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Numerical Simulation of Compressible Gas Flow in Flat Channels in the Narrow Channel Approximation

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Abstract—A compressible gas in plane channels of constant and variable cross sections is numerically simulated using two-dimensional parabolized Navier–Stokes equations. The system of equations is solved numerically using the narrow-channel approximation model. A number of transformations, such as nondimensionalization of the system of equations to reduce the given domain to a square and refinement of computational points with large gradients of gas-dynamic parameters, are described in detail. Pressure gradient is determined from the flow-rate conservation condition. An efficient method is given for simultaneously determining the pressure gradient and longitudinal velocity, followed by other gas-dynamic parameters of stability for subsonic and supersonic flows, as well as a method for determining the critical flow rate for solving Laval nozzle problems. The results of methodical calculations are presented to validate the calculation methodology and confirm the reliability of the results by comparing them with data obtained by other authors.

Keywords: internal flow, narrow-channel approximation, Laval nozzles, modeling, flow, flow rate, numerical solution, pressure gradient, Mach number

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INTRODUCTION

The advent of modern powerful computing technologies has made it possible to comprehensively study internal flows using mathematical modeling [1–4].

The relevance of these studies is directly related to the application of their results to the creation of various mixing and combustion devices, combustion chambers of various power facilities, and internal combustion engines, which are directly associated with the steady increase in the share of natural gas in the fuel balance of the world, with the progressive growth in the unit power of thermal aggregates, as well as to the environmental pollution and other problems. The development of rocket and space technologies and the creation of high-power continuous gas lasers has especially provoked much interest in studying the flow of gases and gas mixtures in flat channels and nozzles when the effect of viscosity is significant. The most complete picture of the flow in narrow channels (nozzles) can be obtained by solving the complete system of Navier–Stokes equations [2–7]. However, since this is associated with major mathematical difficulties and large computer times, simpler approaches are appropriate in some cases. The first approach is to traditionally separate the flow into an inviscid core and a flow in the boundary layer [5–10], which is very justified and gives good results for relatively thin boundary layers. However, if boundary layers cover a significant part or the entire height of the nozzle cross section (channel), the so-called narrow-channel approximation [2], [11–15] and a model based on parabolized Navier–Stokes equations are normally used.

The use of these models is justified to describe real flows of viscous gases and gas mixtures, when there is a certain dominant motion direction in the flow and the upstream propagation of disturbances along this direction is insignificant. The second condition for the applicability of models of this group is $\mathrm{Re}^{-1} \ll 1$ [2].

The narrow-channel approximation model is designed exclusively for describing internal flows in channels with a size in the major flow direction much larger than its size in cross sections that are perpendicular to this direction, i.e., $f_0 \ll L$ (where f_0 is the inlet cross section and L is the length of the channel).

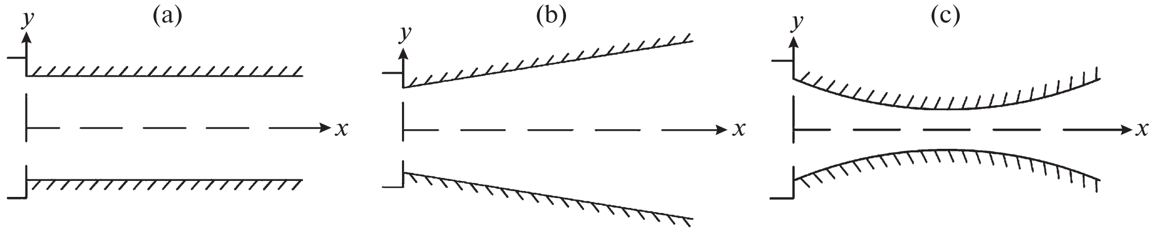

Fig. 1.

The narrow-channel equations are obtained from the Navier–Stokes equations under the assumption that the ratio of the longitudinal to transverse velocity component and the ratio of the longitudinal to transverse gradient are of the order of the relative elongation of the channel (nozzle) [2], [13].

1. STATEMENT OF THE PROBLEM

Assume that a gas flow with certain physical parameters going from a chamber into a flat axisymmetric channel (Figs. 1a–1c) with an inlet cross-section f_0 and length L is viscous, heat-conducting, and two-dimensional under no-gravity conditions.

Under the assumptions formulated above, the system of equations describing the given process can be represented in physical coordinates as [2], [11–14]:continuity equation

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{1}$$

equation of motion along the x axis

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{2}$$

equation of energy

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\frac{\mu}{P_\Gamma}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

equation of state

$$\frac{P}{\rho} = \frac{RT}{M}, \tag{4}$$

where x and y are the longitudinal and transverse coordinates, respectively; ρ is density; P is pressure; $h = C_pT$ is enthalpy; C_p is the heat capacity at constant pressure; T is temperature; R is the universal gas constant; M is the molecular weight; μ and λ are the viscosity and thermal conductivity coefficients, respectively; and $P_\Gamma = \mu C_p/\lambda$ is the Prandtl number.

The values of μ and λ were calculated from formulas presented in [16] and the heat capacity was approximated by a quadratic dependence tabulated in [17].

2. BOUNDARY AND INITIAL CONDITIONS

To set boundary conditions, we assume that the channel of constant and variable cross sections (nozzle) is symmetrical (Fig. 1). In this case, we can limit ourselves to considering the flow in the region between the axis of symmetry and one of the channel walls. The reduced system of differential equations (1)–(4) can be solved using the following boundary and initial conditions:

$$\begin{gathered} x = 0, \quad 0 \le y < f(x): \quad u = u_0, \quad p = p_0, \quad h = h_0, \quad v = 0; \\ y = f(x): \quad 0 \le x \le L, \end{gathered} \tag{5}$$

$$u = 0, \quad v = 0, \quad h = h_\omega \quad \left(\partial h/\partial y = 0 \ \ or \ \ \frac{\partial h}{\partial y} = q_\omega\right), \tag{6}$$

$$y = 0, \quad 0 < x \le L: \quad \frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} = 0. \tag{7}$$

Under these conditions, the subscript ω refers to parameters on the channel wall, $h_\omega = C_p T_\omega$, or the heat flux $\partial h/\partial y = q_\omega$ in the channel wall.

Condition (5) means that the channel inlets are given by the profiles of gas-dynamic quantities, and (6) and (7) are the boundary condition on the channel wall and the symmetry condition, respectively.

The system of equations (1)–(4) with initial and boundary conditions (5)–(7) is not closed, since there is no equation for the pressure gradient dp/dx.

To specify this equation, we assume flow rate conservation along the channel:

$$\int_0^{f(x)} \rho u dy = Q = \text{const}. \tag{8}$$

Normally, the pressure gradient is obtained from the system of equations (1)–(4) using the shooting method or some of its modifications [7], [18], implying that a certain value of the pressure gradient is specified, equations (1)–(4) are solved, the flow-rate conservation equation is checked, and a new pressure gradient value is selected if the required accuracy is not met. The efficiency of this method is low, since each value of dp/dx requires convergence of the iteration for solving the system of equations (1)–(4); i.e., these equations actually have to be solved many times at each step with respect to the longitudinal coordinate.

A more economical method is the simultaneous determination of the pressure gradient and other gas-dynamic parameters [14]; however, this method is stable only for subsonic gas flows [15]. This paper proposes a modification of the method [14], expanding the scope of application for subsonic and supersonic flows.

3. SOLUTION METHOD

System of equations (1)–(4) with boundary and initial conditions (5)–(7) and additional condition (8) is numerically solved by reducing them to dimensionless equations.

To this end, we take some characteristic values of the length, velocity, temperature, pressure, and molecular weight of $f_0, u_0, T_0, \rho_0 u_0^2$, and M_0 as the scales of these quantities; the values of physical properties of a gas and its transfer coefficients calculated at temperature T_0 and pressure P_0; $\rho_0, C_p, \mu_0, \lambda_0$ as their corresponding scales; and the quantity $C_p T_0$ as the scale of specific enthalpy h_0.

The dimensional and dimensionless parameters are related as

$$\bar{x} = \frac{x}{f_0}, \quad \bar{y} = \frac{y}{f_0}, \quad \bar{u} = \frac{u}{u_0}, \quad \bar{\nu} = \frac{\nu}{\nu_0}, \quad \bar{\rho} = \frac{\rho}{\rho_0}, \quad \bar{p} = \frac{p}{\rho_0 u_0^2}, \quad \bar{\mu} = \frac{\mu}{\mu_0}, \quad \bar{\lambda} = \frac{\lambda}{\lambda_0},$$
$$\bar{T} = \frac{T}{u_0^2/R_m}, \quad \overline{C_p} = \frac{C_p}{R_m}, \quad \bar{L} = \frac{L}{f(0)}. \tag{9}$$

Here, f_0 is the half-height of the channel, and the subscript 0 denotes jet parameters at the channel entrance.

To reduce the problem to a model problem, we transform the given domain using the following coordinate transformation:

$$\xi = \bar{x}/\bar{L}; \quad \eta = \bar{y}/f(\bar{x}), \tag{10}$$

where $f(\bar{x})$ is the dimensionless channel profile.

We use the following formulas to go from physical $(\bar{x}, \bar{y})$ to new (ξ, η) variables:

$$\frac{\partial}{\partial \bar{x}} = \frac{1}{\bar{L}}\frac{\partial}{\partial \xi} - \frac{\eta f'}{f}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial \bar{y}} = \frac{1}{f}\frac{\partial}{\partial \eta}, \quad \frac{\partial^2}{\partial \bar{y}^2} = \frac{1}{f^2}\frac{\partial^2}{\partial \eta^2}. \tag{11}$$

It is well known that the flow in the near-wall region is characterized by large gradients of gas-dynamic parameters; therefore, it makes sense to perform a coordinate transformation allowing the near-wall computational points to be refined in the physical plane and conserving a constant step in the computational plane.

This transformation has the form

$$F(y) = 1 - \frac{\ln[1 + K_y(e-1)\eta]}{\ln[1 + K_y(e-1)]}, \tag{12}$$

where $e \approx 2.71828$.

Also, the stepwise specification of input parameters requires a refinement of computational points in the inlet area of the channel. To this end, we introduce analytical functions that transform the refined computational mesh of the inlet area of the channel:

$$\varphi(x) = 1 - \frac{\ln[1 + K_x(e-1)\xi]}{\ln[1 + K_x(e-1)]}, \tag{13}$$

where k_y, and K_x are coordinate refinement parameters.

For the nozzle problem, it makes sense to introduce a refinement function into the critical cross section [7].

In transformations (12)–(13) and below, the reduced coordinates (x, y) are used out of habit.

Due to transformations (9)–(13), Eqs. (1)–(4) take the following form:continuity equation

$$\frac{F_y}{L}\frac{\partial}{\partial x}\rho u + \frac{\varphi(x)}{f(x)}\frac{\partial}{\partial y}\rho v^* = 0, \tag{14}$$

equation of motion

$$\frac{\rho u F_y}{L}\frac{\partial u}{\partial x} + \rho\frac{v^*\varphi(x)}{f(x)}\frac{\partial u}{\partial y} = -\frac{F_y}{L}\frac{dp}{dx} + \frac{\varphi(x)}{f^2(x)}\frac{\partial}{\partial y}\left(\frac{\mu}{F_y}\frac{\partial u}{\partial y}\right), \tag{15}$$

equation of energy

$$\frac{\rho u F_y}{L}\frac{\partial h}{\partial x} + \rho\frac{v^*\varphi(x)}{f(x)}\frac{\partial h}{\partial y} = \frac{F_y u}{L}\frac{dp}{dx} + \frac{\varphi(x)}{f^2(x)}\frac{\partial}{\partial y}\left(\frac{\mu}{F_y P_r}\frac{\partial h}{\partial y}\right) + \frac{\mu\varphi(x)}{f^2(x)F(y)}\left(\frac{\partial u}{\partial y}\right)^2, \tag{16}$$

equation of state

$$P = \rho T, \tag{17}$$

relationship between enthalpy and temperature

$$h = C_p T, \tag{18}$$

relationship between dynamic viscosity and temperature

$$\mu = \mathrm{const}\, T^{0.64}, \tag{19}$$

equations

$$v^* = (v - uyf_x')/f(x).$$

Dimensionless boundary and initial conditions (5)–(6) are as follows:initial conditions

$$\begin{gathered} x = 0, \quad 0 \le y < 1: \\ u = 1, \quad P = \bar{P}_0, \quad h = \bar{h}_0, \quad \rho = 1, \quad v = 0; \end{gathered} \tag{20}$$

boundary conditions

$$\begin{gathered} y = 1, \quad 0 \le x \le 1: \\ u = 0, \quad v = 0, \quad h = \bar{h}_\omega \quad (\partial h/\partial y = 0), \\ y = 0, \quad 0 < x \le L: \\ \frac{\partial u}{\partial y} = \frac{\partial h}{\partial y} = 0. \end{gathered} \tag{21}$$

Condition (8) for the flow-rate conservation along the channel is

$$f(x)\int_0^1 F_y \rho u dy = Q = \mathrm{const}. \tag{22}$$

To numerically integrate system of equations (14)–(19) with initial and boundary conditions (20)–(21), we use a two-level six-point implicit finite-difference scheme of the Crank–Nicholson type [3].

Introducing the grid function F_j^n (n and j are the point numbers with respect to x and y, respectively), we can write the finite-difference analogues of individual terms in (14)–(16) in the form

$$A\frac{\partial F}{\partial x} = A_j^{n+s}\left(F_{j+1}^{n+1} - F_j^n\right)/\Delta x,$$
$$A\frac{\partial F}{\partial y} = A_j^{n+s}\left(F_{j+1}^{n+s} - F_{j-1}^{n+s}\right)/(2\Delta y), \tag{23}$$
$$\frac{\partial}{\partial y}\left(A\frac{\partial F}{\partial y}\right) = \left[A_{j+1/2}^{n+s}\left(F_{j+1}^{n+s} - F_j^{n+s}\right) - A_{j-1/2}^{n+s}\left(F_j^{n+s} - F_{j-1}^{n+s}\right)\right]/\Delta y^2,$$

where the subscript $n+s$ means $F^{n+1}S + F^n(1-s)$.

This scheme has a second order of approximation with respect to y and x for $S = 1/2$ and a first order with respect to x for $S \neq 1/2$.

Using differential analogues (23), Eqs. (15)–(16) can be written as

$$A_jF_{j-1}^{n+1} + B_jF_j^{n+1} + C_jF_{j+1}^{n+1} + D_jP^{n+1} = E_j, \quad j = \overline{2, N-1}, \tag{24}$$

where $F = u, h$ for the equation of energy and $D_j = 0, N$ is the number of computational points with respect to y.

Boundary conditions (21) in the difference form give equations for $j = 1$ and $j = N$. For example, conditions $\partial F/\partial y|_{j=1} = 0$ can be written with the first

$$F_1 = F_2$$

or second order of accuracy

$$-3F_1 + 4F_2 + F_3 = 0.$$

The conditions on the wall have the form

$$F_N = E_\omega \quad (E_\omega = 0 \ for\ u, \nu;\ h = C_pT_\omega \ or\ h_N = h_{N-1} \ or\ h).$$

Using the standard sweep method, one can easily obtain solutions of (24) in the form

$$F_j^{n+1} = \alpha F_{j+1}^{n+1} + \beta_j + \gamma_jP^{n+1}, \tag{25}$$

where $\alpha_j = -c_j/(A_j\alpha_{j-1} + \beta_j)$,

$$\beta_j = (E_j - A_j\beta_{j-1})/(A_j\alpha_{j-1} + \beta_j),$$
$$\gamma_j = -(D_j + A_j\gamma_{j-1})/(A_j\alpha_{j-1} + \beta_j), \quad j = \overline{2, N-1},$$

and α_1, β_1 and γ_1 are determined from the boundary conditions for $y = 0$.

For further use, it is convenient to represent solution (25) as

$$F_j^{n+1} = \varepsilon_j + \varphi_jP^{n+1}, \tag{26}$$

where

$$\varepsilon_j = \alpha_j\varepsilon_{j+1} + \beta_j, \quad \varphi_j = \alpha_j\varphi_{j+1} + \gamma_j,$$

and ε_N and φ_N are determined from the boundary conditions on the wall. For the equation of energy $\gamma_j = \varphi_j = 0$, the enthalpy at the layer $n+1$ can be obtained directly from (26).

To determine the pressure from the condition of flow-rate conservation, we use the method described in [14] to substitute expression (26) for u^{n+1} $\left(u_j^{n+1} = \varepsilon_j + \varphi_jP^{n+1}\right)$ into formula (22) and express density ρ in terms of the desired pressure P^{n+1} of (17). As a result, we obtain a quadratic equation with respect to P^{n+1}:

$$\left[f(x)\int_0^1 \frac{\varphi}{T}F_ydy\right]\left(P^{n+1}\right)^2 + \left[f(x)\int_0^1 \frac{\varepsilon}{T}F_ydy\right]P^{n+1} - Q = 0. \tag{27}$$

Solving this equation, one can easily find u_j^{n+1} for all $j = \overline{1, N-1}$ using (26).

The criterion for choosing the corresponding root of equation (27) is

$$\int_0^1 \frac{\varepsilon + 2\varphi p}{T} F_y \geq 0.$$

It should be noted that, unlike the well-known analogue by Simuni's method (when the density is taken from the previous iteration), the algorithm that simultaneously determines velocity and pressure is stable in both the subsonic and supersonic ranges, which makes it possible to construct a single economical algorithm for calculating the flow in the entire channel (nozzle).

The calculation algorithm for a channel of constant cross-section and for an expanding channel can be described as follows:

(a) specify the first approximation of desired unknowns on the level $(n+1)$ using the value of the layer (n);

(b) find pressure P^{n+1} from Eq. (27); here, $\int_0^1 \frac{\varphi}{T} F_y dy$ and $\int_0^1 \frac{\varepsilon}{T} F_y dy$ mean the tabulated results of numerical integration of $\frac{\varphi}{T}$ and $\frac{\varepsilon}{T}$, respectively;

(c) find the longitudinal velocity u_j^{n+1} from the equation $u_j^{n+1} = \varepsilon_j + \varphi_j P^{n+1}$;

(d) find enthalpy h_j^{n+1} from Eq. (16); here, u_j^{n+1} is taken to be the value of longitudinal velocity;

(e) find E_j^{n+1} from relation (18);

(f) find density ρ_j^{n+1} from (17);

(g) calculate viscosity μ_j using formula (19);

(h) find v_j^* by integrating continuity equations (14):

$$\rho_j^{n+1}(v_j^*)^{n+1} = -\frac{f(x)}{L\varphi_x}\int_0^1 F_y \frac{\partial}{\partial x}\rho u dy.$$

The iterations continue until the difference of two successive approximations converges with a given accuracy.

The algorithm for calculating the flow in the nozzle is much different due to the need of finding the critical flow rate Q^*. An acceptable integration accuracy in the longitudinal direction in the entire computational domain is ensured using a scheme with automatic step selection, which is based on the following criteria.

(1) The relative change in pressure cannot exceed a given value, i.e., $|\Delta P/P| \leq \delta$ (where δ is a small number). If this condition is not met, the step is halved.

(2) The number of iterations K required to for the convergence of the iterative process with respect to
1 temperature and pressure cannot exceed the givem value $K_{\max}$.

(3) If at three successive steps $\Delta x_{n-2} = \Delta x_{n-1} = \Delta x_n$ the iteration converges at $K < K_{\max}$, then the step is doubled.

The last criterion increases the integration step in domains with small longitudinal gradients. The calculation of the flow in nozzles becomes complicated due to the determination of the critical flow rate and the transition of the saddle point.

In this study, like in [12], the critical flow rate is calculated using the shooting method; however, the cases $Q < Q^*$ and $Q > Q^*$ are diagnosed differently. A certain flow rate Q is specified and a parabolic sys-
2 tem of equations is solved numerically (using the abovementioned algorithm). If the region behind the minimum cross section of the nozzle is characterized by increased pressure in several successive located computational layers, this situation is diagnosed as $Q < Q^*$. The case of $Q > Q^*$ is essentially based on the properties of the constructed scheme with automatic selection of the longitudinal integration step. The next step is chosen from its comparison with $\Delta x_{\min}$; if $\Delta x \ll \Delta x_{\min}$, this situation is diagnosed as $Q > Q^*$. The quantity $\Delta x_{\min}$ is the minimum value of the step along the longitudinal coordinate, chosen by the scheme to calculate the subsonic flow regime with the maximum of the already calculated test flow rates.

The scheme parameters (the accuracy of iteration convergence, the maximum permissible number of iterations, and the magnitude of the relative pressure change at one step) are chosen so that the value of Δx in the subsonic regime is determined by the last criterion, when the case $\Delta x \ll \Delta x_{\min}$ corresponds precisely to the nozzle blocking.

When Q is chosen with a required accuracy, the above algorithm is used to make a final calculation of the flow in the nozzle with a maximum subsonic flow rate. The transition through the saddle point is carried out by pressure extrapolation.

4. IMPLEMENTATION OF THE METHOD AND NUMERICAL RESULTS

2 The abovementioned method was implemented as a program. This program can be used to calculate
3 laminar and turbulent flows in flat channels of constant and variable (in Laval nozzles) crosss sections.

It should be noted that all variants of initial and boundary conditions for temperature (of the first or second kind), velocity on the wall at the entrance of specific computational points along x and y axes, the refinement coefficients K_y and K_x, and the specification of the channel shape (size) are included in the set of initial data; therefore, the solution of a wide range of specific problems avoids the need to make any
4 changes in the program, and the sprogram is versatile in this sense.

Next, we present the results of methodological calculations aimed at verifying the efficiency of the calculation methodology and confirming the reliability of the results by comparing them with data obtained by other authors.

The first step in verifying the methodology was to analyze the convergence of the scheme when the difference grid is refined along the longitudinal and transverse coordinates for the flow in a flat channel of constant and expanding cross section. To this end, we used analytical transformations (12) and (13), refining in the mesh of the near-wall region and at the channel entrance, and performed a number of calculations with varying number of nodes N_x and N_y N_x and N_y are the numbers of calculation points along the longitudinal and transverse coordinates, respectively) and mesh refinement coefficients K_y and K_x.

For these purposes, we considered a flat channel (Fig. 1a) with a ratio of length to half-height equal to five and an expanding channel of dimensionless length $L = 10$ with an expansion degree of two ($f(x) = x + 1, \mathrm{Re} = 34$); the Reynolds number $\mathrm{Re} = \rho_0 u_0 f_0/\mu_0$ was calculated from initial values of density, velocity, half-height of the channel, and dynamic viscosity, with a given and power-law dependence on T according to formula (19).

In both cases, the results of calculations performed in different computational grids ($Nx \times Ny = 11 \times 11$; 21×21; 31×31; 41×41) indicate good convergence starting from the grid $N_x = N_y = 21$.

The results obtained on the grid $N_x = N_y = 16$ with refinement coefficients $K_y = -0.4722$ and $K_x = 2.6041$ coincide with the results obtained for sufficiently large grids ($N_x, N_y \geq 31$). Figure 2 shows the longitudinal velocity profiles at the exit of a channel of constant cross-section obtained in the narrow-channel approximation and based on complete Navier–Stokes equations [7].

5 A significant difference in the results for small compuational points is observed near the inlet and near-wall regions of the contour under consideration. Serial calculations have shown that these cases require calculations with a sufficiently large number of computational points or a refinement with the selection of refinement coefficients.

To check the efficiency of the method developed, we calculated the air flow in a flat wedge-shaped nozzle (its main geometric characteristics are shown in Fig. 3).

Figure 4 shows the distribution of pressure (1), average flow rate (2), and temperature (3) along the nozzle.

Figure 5 shows the Mach number profiles in various sections of the nozzle (1 for $x/L = 0, 1$; 2 for $x/L = 0, 4$; and 3 for $x/L = 1$; where L is the nozzle length), illustrating the development of the boundary layer along the nozzle contour. These results indicate an increased mean-flow rate and sharply decreased pressure and temperature along the nozzle, which exist for flows in Laval nozzles.

The model presented here and the calculation method make it possible to change the gas-dynamic parameters of the flow in a wide range and conduct numerical studies of laminar and turbulent internal flows in flat channels.

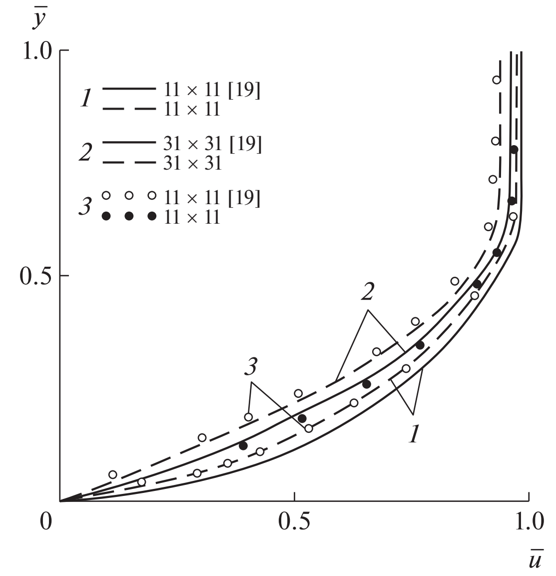

Fig. 2.

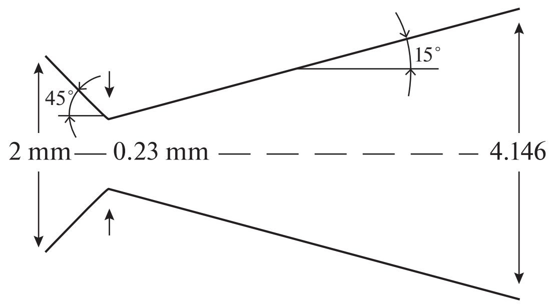

Fig. 3.

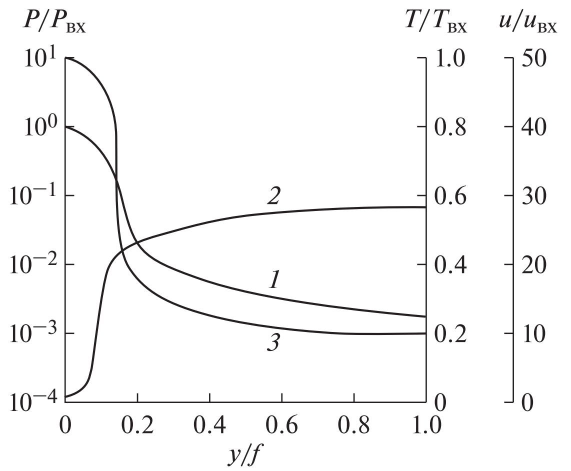

Fig. 4.

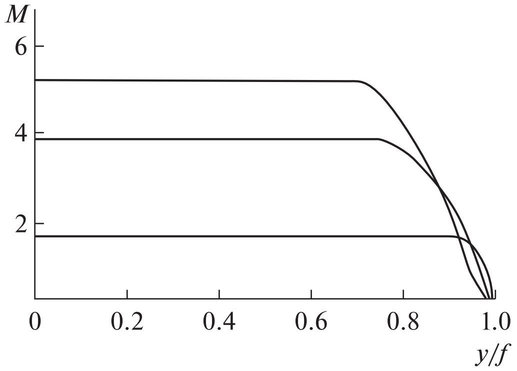

Fig. 5.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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SPELL: 1. givem, 2. abovementioned, 3. crosss, 4. sprogram, 5. compuational