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Abstract	This paper considers the inverse problem of determining the time-dependent coefficient in the fractional wave equation with Hilfer derivative. In this case, the direct problem is initial-boundary value problem for this equation with Cauchy type initial and nonlocal boundary conditions. As overdetermination condition nonlocal integral condition with respect to direct problem solution is given. By the Fourier method, this problem is reduced to equivalent integral equations. Then, using the Mittag-Leffler function and the generalized singular Gronwall inequality, we get apriori estimate for solution via unknown coefficient which we will need to study of the inverse problem. The inverse problem is reduced to the equivalent integral of equation of Volterra type. The principle of contracted mapping is used to solve this equation. Local existence and global uniqueness results are proved. The stability estimate is also obtained.

Keywords (separated by '-')	Fractional derivative - Riemann–Liouville fractional integral - Inverse problem - Integral equation - Fourier series - Banach fixed point theorem
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ORIGINAL RESEARCH

Inverse Coefficient Problem for Fractional Wave Equation with the Generalized Riemann–Liouville Time Derivative

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Abstract This paper considers the inverse problem of determining the time-dependent coefficient in the fractional wave equation with Hilfer derivative. In this case, the direct problem is initial-boundary value problem for this equation with Cauchy type initial and nonlocal boundary conditions. As overdetermination condition nonlocal integral condition with respect to direct problem solution is given. By the Fourier method, this problem is reduced to equivalent integral equations. Then, using the Mittag-Leffler function and the generalized singular Gronwall inequality, we get apriori estimate for solution via unknown coefficient which we will need to study of the inverse problem. The inverse problem is reduced to the equivalent integral of equation of Volterra type. The principle of contracted mapping is used to solve this equation. Local existence and global uniqueness results are proved. The stability estimate is also obtained.

Keywords Fractional derivative · Riemann–Liouville fractional integral · Inverse problem · Integral equation · Fourier series · Banach fixed point theorem

1 Introduction

We consider the following fractional wave equation in the domain $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$

$$\left(D_{0+,t}^{\alpha,\beta} u\right)(x,t) - u_{xx} + q(t)u(x,t) = f(x,t), \tag{1}$$

with the initial conditions of Cauchy type

$$I_{0+,t}^{(2-\alpha)(1-\beta)} u(x,t)\big|_{t=0} = \varphi_1(x),$$
$$\frac{\partial}{\partial t}\left(I_{0+,t}^{(2-\alpha)(1-\beta)} u\right)(x,t)\big|_{t=0} = \varphi_2(x), \quad x \in [0,1] \tag{2}$$

Communicated by Prof. Rahul Roy.

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uncorrected proof

and the boundary conditions

$$u(0,t)=u(1,t),\quad u_x(1,t)=0,\quad 0\le t\le T. \tag{3}$$

Here the generalized Riemann–Liouville (Hilfer) fractional differential operator $D^{\alpha,\beta}_{0+,t}$ of the order $1<\alpha<2$ and type $0\le\beta\le 1$ is defined as follows [1, pp. 112–118], [2, pp. 62–65]:

$$D^{\alpha,\beta}_{0+,t}u(\cdot,t)=\left(I^{\beta(2-\alpha)}_{0+,t}\frac{\partial^2}{\partial t^2}\left(I^{(1-\beta)(2-\alpha)}_{0+,t}u\right)\right)(\cdot,t),$$

$$I^{\gamma}_{0+,t}u(x,t)=\frac{1}{\Gamma(\gamma)}\int\limits_0^t\frac{u(x,\tau)}{(t-\tau)^{1-\gamma}}d\tau,\quad \gamma\in(0,1)$$

is the Riemann–Liouville fractional integral of the function $u(x,t)$ with respect to t [3], [4, pp. 69–72], $\Gamma(\cdot)$ is the Euler's Gamma function. The functions $f(x,t)$, $\varphi_1(x)$, $\varphi_2(x)$, are known functions.

In [1, pp. 112–118], [5, pp. 28–37], by R. Hilfer was introduced a generalized form of the Riemann–Liouville fractional derivative of order α and a type $\beta\in[0,1]$, which coincides with the Riemann–Liouville fractional derivative at $\beta=0$ and with Gerasimov-Caputo fractional derivative at $\beta=1$, and the case $\beta\in(0,1)$ interpolates these both fractional derivatives.

For the given functions $f(x,t)$, $\varphi_i(x)$, $i=1,2$, $q(t)$ and numbers $\alpha\in(1,2)$, $\beta\in[0,1]$, the problem of determining the solution to the initial boundary value problem (1)–(3) we call as the *direct problem*.

In *inverse problem* it is required to determine the function $q(t)$, $t\in[0,T]$ in equation (1), if the solution to the problem (1)–(3) satisfies the following overdetermination condition:

$$\int\limits_0^1 w(x)u(x,t)dx=h(t),\quad 0\le t\le T, \tag{4}$$

where $w(x)$ and $h(t)$ are given functions.

Assume that throughout this article, given functions φ_1, φ_2, f, w and h satisfy the following assumptions:

(A1) $\{\varphi_1,\varphi_2\}\in C^3[0,1]$, $\{\varphi_1^{(4)},\varphi_2^{(4)}\}\in L_2[0,1]$, $\varphi_i(0)=\varphi_i(1)=0$, $\varphi''_i(0)=\varphi''_i(1)=0$, $i=1,2$;

(A2) $f(x,\cdot)\in C[0,T]$ *and for* $t\in[0,T]$, $f(\cdot,t)\in C^3[0,1]$, $f(\cdot,t)^{(4)}\in L_2[0,1]$, $f(0,t)=f(1,t)=0$, $f_{xx}(0,t)=f_{xx}(1,t)=0$;

A3) $w(x)\in C^2[0,1]$ *and* $w(0)=w(1)=0$ *and* $w''(0)=w''(1)=0$;

A4) $\left(D^{\alpha,\beta}_{0+,t}h\right)(t)\in C[0,T]$, $|h(t)|\ge h_0>0$, h_0 *is a given number,*

$$\int\limits_0^1 w(x)\varphi_1(x)dx=I^{(2-\alpha)(1-\beta)}_{0+,t}h(t)_{t=0+},$$

$$\int\limits_0^1 w(x)\varphi_2(x)dx=\frac{\partial}{\partial t}\left(I^{(2-\alpha)(1-\beta)}_{0+,t}h\right)(t)_{t=0+}.$$

Fractional Calculus is a new growing field of applied mathematics. Fractional derivative is the generalization of the classical derivative of whole order of applied mathematics. Many problems of visco-elasticity, dynamical processes in self-similar structures, biosciences, signal processing, system control theory, electrochemistry, diffusion processes and etc are more accurately modeled with differential equations of fractional order [4–8].

The identification of the right hand side and the order of time fractional derivative equation in applied fractional modeling plays an important role. In the papers [9–11], an inverse problem for determining these unknowns of time fractional derivative in a subdiffusion equation with an arbitrary second order elliptic differential operator is considered. It is proved that the additional information about the solution at a fixed time instant at a monitoring location, as the observation data, identified uniquely the order of the fractional derivative.

In the works [12–15], the unique solvability of the nonlocal direct problems and inverse source problems for the various fractional differential equations with Caputo and Riemann–Liouville integral-differential operators were investigated.

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Inverse problems for classical integro-differential wave propagation equations have been extensively studied. Nonlinear inverse coefficient problems with various types of overdetermination conditions are often found in the literature (e.g., [16–21] and references therein). In the works [22–28], inverse problems of determining unknown coefficients in Cauchy problem for fractional diffusion-wave equation were investigated. Local existence and uniqueness in whole are proved and estimates of conditional stability are obtained.

In this paper, we investigate the local existence and global uniqueness of an inverse problem of determining time-dependent coefficient in the generalized time fractional wave equation with initial, nonlocal boundary and overdetermination integral conditions.

In the next section, we provide some necessary preliminaries are given.

2 Preliminaries

In this section, we present some useful definitions and results of fractional calculus.

Two parameter Mittag-Leffler function The two parameter M-L function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\alpha)$—denote the real part of the complex number α. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications.

Proposition 1 . *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1 + |z|}, \quad \kappa \le |arg(z)| \le \pi.$$

For the proof, we refer to [4, pp. 40–45], *for example.*

We consider the weighted spaces of continuous functions [4, pp. 4–5, 162–163].

$$C_\gamma[a, b] := \{g : (a, b] \to R : (t-a)^\gamma g(t) \in C[a, b], \quad 0 \le \gamma < 1, \},$$

$$C_\gamma^{\alpha,\beta}(\Omega) = \left\{ g(t) : \ D_{0+,t}^{\alpha,\beta} g(t) \in C_\gamma(0, T]; \ 1 < \alpha \le 2, \quad 0 \le \beta \le 1 \right\},$$

$$C_\gamma^{2,\alpha,\beta}(\Omega) = \Big\{ u(x,t) : \ u(\cdot, t) \in C^2(0, 1); \quad t \in [0, T] \text{ and}$$

$$D_{0+,t}^{\alpha,\beta} u(x, \cdot) \in C_\gamma(0, T]; \ x \in [0, 1], \ 1 < \alpha \le 2, \quad 0 \le \beta \le 1 \Big\},$$

$$C_\gamma^0[a, b] = C_\gamma[a, b],$$

with the norms

$$\|f\|_{C_\gamma} = \|(t-a)^\gamma f(t)\|_C, \quad \|f\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \|f^{(k)}\|_C + \|f^{(n)}\|_{C_\gamma}.$$

Lemma 1 [29, p. 188]. *Suppose $b \ge 0, \ \alpha > 0$ and $a(t)$ nonnegative function locally integrable on $0 \le t < T$ (some $T \le +\infty$) and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + b \int_0^t (t-s)^{\alpha-1} u(s) ds$$

then

$$u(t) \le a(t) + b\Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(b\Gamma(\alpha)(t-s)^\alpha \right) a(s) ds.$$

uncorrected proof

Lemma 2 [29, p. 189] *Suppose* $b \geq 0,\ \alpha > 0,\ \gamma > 0,\ \alpha+\gamma > 1$ *and* $a(t)$ *nonnegative function locally integrable on* $0 \leq t < T$ *and suppose* $t^{\gamma-1}u(t)$ *is nonnegative and locally integrable on* $0 \leq t < T$ *with*

$$u(t) \leq a(t) + b\int\limits_0^t (t-s)^{\alpha-1}s^{\gamma-1}u(s)ds,$$

then

$$u(t) \leq a(t)E_{\alpha,\gamma}\left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}}\, t\right),$$

where

$$E_{\alpha,\gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}, \quad c_0 = 1,\ \frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$$

for $m \geq 0$. *As* $t \to +\infty$ $E_{\alpha,\gamma}(t) = O\left(t^{\frac{1}{2}\frac{\alpha+\gamma-1}{\alpha-\gamma}}\exp\left(\frac{\alpha+\gamma-1}{\alpha}t^{\frac{\alpha+\gamma-1}{\alpha}}\right)\right)$.

From the above, there exist some positive constants $M_i,\ i = 1, 2, 3,$ such that

$$M_1 = \max_{0\leq t\leq T}\left|E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right)\right|, \quad M_2 = \max_{0\leq t\leq T}\left|E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right)\right|,$$
$$M_3 = \max_{0\leq s<t\leq T}\left|E_{\alpha,\alpha}(-\lambda_n^2(t-s)^{\alpha})\right|.$$

3 Existence and uniqueness results for direct problem solution

First, note that for the non-selfadjoint operator $X^{''}(x) + \lambda^2 X(x) = 0$ with $X(0) = X(1),\quad X^{'}(1) = 0$

$$X_0(x) = 2, \quad X_{2k}(x) = 4\cos(2\pi kx),$$
$$X_{2k-1}(x) = 4(1-x)\sin(2\pi kx), \quad k = 0, 1, 2, \ldots \tag{5}$$

and

$$Y_0(x) = x, \quad Y_{2k}(x) = x\cos(2\pi kx),$$
$$Y_{2k-1}(x) = \sin(2\pi kx), \lambda_k = 2\pi k, \quad k = 1, 2, 3, \ldots, \tag{6}$$

which are Riesz bases in $L_2[0; 1]$. For more details, the reader can consult [30–32].

By applying the Fourier method, the solution $u(x, t)$ of the problem (1)–(3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x,t) = X_0(x)u_0(t) + \sum_{n=1}^{\infty} X_{2n-1}(x)u_{2n-1}(t) + \sum_{n=1}^{\infty} X_{2n}(x)u_{2n}(t). \tag{7}$$

The coefficients $u_0(t),\ u_{2n}(t),\ u_{2n-1}(t)$ for $n \geq 1$ are to be found by making use of the orthogonality of the eigenfunctions. Namely, we multiply (1) by the eigenfunctions of (6) and integrate over (0, 1). Recall that the scalar product in $L_2[0, 1]$ is defined by $(f, g) = \int\limits_0^1 f(x)g(x)dx$. Let us note the expansion coefficients of $f(x, t)$ and $\varphi(x)$ in the eigenfunctions of (6) for $n \geq 1$ respectively by

$$\begin{cases} (f(x,t), Y_0(x)) = f_0(t), \\ (f(x,t), Y_{2n-1}(x)) = f_{2n-1}(t), \\ (f(x,t), Y_{2n}(x)) = f_{2n}(t), \end{cases}$$

$$\begin{cases} (\varphi_i(x), Y_0(x)) = \varphi_{0,i}, \\ (\varphi_i(x), Y_{2n-1}(x)) = \varphi_{2n-1,i}, \quad i = 1, 2. \\ (\varphi_i(x), Y_{2n}(x)) = \varphi_{2n,i}, \end{cases}$$

uncorrected proof

In view of (1) for $(u(x,t), Y_0(x)) = u_0(t)$, we obtain the Cauchy type problem

$$\begin{cases} \left(D^{\alpha,\beta}_{0+,t} u_0\right)(t) + q(t)u_0(t) = f_0(t), \\ I^{(2-\alpha)(1-\beta)}_{0+,t} u_0(t)\big|_{t=0} = \varphi_{0,1}, \quad \frac{d}{dt}\left(I^{(2-\alpha)(1-\beta)}_{0+,t} u_0\right)(t)\big|_{t=0} = \varphi_{0,2}. \end{cases} \tag{8}$$

For $u_{2n-1}(t) = (u(x,t), Y_{2n-1}(x))$; $n \geq 1$, in view of (1) we have

$$\begin{cases} \left(D^{\alpha,\beta}_{0+,t} u_{2n-1}\right)(t) + \lambda_n^2 u_{2n-1} + q(t)u_{2n-1}(t) = f_{2n-1}(t), \\ I^{(2-\alpha)(1-\beta)}_{0+,t} u_{2n-1}(t)\big|_{t=0} = \varphi_{2n-1,1}, \\ \frac{d}{dt}\left(I^{(2-\alpha)(1-\beta)}_{0+,t} u_{2n-1}\right)(t)\big|_{t=0} = \varphi_{2n-1,2}. \end{cases} \tag{9}$$

Also, the Cauchy type problem satisfied by $u_{2n}(t) = (u_{2n}(x,t), Y_{2n}(x))$, $n \geq 1$, are

$$\begin{cases} \left(D^{\alpha,\beta}_{0+,t} u_{2n}\right)(t) + \lambda^2 u_{2n}(t) + 2\lambda u_{2n-1}(t) + q(t)u_{2n}(t) = f_{2n}(t), \\ I^{(2-\alpha)(1-\beta)}_{0+,t} u_{2n}(t)\big|_{t=0} = \varphi_{2n,1}, \\ \frac{d}{dt}\left(I^{(2-\alpha)(1-\beta)}_{0+,t} u_{2n}\right)(t)\big|_{t=0} = \varphi_{2n,2}. \end{cases} \tag{10}$$

We solve problems (8)–(10).

Based [33, pp. 61–114], we have that the initial problem (8) is equivalent in the space $C^{\alpha,\beta}_{\gamma}[0,T]$ to the Volterra integral equation of the second kind

$$\begin{aligned} u_0(t) = {} & \frac{t^{(\beta-1)(2-\alpha)}}{\Gamma(1+(\beta-1)(2-\alpha))}\varphi_{0,1} + \frac{t^{1+(\beta-1)(2-\alpha)}}{\Gamma(\alpha+\beta(2-\alpha))}\varphi_{0,2} \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f_0(\tau)d\tau - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} q(\tau)u_0(\tau)d\tau. \end{aligned} \tag{11}$$

First we prove the following assertions for $u_0(t)$:

Lemma 3 . *We have the estimates*

$$\begin{aligned} t^{\gamma}|u_0| \leq {} & \Bigg(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))} \\ & + \frac{\|f_0\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \Bigg) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right); \end{aligned}$$

$$\begin{aligned} t^{\gamma}\left|\left(D^{\alpha,\beta}_{0+,t} u_0\right)(t)\right| \leq {} & \|f_0\|_{C_\gamma[0,T]} \\ & + \|q\|_{C[0,T]} \Bigg(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))} \\ & + \frac{\|f_0\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \Bigg) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right), \quad t \in [0,T], \end{aligned}$$

where $1 > \gamma > (1-\beta)(2-\alpha)$.

Proof The solution of (11) is bounded in $C^{\alpha,\beta}_{\gamma}[0,T]$ in view of (A1), (A2). Multiplying the equation (11) by t^{γ}, we get

$$\begin{aligned} t^{\gamma}|u_0| \leq {} & \frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))} \\ & + \frac{\|f_0\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\alpha\Gamma(\alpha)} + \frac{\|q\|_{C[0,T]} t^{\gamma}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |u_0(\tau)| d\tau, \quad t \in [0,T], \end{aligned} \tag{12}$$

uncorrected proof

where $B(\alpha, 1-\gamma)$ is Euler's beta function.

Next, according to Lemma 2, we get

$$t^{\gamma}|u_0| \le \left(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))}\right.$$
$$\left. + \frac{\|f_0\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right) =: \Psi_0(t), \quad t \in [0, T].$$

We get the second part of the lemma 3, from the equation in problem (8) and the first estimate of lemma 3:

$$t^{\gamma}\left|\left(D^{\alpha,\beta}_{0+,t} u_0\right)(t)\right| \le \|f_0\|_{C_\gamma[0,T]}$$
$$+\|q\|_{C[0,T]}\left(\frac{t^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{t^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))}\right.$$
$$\left. + \frac{\|f_0\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right).$$

From the last two inequalities we immediately obtain the estimates of lemma 3 for any $t \in [0, T]$. □

In view of [33, pp. 61–114], we have that the initial problem (9) is equivalent in the space $C^{\alpha,\beta}_{\gamma}[0, T]$ to the Volterra integral equation of the second kind

$$u_{2n-1}(t) = t^{(\beta-1)(2-\alpha)} E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right) \varphi_{2n-1,1}$$
$$+ t^{1+(\beta-1)(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2 t^{\alpha}\right) \varphi_{2n-1,2}$$
$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^{\alpha}) f_{2n-1}(\tau) d\tau$$
$$- \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^{\alpha}) q(\tau) u_{2n-1}(\tau) d\tau. \tag{13}$$

We prove the following assertions for $u_{2n-1}(t)$:

Lemma 4 . *For fixed $n \in N$ we have the estimates*

$$t^{\gamma}|u_{2n-1}| \le \left(t^{\gamma+(\beta-1)(2-\alpha)} M_1 |\varphi_{2n-1,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)} M_2 |\varphi_{2n-1,2}|\right.$$
$$\left. + \frac{\|f_{2n-1}\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right);$$

$$t^{\gamma}\left|\left(D^{\alpha,\beta}_{0+,t} u_{2n-1}\right)(t)\right| \le \|f_{2n-1}\|_{C_\gamma[0,T]}$$
$$+ \left(\lambda_n^2 + \|q\|_{C[0,T]}\right)\left(t^{\gamma+(\beta-1)(2-\alpha)} M_1 |\varphi_{2n-1,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)} M_2 |\varphi_{2n-1,2}|\right.$$
$$\left. + \frac{\|f_{2n-1}\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}\right) E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t\right),$$

where $1 > \gamma > (1-\beta)(2-\alpha)$.

Proof Multiplying the equation (13) by t^{γ}, we have

$$t^{\gamma}|u_{2n-1}| \le t^{\gamma+(\beta-1)(2-\alpha)} M_1 |\varphi_{2n-1,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)} M_2 |\varphi_{2n-1,2}|$$
$$+ \frac{\|f_{2n-1}\|_{C_\gamma[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)} + \frac{|q|_{C[0,T]} t^{\gamma}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |u_{2n-1}(\tau)| d\tau. \tag{14}$$

uncorrected proof

Further, via lemma 2, we get

$$t^{\gamma}|u_{2n-1}| \le \Big(t^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}| + \frac{\|f_{2n-1}\|_{C_\gamma[0,T]}t^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Big) \times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right) =: \Psi_{2n-1}(t), \quad t\in[0,T].$$

Taking into account the first equation in (9) and the first estimate of lemma 4, we have the second part of the lemma 4:

$$t^{\gamma}\left|\left(D^{\alpha,\beta}_{0+,t}u_{2n-1}\right)(t)\right| \le \|f_{2n-1}\|_{C_\gamma[0,T]} + \left(\lambda_n^2+\|q\|_{C[0,T]}\right)\Big(t^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}| + \frac{\|f_{2n-1}\|_{C_\gamma[0,T]}t^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Big)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}t\right).$$

From the last two inequalities we obtain the estimates of lemma 4 for any $t\in[0,T]$. □

Analogously, we have following integral equation from problem (10):

$$\begin{aligned} u_{2n}(t) = {} & t^{(\beta-1)(2-\alpha)}E_{\alpha,1+(\beta-1)(2-\alpha)}\left(-\lambda^2t^{\alpha}\right)\varphi_{2n,1} \\ & + t^{1+(\beta-1)(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda^2t^{\alpha}\right)\varphi_{2n,2} \\ & + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})f_{2n}(\tau)d\tau \\ & - 2\lambda_n\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})u_{2n-1}(\tau)d\tau \\ & - \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})q(\tau)u_{2n}(\tau)d\tau. \end{aligned} \tag{15}$$

Under (A1), (A2), $u_{2n}(t)$ is bounded in $C^{\alpha,\beta}_{\gamma}[0,T]$ as follows

$$\begin{aligned} t^{\gamma}|u_{2n}| \le {} & t^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}| + t^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}| \\ & + \frac{\|f_{2n}\|_{C_\gamma[0,T]}t^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)} + 2\lambda_n\frac{\|u_{2n-1}\|_{C^{\alpha,\beta}_{\gamma}[0,T]}t^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)} \\ & + \frac{\|q\|_{C[0,T]}t^{\gamma}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u_{2n}(\tau)|d\tau. \end{aligned} \tag{16}$$

According to the generalized Gronoull integral inequality, we get the following estimate

$$t^{\gamma}|u_{2n}| \leq \left(t^{\gamma+(\beta-1)(2-\alpha)} M_1 |\varphi_{2n,1}| \right.$$
$$+t^{1+\gamma+(\beta-1)(2-\alpha)} M_2 |\varphi_{2n,2}| + \frac{\|f_{2n}\|_{C_{\gamma}[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}$$
$$\left. +2\lambda_n \frac{\|u_{2n-1}\|_{C_{\gamma}^{\alpha,\beta}[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \right)$$
$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t \right) =: \Psi_{2n}(t), \quad t \in [0, T].$$

Next, by the second estimate in lemma 4, we have

$$t^{\gamma}\left|\left(D_{0+,t}^{\alpha,\beta} u_{2n}\right)(t)\right| \leq \|f_{2n}\|_{C_{\gamma}[0,T]} + 2\lambda_n |u_{2n-1}|_{C_{\gamma}[0,T]}$$
$$+\left(\lambda_n^2 + \|q\|_{C[0,T]}\right)\left(t^{\gamma+(\beta-1)(2-\alpha)} M_1 |\varphi_{2n,1}|\right.$$
$$+t^{1+\gamma+(\beta-1)(2-\alpha)} M_2 |\varphi_{2n,2}| + \frac{\|f_{2n}\|_{C_{\gamma}[0,T]} t^{\alpha} B(\alpha, 1-\gamma) M_3}{\Gamma(\alpha+1)}$$
$$\left. +2\lambda_n \frac{\|u_{2n-1}\|_{C_{\gamma}^{\alpha,\beta}[0,T]} t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \right)$$
$$\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]} t^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}} t \right), \quad t \in [0, T].$$

Formally, from (7) by term-by-term differentiation we compose the series

$$\left(D_{0+,t}^{\alpha,\beta} u\right)(x,t) = 2\left(D_{0+,t}^{\alpha,\beta} u_0\right)(t) + 4\sum_{n=1}^{\infty}\left(D_{0+,t}^{\alpha,\beta} u_{2n}\right)(t) cos(\lambda_n x)$$
$$+ 4(1-x)\sum_{n=1}^{\infty}\left(D_{0+,t}^{\alpha,\beta} u_{2n-1}\right)(t) sin(\lambda_n x), \tag{17}$$

$$u_{xx}(x,t) = -4\sum_{n=1}^{\infty}\lambda_n^2 u_{2n}(t) cos(\lambda_n x)$$
$$-8\sum_{n=1}^{\infty}\lambda_n u_{2n-1}(t) cos(\lambda_n x) - 4(1-x)\sum_{n=1}^{\infty}\lambda_n^2 u_{2n-1}(t) sin(\lambda_n x). \tag{18}$$

Let us prove the uniform convergence of series (7), (17) and (18) in the domain $\overline{\Omega}$. This series for any $(x,t) \in \overline{\Omega}$ is majorized by

$$2\Psi_0(T) + 4\sum_{n=1}^{\infty}\Psi_{2n-1}(T) + 4\sum_{n=1}^{\infty}\Psi_{2n}(T),$$

$$2\left(\|f_0\|_{C_{\gamma}[0,T]} + \|q\|_{C[0,T]}\Psi_0(T)\right)$$
$$+4\sum_{n=1}^{\infty}\left(\|f_{2n-1}\|_{C_{\gamma}[0,T]} + \left(\lambda_n^2 + \|q\|_{C[0,T]}\right)\Psi_{2n-1}(T)\right)$$
$$+4\sum_{n=1}^{\infty}\left(\|f_{2n}\|_{C_{\gamma}[0,T]} + 2\lambda_n\Psi_{2n-1}(T) + \left(\lambda_n^2 + \|q\|_{C[0,T]}\right)\Psi_{2n}(T)\right),$$

$$4\sum_{n=1}^{\infty}\left(\lambda_n^2 + 2\lambda_n\right)\Psi_{2n-1}(T) + 4\sum_{n=1}^{\infty}\lambda_n^2\Psi_{2n}(T),$$

uncorrected proof

where $\overline{\Omega} := \{(x,t) : 0 \le x \le 1, \quad 0 \le t \le T\}$.

We hold the following auxiliary lemma.

Lemma 5 *If the conditions* (A1), (A2) *are fulfilled then there are equalities*

$$\varphi_{n,i} = \frac{1}{\lambda_n^4}\varphi_{n,i}^{(4)}, \ i = 1, 2, \quad f_n = \frac{1}{\lambda_n^4} f_n^{(4)}, \tag{19}$$

where

$$\varphi_{n,i}^{(4)} = \int_0^1 \varphi_i^{(4)}(x) Y_n(x) dx, \ i = 1, 2, \quad f_n^{(4)} = \int_0^1 f^{(4)}(x) Y_n(x) dx,$$

with the following estimates:

$$\sum_{n=1}^{\infty} \left|\varphi_{n,1}^{(4)}\right|^2 \le \|\varphi_1^{(4)}\|_{L_2[0,l]}, \quad \sum_{n=1}^{\infty} \left|\varphi_{n,2}^{(4)}\right|^2 \le \|\varphi_2^{(4)}\|_{L_2[0,l]},$$
$$\sum_{n=1}^{\infty} \left|f_n^{(4)}\right|^2 \le \|f^{(4)}\|_{L_2[0,T]}. \tag{20}$$

If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ satisfy the conditions of lemma 5, then due to representations (19) and (20) series (7), (17) and (18) converge uniformly in the rectangle $\overline{\Omega}$, therefore, function $u(x,t)$ satisfies relations (1)–(3).

Using the above results, we obtain the following assertion.

Theorem 1 . *Let* $q(t) \in C[0,T]$, (A1), (A2) *are satisfied, then there exists a unique solution of the direct problem* (1)–(3) $u(x,t) \in C_\gamma^{2,\alpha,\beta}(\overline{\Omega})$.

Let us use the topological product Banach spaces $K = \left[C_\gamma^{\alpha,\beta}[0,T]\right]^3$ endowed with its norme to prove the existence and uniquness of the solution under this form $(u_0(t), u_{2n-1}(t), u_{2n}(t)) \in K$. Define the operator $\mathcal{A}$ on K by $\mathcal{A}(u_0(t), u_{2n-1}(t), u_{2n}(t)) = (P_0u_0(t), P_{2n-1}u_{2n-1}(t), P_{2n}u_{2n}(t))$ where the operators P_0, P_{2n-1}, P_{2n} are defined on $C_\gamma^{\alpha,\beta}[0,T]$ by the right side of (11),(13) and (15) respectively. In view of (12), (14) and (16) $\mathcal{A} : K \to K$.

Prove that $\mathcal{A}$ is a contraction on K. So, for each $((u_0(t), u_{2n-1}(t), u_{2n}(t)); (\widetilde{u}_0(t), \widetilde{u}_{2n-1}(t), \widetilde{u}_{2n}(t))) \in K$ we have

$$\begin{aligned}
&\|\mathcal{A}(u_0, u_{2n-1}, u_{2n}) - \mathcal{A}(\widetilde{u}_0, \widetilde{u}_{2n-1}, \widetilde{u}_{2n})\|_K \\
&\quad \le \max\left\{\|P_0u_0 - P_0\widetilde{u}_0\|_{C_\gamma^{\alpha,\beta}[0,T]}, \|P_{2n-1}u_{2n-1} - P_{2n-1}\widetilde{u}_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]}, \|P_{2n}u_{2n} - P_{2n}\widetilde{u}_{2n}\|_{C_\gamma^{\alpha,\beta}[0,T]}\right\}.
\end{aligned}$$

First, we get easily

$$\begin{aligned}
\|P_0u_0(t) - P_0\widetilde{u}_0(t)\|_{C_\gamma^{\alpha,\beta}} &\le \frac{\|q\|_{C[0,T]}t^\gamma}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |u_0(\tau) - \widetilde{u}_0(\tau)| d\tau \\
&\le \frac{|q|_{C[0,T]}T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_0 - \widetilde{u}_0\|_{C_\gamma^{\alpha,\beta}[0,T]}.
\end{aligned}$$

Next, we get the following

$$\|P_0u_0 - P_0\widetilde{u}_0\|_{C_\gamma^{\alpha,\beta}[0,T]} \le \frac{\|q\|_{C[0,T]}T^\alpha B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_0 - \widetilde{u}_0\|_{C_\gamma^{\alpha,\beta}[0,T]}.$$

For P_{2n-1}, we have for each $t \in [0,T]$

$$\begin{aligned}
&\|P_{2n-1}u_{2n-1} - P_{2n-1}\widetilde{u}_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]} \\
&\quad \le \frac{\|q\|_{C[0,T]}T^\gamma B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_{2n-1} - \widetilde{u}_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]},
\end{aligned}$$

uncorrected proof

where $n \geq 1$.

Similarly, for each $t \in [0, T]$

$$\|P_{2n}u_{2n}(t) - P_{2n}\widetilde{u}_{2n}(t)\|_{C_{\gamma}^{\alpha,\beta}[0,T]} \leq \frac{2\lambda_n t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \|u_{2n-1} - \widetilde{u}_{2n-1}\|_{C_{\gamma}[0,T]}$$

$$+ \frac{|q|_{C[0,T]} t^{\gamma}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |u_{2n}(\tau) - \widetilde{u}_{2n}(\tau)| d\tau$$

$$\leq \frac{2\lambda_n t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \|u_{2n-1} - \widetilde{u}_{2n-1}\|_{C_{\gamma}^{\alpha,\beta}[0,T]}$$

$$+ \frac{\|q\|_{C[0,T]} T^{\gamma} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_{2n} - \widetilde{u}_{2n}\|_{C_{\gamma}^{\alpha,\beta}[0,T]},$$

which gives for $n \geq 1$

$$\|P_{2n}u_{2n} - P_{2n}\widetilde{u}_{2n}\|_{C_{\gamma}^{\alpha,\beta}[0,T]} \leq \frac{2\lambda_n t^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \|u_{2n-1} - \widetilde{u}_{2n-1}\|_{C_{\gamma}^{\alpha,\beta}[0,T]}$$

$$+ \frac{\|q\|_{C[0,T]} T^{\gamma} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_{2n} - \widetilde{u}_{2n}\|_{C_{\gamma}^{\alpha,\beta}[0,T]}.$$

As a result

$$\|\mathcal{A}(u_0, u_{2n-1}, u_{2n}) - \mathcal{A}(\widetilde{u}_0, \widetilde{u}_{2n-1}, \widetilde{u}_{2n})\|_K$$

$$\leq \max\left\{\left(\frac{\|q\|_{C[0,T]} T^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_0 - \widetilde{u}_0\|_{C_{\gamma}^{\alpha,\beta}[0,T]},\right.\right.$$

$$\frac{\|q\|_{C[0,T]} T^{\gamma} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_{2n-1} - \widetilde{u}_{2n-1}\|_{C_{\gamma}^{\alpha,\beta}[0,T]},$$

$$\left.\frac{\|q\|_{C[0,T]} T^{\gamma} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} \|u_{2n} - \widetilde{u}_{2n}\|_{C_{\gamma}^{\alpha,\beta}[0,T]}\right)$$

$$\left.+ \frac{2\lambda_n T^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)} \left(0,\ \|u_{2n-1} - \widetilde{u}_{2n-1}\|_{C_{\gamma}^{\alpha,\beta}[0,T]},\ 0\right)\right\}$$

$$\leq \max\left\{\frac{\|q\|_{C[0,T]} T^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} + \frac{2\lambda_n T^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)}\right\}$$

$$\times \|(u_0, u_{2n-1}, u_{2n}) - (\widetilde{u}_0, \widetilde{u}_{2n-1}, \widetilde{u}_{2n})\|_{C_{\gamma}^{\alpha,\beta}[0,T]}.$$

If

$$\max\left\{\frac{\|q\|_{C[0,T]} T^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha)} + \frac{2\lambda_n T^{\alpha} B(\alpha, 1-\gamma)}{\Gamma(\alpha+1)}\right\} < 1$$

then, $\mathcal{A}$ is a contraction on K and has a unique fixed point which is the coefficients (u_0, u_{2n-1}, u_{2n}) of the solution (7). Then, there exists a unique solution of (1)–(3) for arbitrary $q(t)$ bounded in $C_{\gamma}^{\alpha,\beta}[0, T]$.

4 Investigation of the inverse problem (1)–(4)

Multiply (1) by $w(x)$ and integrate in x from 0 to 1. As result we have

$$\int_0^1 w(x)\left\{\left(D_{0+,t}^{\alpha,\beta} u\right)(x,t) - u_{xx} + q(t)u(x,t)\right\}dx = \int_0^1 w(x) f(x,t) dx.$$

uncorrected proof

Future after some easy convertings in wiev of A3), we get

$$\left(D_{0+,t}^{\alpha,\beta}h\right)(t)+q(t)h(t)-\int_0^1 w''(x)u(x,t)dx=\int_0^1 w(x)f(x,t)dx,$$

which yields

$$q(t)=\frac{1}{h(t)}\left(\int_0^1 w(x)f(x,t)dx-\left(D_{0+,t}^{\alpha,\beta}h\right)(t)\right)$$

$$+\frac{1}{h(t)}\int_0^1 w''(x)\left(X_0(x)u_0(t)+\sum_{n=1}^{\infty}X_{2n-1}(x)u_{2n-1}(t)+\sum_{n=1}^{\infty}X_{2n}(x)u_{2n}(t)\right)dx.$$

The functions $u_n(t)$ depend on $q(t)$, i.e. $u_n(t;q)$. After simple converting, we get the following integral equation for determining $q(t)$:

$$q(t)=q_0(t)+\frac{1}{h(t)}\left(w_0u_0(t;q)+\sum_{n=1}^{\infty}w_{2n-1}u_{2n-1}(t;q)+\sum_{n=1}^{\infty}w_{2n}u_{2n}(t;q)\right), \tag{21}$$

where

$$q_0(t)=\frac{1}{h(t)}\left(\int_0^1 w(x)f(x,t)dx-\left(D_{0+,t}^{\alpha,\beta}h\right)(t)\right),\quad w_0=\int_0^1 w''(x)X_0(x)dx,$$

$$w_{2n-1}=\int_0^1 w''(x)X_{2n-1}(x)dx,\quad w_{2n}=\int_0^1 w''(x)X_{2n}(x)dx.$$

$u_0,\ u_{2n-1},\ u_{2n}$ are determined by right sides of (11), (13), (15) respectively.

We introduce an operator F defining it by the right hand side of (21):

$$F[q](t)=q_0(t)+\frac{1}{h(t)}\left(w_0u_0(t;q)+\sum_{n=1}^{\infty}w_{2n-1}u_{2n-1}(t;q)+\sum_{n=1}^{\infty}w_{2n}u_{2n}(t;q)\right). \tag{22}$$

Then, the equation (22) is written in more convenient form as

$$F[q](t)=q(t). \tag{23}$$

Let

$$q_{00}:=\max_{t\in[0;T]}|q_0(t)|=\left\|\frac{1}{h(t)}\left(\int_0^1 w(x)f(x,t)dx-\left(D_{0+,t}^{\alpha,\beta}h\right)(t)\right)\right\|_{C[0,T]}.$$

Fix a number $\rho>0$ and consider the ball

$$B(q_0,\rho):=\{q(t):q(t)\in C[0,T],\ \|q-q_0\|\le\rho\}.$$

Theorem 2 . *Let* (A1)–(A4) *are satisfied. Then there exists a number $T^*\in(0;T)$, such that there exists a unique solution $q(t)\in C[0,T^*]$ of the inverse problem (1)–(4).*

Proof Let us first prove that for an enough small $T > 0$ the operator F maps the ball $B(q_0, \rho)$ implies that $F[q](t) \in B(q_0, \rho)$. Indeed, for any continuous function $q(t)$, the function $F[q](t)$ calculated using formula (22) will be continuous. Moreover, estimating the norm of the differences, we find that

$$
\begin{aligned}
\|F[q](t) - q_0(t)\| \le \frac{w_0}{h_0}\Bigg(& \frac{T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))} \\
&+\frac{\|f_0\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right) \\
&+\frac{w_0}{h_0}\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}| \\
&+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}| \\
&+\frac{\|f_{2n-1}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right) \\
&+\frac{w_0}{h_0}\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}| \\
&+\frac{\|f_{2n}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)} \\
&+2\lambda_n\frac{|u_{2n-1}|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).
\end{aligned}
$$

Here we have used the estimate for $u_0,\ u_{2n-1},\ u_{2n}$ given in (11), (13), (15). In view of the above lemmas last series is convergent series. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $q(t)$ belongs to the ball $B(q_0, \rho)$ implies the inequality $\|q\| \le \|q_0\| + \rho$. Therefore, we only strengthen the inequality if we replace $\|q\|$ in this inequality with the expression $\|q_0\| + \rho$. Performing these replacements, we obtain the estimate

$$
\begin{aligned}
\|F[q](t) - q_0(t)\| \le \frac{w_0}{h_0}\Bigg(& \frac{T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))} + \frac{T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))} \\
&+\frac{\|f_0\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right) \\
&+\frac{w_0}{h_0}\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}| \\
&+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}| \\
&+\frac{\|f_{2n-1}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right) \\
&+\frac{w_0}{h_0}\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}| \\
&+\frac{\|f_{2n}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)} \\
&+2\lambda_n\frac{|u_{2n-1}|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left(\left(\|q_0\|+r\right)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right).
\end{aligned}
$$

Let T_1 be a positive root of the equation

uncorrected proof

Therefore if by T_1 we denote the positive root of the equation (for T)

$$
\begin{aligned}
&\frac{w_0}{h_0}\Bigg(\frac{T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))}+\frac{T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))}\\
&+\frac{\|f_0\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left((\|q_0\|+r)\,T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}\,T\right)\\
&+\frac{w_0}{h_0}\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}|\\
&+\frac{\|f_{2n-1}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left((\|q_0\|+r)\,T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}\,T\right)\\
&+\frac{w_0}{h_0}\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}|\\
&+\frac{\|f_{2n}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\\
&+2\lambda_n\frac{\|u_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]}T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,\gamma}\left(\left((\|q_0\|+r)\,T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}\,T\right)=\rho,
\end{aligned}
$$

then $\|F[q](t)-q_0(t)\|\le\rho$ for $T\le T_1$; i.e. $F[q](t)\in B(q_0,\rho)$.

Now let us take any functions $q(t),\ \widetilde{q}(t)\in B(q_0,\rho)$ and estimate the distance between their images $F[q](t)$ and $F[\widetilde{q}](t)$ in the space $C[0,T]$. The function $\widetilde{u}_n(t)$ corresponding to $\widetilde{q}(t)$ satisfies the integral equations (11),(13) and (15) with $\varphi_{n,1}=\widetilde{\varphi}_{n,1},\ \varphi_{n,2}=\widetilde{\varphi}_{n,2}$ and $f_n=\widetilde{f}_n$. Composing the difference $F[q](t)-F[\widetilde{q}](t)$ with the help of equations (8)–(10) and then estimating its norm, we obtain

$$
\begin{aligned}
\|F[q](t)-F[\widetilde{q}](t)\|\le{}&\frac{w_0\|q\|_{C[0,T]}T^\gamma}{h_0\Gamma(\alpha)}\Bigg[\frac{T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))}\\
&+\frac{T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))}+\frac{\|f_0\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\\
&+\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}|\\
&+\frac{\|f_{2n-1}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)\\
&+\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}|\\
&+\frac{\|f_{2n}\|_{C_\gamma[0,T]}T^\alpha B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+2\lambda_n\frac{\|u_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]}T^\alpha B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)\\
&\times E_{\alpha,\gamma}\left(\left(\|q\|_{C[0,T]}T^\gamma\right)^{\frac{1}{\alpha+\gamma-1}}\,T\right)\Bigg]\|q-\widetilde{q}\|_{C[0,T]}. \qquad (24)
\end{aligned}
$$

The functions $q(t)$ and $\widetilde{q}(t)$ belong to the ball $B(q_0,\rho)$, and hence for each of these functions one has inequality $\|q\|\le\|q_0\|+\rho$. Note that the function on the right-hand side in inequality (24) at the factor $\|q\|-\|\widetilde{q}\|$ is monotone increasing with $\|q\|$, $\|\widetilde{q}\|$ and T. Consequently, replacing $\|q\|$ and $\|\widetilde{q}\|$ in inequality (24)

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with $\|q\|+\rho$ will only strengthen the inequality. This, we have

$$\|F[q](t)-F[\widetilde{q}](t)\| \le \frac{w_0\|q\|_{C[0,T]}T^{\gamma}}{h_0\Gamma(\alpha)}\Bigg[\frac{T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))}$$
$$+\frac{T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))}+\frac{\|f_0\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}$$
$$+\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}|$$
$$+\frac{\|f_{2n-1}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)$$
$$+\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}|$$
$$+\frac{\|f_{2n}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+2\lambda_n\frac{\|u_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)$$
$$\times E_{\alpha,\gamma}\left(\left((\|q_0\|+\rho)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg]\|q-\widetilde{q}\|_{C[0,T]}.$$

Therefore, if T_2 is the positive root of the equation (for T)

$$\frac{w_0\|q\|_{C[0,T]}T^{\gamma}}{h_0\Gamma(\alpha)}\Bigg[\frac{T^{\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,1}|}{\Gamma(1+(\beta-1)(2-\alpha))}+\frac{T^{1+\gamma+(\beta-1)(2-\alpha)}|\varphi_{0,2}|}{\Gamma(\alpha+\beta(2-\alpha))}$$
$$+\frac{\|f_0\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}+\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n-1,1}|$$
$$+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n-1,2}|+\frac{\|f_{2n-1}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}\Bigg)$$
$$+\sum_{n=1}^{\infty}\Bigg(T^{\gamma+(\beta-1)(2-\alpha)}M_1|\varphi_{2n,1}|+T^{1+\gamma+(\beta-1)(2-\alpha)}M_2|\varphi_{2n,2}|$$
$$+\frac{\|f_{2n}\|_{C_\gamma[0,T]}T^{\alpha}B(\alpha,1-\gamma)M_3}{\Gamma(\alpha+1)}+2\lambda_n\frac{\|u_{2n-1}\|_{C_\gamma^{\alpha,\beta}[0,T]}T^{\alpha}B(\alpha,1-\gamma)}{\Gamma(\alpha+1)}\Bigg)$$
$$\times E_{\alpha,\gamma}\left(\left((\|q_0\|+\rho)T^{\gamma}\right)^{\frac{1}{\alpha+\gamma-1}}T\right)\Bigg]=1$$

then for $T\in(0,T_2)$ the operator F contracts the distance between the elements $q(t),\ \widetilde{q}(t)\in B(q_0,\rho)$. Consequently, if we choose $T^*<\min(T_1,T_2)$ then the operator F is a contraction in the ball $B(q_0,\rho)$. However, in accordance with the Banach theorem [34, pp. 87–97], the operator F has unique fixed point in the ball $B(q_0,\rho)$ i.e., there exists a unique solution of equation (23). Theorem 2 is proven. □

Let T be positive fixed number. Consider the set D_{μ_0} of the given functions $(\varphi_1,\ \varphi_2,\ h,\ f)$ for which all conditions from (A1)–(A4) are fulfilled and

$$\max\{\|\varphi_1\|_{C^4[0,1]},\ \|\varphi_2\|_{C^4[0,1]},\ \|h\|_{C_\gamma^{\alpha,\beta}[0,T]},\ \|f\|_{C_\gamma(\overline{\Omega})}\}\le\mu_0.$$

We denote by G_{ν_1} the set of function $q(t)$ that for some $T>0$ satisfy the following condition $\|q\|_{C[0,T]}\le\mu_1,\ \mu_1>0$.

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Theorem 3 . *Let* $(\varphi_1,\ \varphi_2,\ h,\ f) \in D_{\nu_0}$, $(\widetilde{\varphi}_1,\ \widetilde{\varphi}_2,\ \widetilde{h},\ \widetilde{f}) \in D_{\mu_0}$ *and* $q,\ \widetilde{q} \in G_{\mu_1}$. *Then, for solution of the inverse problem (1)–(4) the following stability estimate holds:*

$$\|q-\widetilde{q}\|_{C[0,T]} \le r\Big[\|\varphi_1-\widetilde{\varphi}_1\|_{C[0,1]} + \|\varphi_2-\widetilde{\varphi}_2\|_{C[0,1]} + \|h-\widetilde{h}\|_{C_\gamma^{\alpha,\beta}[0,T]} + \|f-\widetilde{f}\|_{C_\gamma(\overline{\Omega})}\Big], \quad (25)$$

where the constant r *depends only on* $\mu_0,\ \mu_1,\ T,\ \alpha,\ \beta$, *and* $\Gamma(\alpha),\ B(\alpha, 1-\gamma)$.

Proof . To prove this theorem, using (21) we write down the equations for $\widetilde{q}(t)$ and compose the difference $\widehat{q} = q(t) - \widetilde{q}(t)$. Then after evaluating this expression and using estimates $u_n(t)$,$\widehat{u}_n(t)$, we obtain following estimates

$$\begin{aligned}
\|q-\widetilde{q}\|_{C[0,T]} \le \max_{0\le t\le T} \Bigg|\frac{1}{h(t)}\Bigg(\int_0^1 w(x)f(x,t)dx - \left(D_{0+,t}^{\alpha,\beta}h\right)(t) \\
+ \int_0^1 w''(x)\Bigg(X_0(x)u_0(t) + \sum_{n=1}^{\infty} X_{2n-1}(x)u_{2n-1}(t) + \sum_{n=1}^{\infty} X_{2n}(x)u_{2n}(t)\Bigg)dx\Bigg) \\
- \frac{1}{\widetilde{h}(t)}\Bigg(\int_0^1 w(x)\widetilde{f}(x,t)dx - \left(D_{0+,t}^{\alpha,\beta}\widetilde{h}\right)(t) + \int_0^1 w''(x)\Bigg(X_0(x)\widetilde{u}_0(t) \\
+ \sum_{n=1}^{\infty} X_{2n-1}(x)\widetilde{u}_{2n-1}(t) + \sum_{n=1}^{\infty} X_{2n}(x)\widetilde{u}_{2n}(t)\Bigg)dx\Bigg)\Bigg| \\
\le \max_{0\le t\le T}\left\{\frac{w_0}{h_0^2}\left|\int_0^1 \Big[h(t)\left(f(x,t)-\widetilde{f}(x,t)\right) + \widetilde{f}(x,t)\left(h(t)-\widetilde{h}(t)\right)\Big]dx\right.\right. \\
\left.\left. + \widetilde{h}(t)\left(\left(D_{0+,t}^{\alpha,\beta}h\right)(t) - \left(D_{0+,t}^{\alpha,\beta}\widetilde{h}\right)(t)\right) + \left(D_{0+,t}^{\alpha,\beta}\widetilde{h}\right)(t)\left(h(t)-\widetilde{h}(t)\right)\right|\right\} \\
+ \max_{0\le t\le T}\left\{\frac{w_0}{h_0^2}\Big|2\left[\widetilde{h}(t)\left(u_0(t)-\widetilde{u}_0(t)\right) + \widetilde{u}_0(t)\left(h(t)-\widetilde{h}(t)\right)\right]\right. \\
+ 4\left[\widetilde{h}(t)\left(u_{2n-1}(t)-\widetilde{u}_{2n-1}(t)\right) + \widetilde{u}_{2n-1}(t)\left(h(t)-\widetilde{h}(t)\right)\right] \\
\left. + 4\left[\widetilde{h}(t)\left(u_{2n}(t)-\widetilde{u}_{2n}(t)\right) + \widetilde{u}_{2n}(t)\left(h(t)-\widetilde{h}(t)\right)\right]\Big|\right\} \\
\le r_0\Bigg(\|\varphi_1-\widetilde{\varphi}_1\| + \|\varphi_2-\widetilde{\varphi}_2\| + \|f-\widetilde{f}\| + + \left\|\left(D_{0+,t}^{\alpha,\beta}h\right) - \left(D_{0+,t}^{\alpha,\beta}\widetilde{h}\right)\right\| \\
+ \|h-\widetilde{h}\|\Bigg) + r_1\int_0^t (t-\tau)^{\alpha-1}\|q(\tau)-\widetilde{q}(\tau)\|_{C[0,T]}d\tau, \quad t\in[0,T], \quad (26)
\end{aligned}$$

where $r_0,\ r_1$ depends only on $\mu_0,\ \mu_1,\ T,\ \alpha$, and $\Gamma(\alpha),\ B(\alpha, 1-\gamma)$. From (26) using lemma 1, we get the estimate

$$\|q-\widetilde{q}\|_{C[0,T]} \le r_0\Bigg(\|\varphi_1-\widetilde{\varphi}_1\|_{C[0,1]} + \|\varphi_2-\widetilde{\varphi}_2\|_{C[0,1]} + \|f-\widetilde{f}\|_{C_\gamma(\overline{\Omega})} + \|h-\widetilde{h}\|_{C_\gamma^{\alpha,\beta}[0,T]}\Bigg)E_{\alpha,1}\left(r_1\Gamma(\alpha)t^\alpha\right),\ t\in[0,T]. \quad (27)$$

This inequality implies the estimate (25), if we set $r = r_0 E_{\alpha,1}\left(r_1\Gamma(\alpha)t^\alpha\right)$.

From theorem 3 follows also the next assertion on uniqueness in whole for solution to the inverse problem.

□

Theorem 4 . *Let the functions* $\varphi_1,\ \varphi_2,\ h,\ f$ *and* $\widetilde{\varphi}_1,\ \widetilde{\varphi}_2,\ \widetilde{h},\ \widetilde{f}$ *have the same meaning as in Theorem* 3 *and conditions* (A1)–(A4). *Moreover, if* $\varphi_1 = \widetilde{\varphi}_1$, $\varphi_2 = \widetilde{\varphi}_2$, $h = \widetilde{h},\ f = \widetilde{f}$, *for* $t \in [0, T]$ *then* $q(t) = \widetilde{q}(t)$ $t \in [0, T]$.

5 Conclusion

In this work, the solvability of a nonlinear inverse problem of the time-dependent source coefficient for a time fractional wave equation with initial-nonlocal boundary and integral overdetermination conditions was studied. Firsty we investigated solvability the initial-nonlocal boundary conditional problem (1)–(3). The problem replaced by an equivalent of integral equation. Existence and uniqueness of direct problem solution were proven. The nonlocal boundary conditions, the Hilfer fractional derivative and the control coefficient made our problem more difficult. The conditions for the existence, uniqueness and continuous dependence upon the data of the problem have been established by using the Fourier method with some bi-orthogonal system, an associated Hilfer fractional derivative which contains an initial data and the Banach fixed point theorem for a product of Banach spaces.

Uncorrected Proof

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