

From this and formulas (5.2)–(5.6) it follows

$$\|U\psi-\psi^0\|_\sigma=\max\Bigg\{\max_{1\le i,l\le n}\sup_{(x,t)\in D(\mu)}\left|\left(U^1_{il}\psi-\psi^{10}_{il}\right)e^{-\sigma t}\right|,\ \max_{1\le i,l\le n}\sup_{t\in\left[0,\frac{H}{\mu}\right]}\left|\left(U^2_{il}\psi-\psi^{20}_{il}\right)e^{-\sigma t}\right|,$$
$$\max_{1\le i,l\le n}\sup_{t\in\left[0,\frac{H}{\mu}\right]}\left|\left(U^3_{il}\psi-\psi^{30}_{il}\right)e^{-\sigma t}\right|\Bigg\}\le\frac{1}{\sigma}\beta_0,$$

where $\beta_0:=\max(\beta_1,\ \beta_2,\ \beta_3)$. Choosing $\sigma>(1/\rho)\beta_0$, we see that the operator $U$ takes the set $S\left(\psi^0,\rho\right)$ into itself.

Now we take any two functions $\psi,\widetilde{\psi}\in S\left(\psi^0,\rho\right)$ and estimate the norm of the difference $U\psi-U\widetilde{\psi}$. Using the obvious inequality

$$\left|\psi^k_{il}\psi^s_{il}-\widetilde{\psi}^k_{il}\widetilde{\psi}^s_{il}\right|e^{-\sigma t}\le|\psi^s_{il}|\left|\psi^k_{il}-\widetilde{\psi}^k_{il}\right|e^{-\sigma t}+\left|\widetilde{\psi}^k_{il}\right|\left|\psi^s_{il}-\widetilde{\psi}^s_{il}\right|e^{-\sigma t}\le 2\rho_0\|\psi-\widetilde{\psi}\|_\sigma$$

and estimates for integrals similar to those given above, we obtain

$$\left|\left(U^1_{il}\psi-U^1_{il}\widetilde{\psi}\right)e^{-\sigma t}\right|=$$
$$=\left|\int\limits_{t^i_0}^{t}\sum_{j=1}^{n}\left(\psi^2_{ij}(\tau)-\widetilde{\psi}^2_{ij}(\tau)\right)e^{-\sigma\tau}\varphi^l_j(x+\lambda_i(\tau-t))e^{-\sigma(t-\tau)}d\tau+\right.$$
$$+\int\limits_{t^i_0}^{t}e^{-\sigma(t-\tau)}\int\limits_0^{\tau}\sum_{j=1}^{n}\Big[\psi^2_{ij}(\alpha)e^{-\sigma\alpha}\psi^1_{jl}(x+\lambda_i(\tau-t),\tau-\alpha)e^{-\sigma(\tau-\alpha)}-$$
$$\left.-\widetilde{\psi}^2_{ij}(\alpha)e^{-\sigma\alpha}\widetilde{\psi}^1_{jl}(x+\lambda_i(\tau-t),\tau-\alpha)e^{-\sigma(\tau-\alpha)}\Big]d\alpha d\tau\right|\le$$
$$\le n\left[\varphi_0\|\psi-\widetilde{\psi}\|_\sigma+2\tau\rho_0\|\psi-\widetilde{\psi}\|_\sigma\right]\int_0^t e^{-\sigma(t-\tau)}d\tau\le$$
$$\le\frac{1}{\sigma}n\left(\varphi_0+\frac{2H}{\mu}\rho_0\right)\|\psi-\widetilde{\psi}\|_\sigma=:\frac{1}{\sigma}\gamma_1\|\psi-\widetilde{\psi}\|_\sigma,$$

similarly, we obtain the following estimates

$$\left|\left(U^2_{il}\psi-U^2_{il}\widetilde{\psi}\right)e^{-\sigma t}\right|\le$$
$$\le\frac{1}{\sigma}\frac{n^2\Psi_0}{\Phi_0}\left[\Gamma_0+\lambda_0\varphi_0+\frac{2H}{\mu}\left(1+n\varphi_0\right)\rho_0\right]\|\psi-\widetilde{\psi}\|_\sigma=:\frac{1}{\sigma}\gamma_2\|\psi-\widetilde{\psi}\|_\sigma,$$

$$\left|\left(U_{il}^3\psi - U_{il}^3\widetilde{\psi}\right)e^{-\sigma t}\right| \leq$$

$$\leq \frac{1}{\sigma}n\left[\Gamma_0 + \varphi_0 + \frac{2H}{\mu}\left(1 + \frac{n\varphi_0}{\mu}\right)\rho_0\right]\|\psi - \widetilde{\psi}\|_\sigma =: \frac{1}{\sigma}\gamma_3\|\psi - \widetilde{\psi}\|_\sigma.$$

Where we have the following

$$\|U\psi - U\widetilde{\psi}\|_\sigma = \max\left\{\max_{1\leq i,l\leq n}\sup_{(x,t)\in D(\mu)}\left|\left(U_{il}^1\psi - U_{il}^1\widetilde{\psi}\right)e^{-\sigma t}\right|,\right.$$

$$\max_{1\leq i,l\leq n}\sup_{t\in\left[0,\frac{H}{\mu}\right]}\left|\left(U_{il}^2\psi - U_{il}^2\widetilde{\psi}\right)e^{-\sigma t}\right|,$$

$$\left.\max_{1\leq i,l\leq n}\sup_{t\in\left[0,\frac{H}{\mu}\right]}\left|\left(U_{il}^3\psi - U_{il}^3\widetilde{\psi}\right)e^{-\sigma t}\right|\right\} \leq \frac{1}{\sigma}\gamma_0\|\psi - \widetilde{\psi}\|_\sigma,$$

here $\gamma_0 := \max(\gamma_1,\ \gamma_2,\ \gamma_3)$. Choosing now $\sigma > \gamma_0$, we get that the operator $U$ contracts the distance between the elements $\psi, \widetilde{\psi}$ by $S\left(\psi^0, \rho\right)$.

As follows from the estimates made, if the number $\sigma$ is chosen from the condition $\sigma > \sigma^* := max\{\beta_0, \gamma_0\}$, then the operator $U$ is contracting on $S\left(\psi^0, \rho\right)$. By the Banach fixed-point theorem [[20], p. 87-97], Eq. (5.1) is then solvable and has a unique solution in $S\left(\psi^0, \rho\right)$ for any fixed $H > 0$. Theorem 5.1 is proved.

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