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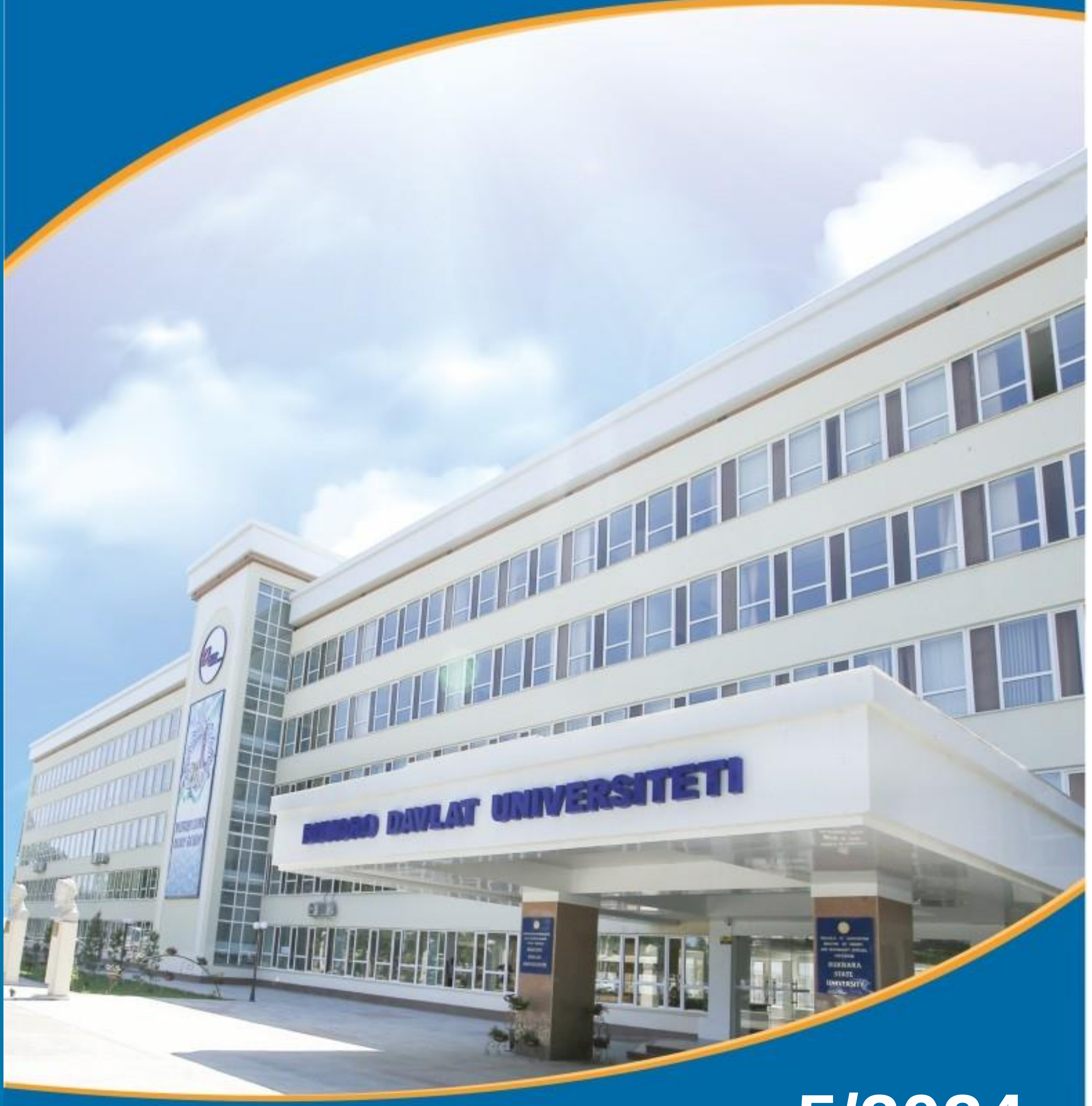

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MUNDARIJA *** СОДЕРЖАНИЕ *** CONTENTS

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DARAJALI YIG'INDILAR VA BERNULLI SONLARI

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Annotatsiya. *Maqolada chekli sondagi dastlabki n ta natural sonlarning chekli m-darajalari yig'indisini hisoblash usullari bayon qilingan. Maqola ilmiy-metodik xarakterda bo'lib, unda darajali yig'indilarni elementar algebra, oliy algebra, matematik analiz usullari yordamida hisoblash yo'llari yoritilgan.*

Shuningdek, hosil qilingan formulalarning ba'zi yig'indilarni hisoblashda tatbiqlari natija sifatida keltirilgan.

Kalit so'zlar: *Daraja, yig'indi, elelmentar, algebra, son.*

СУММЫ УРОВНЕЙ И ЧИСЛА БЕРНУЛИ

Аннотация. *В статье изложены способы вычисления суммы конечных m-ых степеней конечного количества первоначальных n натуральных чисел. Статья научно - методического характера, в ней освещены пути исползования методов элементарной алгебры, высщей алгебры, математического анализа при вычеслении степенных сумм. Также, в качестве следствии приведены применения полученных формул для вычислении некоторых сумм.*

Ключевые слова: *Степень, сумма, элементар, алгебра, число.*

LEVEL SUMS AND BERNOULLI'S NUMBERS

Abstract. *The article describes the methods of calculating the sum of finite m-levels of the first n natural numbers in a finite number. The article is of a scientific-methodical nature, and it describes the ways of calculating the level sums using elementary algebra, higher algebra, and mathematical analysis methods.*

Also, applications of the generated formulas in the calculation of some sums are given as a result.

Key words: *Level, sum, elementary, algebra, number.*

n ixtiyoriy chekli natural son bo'lib, m=0,1,2,.., n bo'lsin. Maqola tahliliy xarakterga ega bo'lib, unda dastlabkini n ta natural sonlarning manfiy bo'lmagan butun darajalari yig'indisi $S_m(n)$ ni hamda 1 dan n gacha bo'lgan natural sonlardan mumkin bo'lgan barcha k ko'paytmalaridan tuzilgan $T_k(n)$ yig'indini hisoblash formulalarining isbotlari keltiriladi. $S_m(n)$ yig'indi bilan Bernulli sonlari orasida bog'lanish ko'rsatiladi.

1. $S_m(n) = 1^m + 2^m + 3^m + ... + n^m$ (1)

yig'indini hisoblash formulalarini keltirib chiqarish usullarini qaraymiz.

1 – yo'l. Ravshanki $S_0(n) = 1+1+1+...+1 = n$ (2)

$$S_1(n) = 1+2+3+...+n = \frac{n(n+1)}{2} \quad (3)$$

$S_2(n)$ ni hisoblash uchun $(a+1)^3 = a^3 + 3a^2 + 3a + 1$ tenglikdan a ning o'rniga navbat bilan 1,2,3,…. , n sonlarni qo'yib, ushbu sonli tengliklarni hosil qilamiz.

$2^3 = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1$
$3^3 = 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1$
……………………………
$(n+1)^3 = n^3 + 3n^2 + 3n + 1$

Bu tengliklarni hadma – had qo'shib, hosil bo'ladigan tenglikda teng qo'shiluvchilarni ixchamlaymiz.

$$2^3+3^3+\ldots+n^3+(n+1)^3=1^3+2^3+3^3+\ldots+n^3+3\cdot(1^2+2^2+3^2+\ldots+n^2)+3\cdot(1+2+3+\ldots+n)+n$$

$$(n+1)^3=3\cdot S_2(n)+3\cdot S_1(n)+(n+1)$$

$$3\cdot S_2(n)=(n+1)^3-(n+1)-3\cdot S_1(n)$$

$$S_2(n)=1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6} \qquad (4)$$

$(a+1)^4=a^4+4a^3+6a^2+4a+1$ tenglikdan foydalanib, yuqoridagiga o'xshab hisoblashlarni takrorlab, ushbu hosil qilinadi.

$$S_3(n)=1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Endi $S_1(n), S_2(n),\ldots\ldots,S_{m-1}(n)$ yig'indilar uchun formulalar ma'lum bo'lganda, $S_m(n)$ ni topish mumkin bo'lgan formulani keltirib chiqaramiz. Buning uchun $(a+1)^{m+1}=a^{m+1}+C_{m+1}^1\cdot 1^m+C_{m+2}^2\cdot 1^{m-1}+\ldots+C_{m+1}^m\cdot a+1$ formulada a ning o'rniga navbat bilan 1,2,3,…., n sonlarni qo'yib, hosil bo'ladigan n ta

$$2^{m+1}=1^{m+1}+C_{m+1}^1\cdot 1^m+C_{m+2}^2\cdot 1^{m-1}+\ldots+C_{m+1}^m\cdot 1+1$$

$$3^{m+1}=2^{m+1}+C_{m+1}^1\cdot 2^m+C_{m+2}^2\cdot 2^{m-1}+\ldots+C_{m+1}^m\cdot 2+1$$

$$(n+1)^{m+1}=n^{m+1}+C_{m+1}^1\cdot n^m+C_{m+2}^2\cdot n^{m-1}+\ldots+C_{m+1}^m\cdot n+1$$

tengliklarni hadma – had qo'shib, soddalashtirib

$(n+1)^{m+1}=(n+1)+C_{m+1}^1\cdot S_m(n)+C_{m+2}^2\cdot S_{m-1}(n)+\ldots+C_{m+1}^m\cdot S_1(n)$ tenglikni hosil qilamiz, bu tenglikdan $S_m(n)$ uchun rekkurent formula topiladi.

$$S_m(n)=\frac{(n+1)((n+1)^m-1)-C_{m+1}^2\cdot S_{m-1}(n)-\ldots-C_{m+1}^m\cdot S_1(n)}{C_{m+1}^1} \qquad (5)$$

2 – yo'l. Nyuton binomi

$(a+b)^m=a^m+C_m^1\cdot a^{m-1}\cdot b+C_m^2\cdot a^{m-2}\cdot b^2+\ldots+C_m^{m-1}\cdot a\cdot b^{m-1}+b^m$ formulasida $b=-1$ ni qo'yib, m ni m+1 bilan almashtirib,

$(a-1)^{m+1}-a^{m+1}=-C_{m+1}^1\cdot a^m+C_{m+1}^2\cdot a^{m-1}-\ldots+(-1)^m C_{m+1}^m\cdot a+(-1)^m$ tenglik hosil qilinadi. Bu tenglikda a ning o'rniga navbat bilan 1,2,3,…. ,n sonlarni qo'yib, hosil boladigan n ta tenglikarni qo'shib, so'ngra soddalashtirib,

$$S_m(n)=\frac{n^{m+1}+C_{m+1}^2\cdot S_{m-1}(n)+\ldots(-1)^m C_{m+1}^1\cdot S_1(n)+(-1)^{m+1}\cdot S_0(n)}{C_{m+1}^1} \qquad (6)$$

rekkurent formula hosil qilinadi.

3 – yo'l. $t^n-1=((t-1)+1)^n-1$ tenlikka Nyuton binomi formulasini qo'llaymiz.

$$t^n-1=(t-1)^n+C_n^1(t-1)^{n-1}+C_n^2(t-1)^{n-2}+\ldots+C_n^{n-2}(t-1)^2+C_n^{n-1}(t-1)$$

Bu tenglikka $t\neq 1$ deb olib, uning har ikkala tomonini $t-1$ ga bo'lamiz:

$$\frac{t^n-1}{t-1}=1+t+t^2+\ldots+t^{n-2}+t^{n-1}=C_n^1+C_n^2(t-1)+\ldots+C_n^{n-1}(t-1)^{n-2}+(t-1)^{n-1}$$

Endi $t^m=a_m$ $(m=0,1,\ldots,n-1)$ almashtirish bajaramiz:

$$a_0+a_1+a_2+\ldots+a_{n-1}=C_n^1+(a_1-a_0)C_n^2+(a_2-2a_1+a_0)C_n^3+\ldots$$

$$\ldots+(a_{n-1}-C'^1_n\,a_{n-2}+C'^2_n\,a_{n-3}-\ldots+(-1)^{n-2})C_n^n$$

Bu tenglikda $a_l = (l+1)^m,\ (l = 0,1,..n-1)$ deb olib, ushbu formula hosil qilinad:

$$S_m(n) = C_n^1 + (2^m - 1)C_n^2 + (3^m - 2\cdot 2^m + 1)C_n^3 + \dots$$
$$\dots + \left[(m+1)^m - C_m^1 m^m + C_m^2 (m-1)^m - \dots + (-1)^m\right]\cdot C_n^{m+1} \qquad (7)$$

4 – yo'l.

$S_m(n) = 1^m + 2^m + \dots + n^m$ yig'indi $S_m(n) = x_1{}^m + x_2{}^m + \dots + x_n{}^m$ (8)

simmetrik ko'phadning $x_k = k\ (k = 1,2,\dots,n)$ bo'lgandagi qiymatidan iborat.

Ma'lumki, S_m simmetrik ko'phad elementar simmetrik $\sigma_1, \sigma_2, \dots\dots, \sigma_n$ ko'phadlar orqali ifodalanadi. Shuningdek, $S_1, S_2, \dots\dots, S_k$ simmetrik ko'phadlar $\sigma_1, \sigma_2, \dots\dots, \sigma_n$ elementar simmetrik ko'phadlar orqali ifodalanadi va aksincha. Bu esa S_m ni $S_1, S_2, \dots\dots, S_{m-1}$ lar orqali ifodalash formulasini topishga imkon beradi. Buning uchun ushbu

$$S_m = S_{m-1}\sigma_1 - S_{m-2}\sigma_2 + \dots + (-1)^{m-1}S_1\sigma_{m-1} + (-1)^m\cdot m\cdot\sigma_m \quad (m \le n) \qquad (9)$$

$$S_m = S_{m-1}\sigma_1 - S_{m-2}\sigma_2 + \dots + (-1)^n S_{m-n}\sigma_n \quad (m > n) \qquad (10)$$

Nyuton formulalaridan foydalaniladi.

$S_m(n)$ yig'indi uchun (6) formuladan foydalanib, Bernulli sonlarini topamiz. U

$$C_{2m+1}^2\cdot B_1 - C_{2m+1}^4\cdot B_2 + C_{2m+1}^6\cdot B_3 - \dots + (-1)^{m-1}C_{2m+1}^{2m}\cdot B_m = m - \frac{1}{2} \qquad (11)$$

tenglik bilan aniqlanadigan $B_1, B_2, \dots\dots, B_m$ sonlar Bernulli sonlari deyiladi.

$S_m(n)$ ozod hadi 0 ga teng ko'phad bo'lganligi uchun $S_m(n) : n$ ham ko'phad bo'lib, uni $P_m(n)$ bilan belgilab, ushbu ko'phad hosil qilinadi.

$$P_m(n) = \frac{1}{m+1}\left[n^m + C_{m+1}^2\cdot\frac{S_{m-1}(n)}{n} - C_{m+1}^3\cdot\frac{S_{m-2}(n)}{n} + \dots + (-1)^m\cdot C_{m+1}^m\cdot\frac{S_1(n)}{n} + (-1)^{m+1}\cdot\frac{S_{01}(n)}{n}\right]$$

$P_m(n)$ ko'phadda n=0 bo'lganda $B_m = P_m(0)$ Bernulli soni hosil bo'ladi.

$$P_m(0) = B_m = \frac{1}{m+1}\left[C_{m+1}^2\cdot B_{m-1} - C_{m+1}^3\cdot B_{m-2} + \dots + (-1)^m\cdot C_{m+1}^m\cdot B_1 - (-1)^m\cdot B_0\right] \qquad (12)$$

(12) formula yordamida dastlabki bir necha Bernulli sonlarini topamiz. $B_0 = 1$ deb olinadi

$B_1 = \frac{1}{2};\ B_2 = \frac{1}{6};\ B_3 = 0;\ B_4 = -\frac{1}{30};\ B_5 = 0;\ B_6 = \frac{1}{42}$ hisoblashlardan ko'rinadiki B_1 dan boshqa barcha toq nomerli Bernulli sonlari 0 ga teng. Shu bois ayrim hollarda toq nomerli Bernulli sonlari nazarga olinmay va $(-1)^{m-1}\cdot B_m$ sonlar o'rniga B_m sonlar olinib, ularni yangidan nomerlab,

$B_1 = \frac{1}{6};\ B_2 = \frac{1}{30};\ B_3 = \frac{5}{66};\ B_4 = \frac{691}{2730};\dots$ Bernulli sonlari hosil qilinadi. Bernulli sonlari ta'rifi bo'yicha aynan shu sonlar hosil bo'ladi.

Shu tarzdagi Bernulli sonlari natural sonlarning m – darajali yig'indisi formulasida qatnashgan bo'ladi:

$$\sum_{k=1}^{n}k^m = \frac{n^{m+1}}{m+1} - \frac{n^m}{2} + \frac{m}{2!}\cdot B_1\cdot n^{m-1} - \frac{m(m-1)(m-2)}{4!}\cdot B_2\cdot n^{m-3} + \frac{m(m-1)(m-2)(m-3)(m-4)}{6!}\cdot B_3\cdot n^{m-3} - \dots$$

m ning juft va toqligiga qarab o'ng tomonda n yoki n^2 hadgacha yoziladi.

Misol. m = 1 uchun $\sum_{k=1}^{n}k = \frac{n^{1+1}}{1+1} + \frac{n^1}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$,

m = 2 uchun $\sum_{k=1}^{n}k^2 = \frac{n^2}{3} + \frac{n^2}{2} + \frac{2}{2!}\cdot B_1\cdot n = \frac{n^2}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$,

$m = 3$ uchun $\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{3}{2!} \cdot B_1 \cdot n^2 = \frac{n^2}{4} + \frac{n^2}{2} + \frac{3}{2} \cdot \frac{1}{6} \cdot n = \frac{n^2(n+1)^2}{4}$.

Bernulli sonlari darajali yig'indilar bilan bog'liq bo'lgan holdamatematikaning boshqa problemalarini yechishda ham tatbiq qilinadi. Jumladan Ferma teoremasining ba'zi xususiy hollari uchun o'rinli bo'lishini isbotlashda, Varing problemasini yechish yo'lida Bernulli sonlaridan foydalaniladi. Shuningdek $E_0 + C_{2m}^2 E_2 + C_{2m}^4 E_4 + C_{2m}^6 E_6 + ... + C_{2m}^{2m-2} E_{22m-2} = 0,\ E_0 = 1; E_2 = -1; E_4 = 5;...$ tenglik bilan aniqlanadigan E_{2m} Eyler sonlari bilan Bernulli sonlari orasida bog'lanish mavjud.

Endi maqolada nazarda tutilgan ikkinchi masalani qaraymiz. 1 dan *n* gacha bo'lgan natural sonlardan m tadan olib tuzilishi mumkin bo'lgan barcha ko'paytmalar yig'indisi $T_m(n)$ uchun formulani keltirib chiqarish bilan shug'ullanamiz. Bunda darajali yig'indilar uchun hosil qilingan formulalardan foydalanamiz.

$$T_1(n) = \sum_{k=1}^{n} k = S_1(n) = \frac{n(n+1)}{2}, \quad T_n(n) = n!$$

$$T_2(n) = \sum_{1 \le i \le j \le n} i \cdot j = 1 \cdot 2 + 2 \cdot 3 + ... + 1 \cdot n + 2 \cdot 3 + 2 \cdot 4 + ... + 2 \cdot n + ... + (n-1) \cdot n$$

yig'indini hisoblash formulasini keltirib chiqarish uchun

$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i<j} x_i x_j$ ayniyatda barcha *x* larni ularning indekslari bilanalmashtirishdan hosil bo'ladigan $\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^2 + 2\sum_{i<j} ij$ tenglikda oxirgi yig'indini topamiz:

$$T_2(n) = \sum_{1 \le i \le j \le n} i \cdot j = \frac{1}{2}\left(\sum_{i=1}^{n} i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} i^2 = \frac{1}{2} S_1^{\ 2}(n) - \frac{1}{2} S_2^{\ 2}(n) = \frac{(n-1)n(n+1)(3n+2)}{24}$$

$T_3(n) = \sum_{1 \le i \le j \le n} ijk$ yig'indini hisoblash formulasini topish uchun

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^{\ 3} + 3\sum_{\substack{i \ne j \\ i,j=1}} x_i x_j^{\ 2} + 6\sum_{1 \le i < j < k \le n} x_i x_j x_k \quad \text{va} \quad \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j^{\ 2}\right) = \sum_{i=1}^{n} x_i^{\ 3} + \sum_{i \ne j} x_i x_j^{\ 2}$$

ayniyatlardan foydalanib, ushbuni topamiz:

$$\sum_{1 \le i < j < k \le n} x_i x_j x_k = \frac{1}{6}\left(\sum_{i=1}^{n} x_i\right)^3 - \frac{1}{6}\sum_{i=1}^{n} x_i^3 - \frac{1}{2}\left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{i=1}^{n} x_i^{\ 2}\right) + \frac{1}{2}\sum_{i=1}^{n} x_i^{\ 3} = \frac{1}{6}\left[\left(\sum_{i=1}^{n} x_i\right)^3 + 2\left(\sum_{i=1}^{n} x_i^{\ 3}\right) - 3\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j^{\ 3}\right)\right]$$

Barcha *x* larni ularning indekslari bilan almashtiramiz, so'ngra darajali yig'indilardan foydalanamiz:

$$T_3(n) = \sum_{1 \le i < j \le n} ijk = \frac{1}{6}\left[\left(\sum_{i=1}^{n} i\right)^3 + 2\sum_{i=1}^{n} i^3 - 3\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j^2\right)\right] = \frac{1}{6}\left[S_1^{\ 3}(n) + 2S_3(n) - 3S_1(n)S_2(n)\right] = \frac{(n-2)(n-1)n(n+1)^2}{48}$$

Umumiy holda $T_m(n)$ ni hisoblash formulasini keltirib chiqarishda $\left(\sum_{i=1}^{n} x_i\right)^m$ uchun polinomial formuladan foydalanish ancha murakkab ifodalar bilan ishlashga olib keladi. Shu sababli $T_m(n)$ ni ancha qulay ushbu yo'l bilan aniqlanadi. $T_m(n)$ yig'indi qo'shiluvchilarini ikki guruhga ajratiladi. Birinchi guruhga *n* qatnashaydigan qo'shiluvchilar olinadi. Ularning yig'indisi $T_m(n-1)$ bo'ladi. Ikkinchi guruhga n qatnashadigan qo'shiluvchilar olinadi. Agar bunday qo'shiluvchilar yig'indisidan umumiy n ko'paytuvchi qavsdan tashqariga chiqarilsa, natijada $n \cdot T_m(n-1)$ bo'ladi. Shunday qilib,

$$T_m(n) = T_m(n-1) + n \cdot T_{m-1}(n-1) \quad (13)$$

Bu rekkurent formula yordamida $n \geq m$ uchun $T_{m-1}(n)$ ma'lum bo'lganda $T_m(n)$ topiladi.

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