2x2 operator matrix with real parameter and its spectrum

Elyor B. Dilmurodov[1,*], *Nargiza* A. Tosheva[1,2], *Nabiya* A. Turayeva[1], *Bolatbek* T. Karamatov[3], and *Durdona* Sh. Shokirova[3]

[1]Bukhara State University, 200100 Bukhara, Uzbekistan
[2]Bukhara State Pedagogical Institute, Bukhara City, Piridastgir, 2, 200100, Uzbekistan
[3]Navoi State Pedagogical Institute, 210100 Navoi, Uzbekistan

Abstract. In the present paper we consider a linear bounded self-adjoint 2×2 block operator matrix A_μ (so called generalized Friedrichs model) with real parameter $\mu \in R$. It is associated with the Hamiltonian of a system consisting of at most two particles on a d-dimensional lattice Z^{d}, interacting via creation and annihilation operators. A_μ is linear bounded self-adjoint operator acting in the two-particle cut subspace of the Fock space, that is, in the direct sum of zero-particle and one-particle subspaces of a Fock space. We find the essential and discrete spectra of the block operator matrix A_μ. The Fredholm determinant and resolvent operator associated to A_μ are constructed. The spectrum of A_μ plays an important role in the study of the spectral properties of the Hamiltonians associated with the energy operator of a lattice system describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles on a lattice.

1 Introduction

It is known that block operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces [1, 2]. One special class of block operator matrices are the energy operators associated with systems of non-conserved number of quasi-particles. In these systems the number of quasi-particles can be unbounded as in the case of spin-boson models [3, 4] or bounded as in the case of "truncated" spin-boson models [5, 6, 7, 8]. They arise, for example, in the theory of solid-state physics [9], quantum field theory [10] and statistical physics [5, 11].

The essential and discrete spectrum, as well as the isolated and embedded eigenvalues of the block operator matrices, in particular, Hamiltonians on a Fock space is one of the most actively studied objects in spectral theory of the self-adjoint operators, in many problems of mathematical physics and other related fields [12-38].

* Corresponding author: elyor.dilmurodov@mail.ru, e.b.dilmurodov@buxdu.uz

In the present paper we consider the 2×2 operator matrix A_{μ} (so called generalized Friedrichs model) with real parameter $\mu \in R$. We notice that this operator matrix is associated with the Hamiltonian of a system consisting of at most two particles on a d-dimensional lattice Z^{d}, interacting via creation and annihilation operators. It is linear bounded self-adjoint operator acting in the two-particle cut subspace of the Fock space, that is, in the direct sum of zero-particle and one-particle subspaces of a Fock space. The main goal of the paper is to study the spectrum and resolvent operator of the block operator matrix A_{μ}.

2 Literature review

In this section we give an information about spectral properties of the operator matrices and lattice models.

In [12], a special type of soluble model, corresponding to a coupled molecular and nuclear Hamiltonians H_1, the so-called generalized Friedrichs model, is considered. The most important properties of the well-known Faddeev operator corresponding to H_1 related with the number of discrete eigenvalues are determined and proved. A formula for counting the multiplicity of discrete eigenvalues of H_1 is derived.

In the paper [13] a 2×2 operator matrix H_2 is considered. An analog of the well-known Faddeev equation for the eigenvectors of H_2 is constructed and some important properties of this equation, related with the number of eigenvalues, are studied. In particular, the Birman-Schwinger principle for H_2 is proved.

In [14] a block operator matrix A_{μ} ($\mu > 0$ is a coupling constant) of order 2, acting in a direct sum of one-particle and two-particle subspaces of the bosonic Fock space is discussed. It is shown that there exist the critical values of the coupling constant that the operator matrix A_{μ} has infinitely many eigenvalues on the l.h.s. (r.h.s.) of the its essential spectrum.

In the paper [15] a lattice spin-boson model A_2 with fixed atom and at most two photons is considered. The first Schur complement $.S_1(\lambda).$ with spectral parameter λ corresponding to A_2 is constructed. The Birman-Schwinger principle for A_2 with respect to $S_1(\lambda)$ is proved. An important properties of $S_1(\lambda)$ related with the number of eigenvalues of A_2 is studied.

In [16], [17] a 2×2 operator matrix is considered. The critical value of the coupling constant for which operator matrix has an infinite number of eigenvalues is find. These eigenvalues accumulate at the lower and upper bounds of the essential spectrum. An asymptotic formula for the number of such eigenvalues in both the left and right parts of the essential spectrum is obtained. In [18] a 2×2 operator matrix, related to the lattice systems describing three particles in interaction, without conservation of the number of particles on a d-dimensional lattice is considered. The two-particle and three-particle branches of the essential spectrum this operator matrix is described and it is shown that the essential spectrum consist of the union of at most three bounded closed intervals.

In [19], [20] a family of 2×2 operator matrices $\mathrm{A}_{\mu}(k)$, $k \in T^3 := (-\pi,\pi]^3$, $\mu > 0$, associated with the Hamiltonian of a system consisting of at most two particles on a three-dimensional lattice Z^3, interacting via creation and annihilation operators is considered. It is proven that there is a value μ_0 of the parameter μ such that only for $\mu=\mu_0$ the operator $\mathrm{A}_\mu(\bar{0})$ and $\mathrm{A}_\mu(\bar{\pi})$ has virtual level at the point $z=0=\min\sigma_{\mathrm{ess}}(\mathrm{A}_\mu(\bar{0}))$ and $z=18=\max\sigma_{\mathrm{ess}}(\mathrm{A}_\mu(\bar{\pi}))$, respectively, where $\bar{0}:=(0,0,0), \bar{\pi}:=(\pi,\pi,\pi)\in T^3$. The absence of the eigenvalues of $\mathrm{A}_\mu(k)$ for all values of k under the assumption that $\mu=\mu_0$ is shown. The threshold energy expansions for the Fredholm determinant associated to $\mathrm{A}_\mu(k)$ are obtained.

In [21], [22], [23] a family of operator matrices $H(K),\ K\in T^3:=(-\pi,\pi]$ of order three is considered. They arise in the spectral analysis problem of the so called lattice truncated spin-boson Hamiltonian with at most two bosons. In [21] the position and structure of two-particle as well three-particle branches (subsets) of essential spectrum of operator matrix $H(K)$ are investigated. In [22], it is find a finite set $\Lambda\subset T^3$ to prove the existence of infinitely many eigenvalues of $H(K)$ for all $K\in\Lambda$ when the associated Friedrichs model has a zero energy resonance. It is shown that for every $K\in\Lambda$, the number $N(K,z)$ of eigenvalues of $H(K)$ lying on the left of $z, z<0$, satisfies the asymptotic relation

$$\lim_{z\to-0}\frac{N(K,z)}{|\log|z||}=U_0$$

with $0<U_0<\infty$, independently on the cardinality of Λ. In [23] an analogue of the Faddeev equation for the eigenfunctions of $H(K)$ is obtained. An analytic description of the essential spectrum of $H(K)$ is established. Further, it is shown that the essential spectrum of $H(K)$ consists the union of at most three bounded closed intervals.

In [24] a matrix model A related to a system describing two identical fermions and one particle another nature on a lattice, interacting via annihilation and creation operators is considered. The problem of the study of the spectrum of a block operator matrix A is reduced to the investigation of the spectrum of block operator matrices of order three with a discrete variable, and the relations for the spectrum, essential spectrum, and point spectrum are established. Two-particle and three-particle branches of the essential spectrum of the block operator matrix A are singled out.

The paper [25] deals with the studies of the spectrum of a 4×4 block operator matrix A_3, which is a non-symmetric version of the Hamiltonian related with the lattice truncated spin boson matrix with at most 3 photons. The position of the $\sigma_{\mathrm{ess}}(\mathrm{A}_3)$ is described. Its two-particle, three-particle and four-particle branches are analyzed. Moreover, a formula for $\sigma_{\mathrm{p}}(\mathrm{A}_3)$ is derived. The connections for $\sigma_{\mathrm{disc}}(\mathrm{A}_3)$ are obtained.

In [26] an operator matrix A of order four is considered. This operator is corresponding to the Hamiltonian of a system with non conserved number and at most four particles on a lattice. It is shown that the operator matrix A is unitarily equivalent to the diagonal matrix, the diagonal elements of which are operator matrices of order four. The location of the essential spectrum of the operator A is described, that is, two-particle, three-particle and four-particle branches of the essential spectrum of the operator A are singled out. It is established that the essential spectrum of the operator matrix A consists of the union of closed intervals whose number is not over 14. A Fredholm determinant is constructed such that its set of zeros and the discrete spectrum of the operator matrix A

coincide, moreover, it was shown that the number of simple eigenvalues of the operator matrix A lying outside the essential spectrum does not exceed 16. The same properties for the model operators, corresponding to the two-particle and three-particle systems on a lattice are discussed by many authors, see for example [27], [28], [29], [30].

In [27] the model Hamiltonian operator $H_{\mu,\lambda}$, $\mu,\lambda>0$, related to the three-particle system on a 1D lattice, interacting via non-local potentials is analyzed. The new branches of the essential spectrum of operator $H_{\mu,\lambda}$ are studied.

In the paper [28] a Friedrichs model $\mathrm{A}(\mu_1,\mu_2)$, $\mu_1,\mu_2>0$ with rank two perturbation is considered. It is related with a two quantum particle system on 3D integer lattice. The number and location of the discrete eigenvalues of $\mathrm{A}(\mu_1,\mu_2)$ are investigated. The sufficient and necessary conditions which guarantees the equality of the spectrum of $\mathrm{A}(\mu_1,\mu_2)$ and its field of values (or numerical range) is given. The relation of the threshold eigenvalues and virtual levels with the numerical range of $\mathrm{A}(\mu_1,\mu_2)$ are established.

In [29] a three-particle lattice model $H_{\mu,\lambda}$ $\mu,\lambda>0$ by making use nonlocal potential is presented. This Hamiltonian acts as a tensor sum of two Friedrichs models $h_{\mu,\lambda}$ with rank 2 perturbation associated with a system of three quantum particles on a d-dimensional lattice. The number of eigenvalues of $h_{\mu,\lambda}$ is investigated. The suitable conditions on the existence of eigenvalues localized inside, in the gap and below the bottom of the essential spectrum of $H_{\mu,\lambda}$ is provided. In [30] a tensor sum $H_{\mu,\lambda}$, $\mu,\lambda>0$ of two Friedrichs models $h_{\mu,\lambda}$ with rank two perturbation is considered. The Hamiltonian $H_{\mu,\lambda}$ is associated with a system of three quantum particles on one-dimensional lattice. The number and location of the eigenvalues of $H_{\mu,\lambda}$ is investigated. The existence of eigenvalues located respectively inside, in the gap, and below of the bottom of the essential spectrum of $H_{\mu,\lambda}$ is proved.

3 Description of the 2x2 operator matrix and its spectrum

From the operator theory it is known that if the operator A is a bounded linear operator in a Hilbert space H and a decomposition $\mathrm{H}=\mathrm{H}_0\oplus\mathrm{H}_1$ is given, then the operator A always admits a block operator matrix representation

$$\mathrm{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with linear operators

$$A:\mathrm{H}_0\to\mathrm{H}_0,\quad B:\mathrm{H}_1\to\mathrm{H}_0,\quad C:\mathrm{H}_0\to\mathrm{H}_1,\quad D:\mathrm{H}_1\to\mathrm{H}_1.$$

It evident that the operator A is a self-adjoint if and only if $A=A^*$, $D=D^*$ and $C=B^*$.

This paper is devoted to the following case: $\mathrm{H}_0:=C$ is the field of complex numbers and $\mathrm{H}_1:=L_2(T^{\mathrm{d}})$ is the Hilbert space of square-integrable (complex-valued) functions defined on the three-dimensional torus T^{d} (the cube $(-\pi,\pi]^{\mathrm{d}}$ with appropriately identified sides equipped with its Haar measure). The space H_0 is called a zero-particle subspace of a Fock space, the space H_1 is called an one-particle subspace of a Fock space and the space $\mathrm{H} := \mathrm{H}_0 \oplus \mathrm{H}_1$ is called two-particle subspace of the Fock space. The space

$$C \oplus L_2(T^{\mathrm{d}}) \oplus L_2((T^{\mathrm{d}})^2) \oplus \cdots$$

is called the Fock space over $L_2(T^{\mathrm{d}})$. Every element f of the space H can be represented as a vector-function $f = (f_0, f_1)$, where $f_0 \in \mathrm{H}_0$ and $f_1 \in \mathrm{H}_1$. The norm of the element $f = (f_0, f_1) \in \mathrm{H}$ is defined by

$$\| f \| = \left(| f_0 |^2 + \int_{T^{\mathrm{d}}} | f_1(x) |^2 \, dx \right)^{1/2}.$$

The scalar product of the two elements $f = (f_0, f_1) \in \mathrm{H}$ and $g = (g_0, g_1) \in \mathrm{H}$ is defined by

$$(f, g) = f_0 \cdot \overline{g_0} + \int_{T^{\mathrm{d}}} f_1(x) \overline{g_1(x)} dx.$$

In the Hilbert space H we consider the following 2×2 operator matrix

$$\mathrm{A}_\mu := \begin{pmatrix} A_{00} & \mu A_{01} \\ \mu A_{01}^* & A_{11} \end{pmatrix} \tag{1}$$

with real parameter $\mu \in R$, where the matrix elements $A_{ij} : \mathrm{H}_j \to \mathrm{H}_i$, $i = 0,1$, $i \le j$ are defined by the rules

$$A_{00} f_0 = a f_0, \quad A_{01} f_1 = \int_{T^{\mathrm{d}}} v(t) f_1(t) dt, \quad (A_{11} f_1)(x) = (a + w(x)) f_1(x).$$

Here $f_i \in \mathrm{H}_i$, $i = 0,1$, $a \in R$, the functions $v(\cdot)$ and $w(\cdot)$ are real-valued continues functions defined on T^{d}. A_{01}^* denotes the adjoint operator to A_{01}. It is clear that

$$(A_{01}^* f_0)(x) = f_0, \quad f_0 \in \mathrm{H}_0.$$

Under these assumptions the operator matrix A_μ is a bounded and self-adjoint in .

An operator matrix A_μ is called the generalized Friedrichs model and it is associated with the Hamiltonian of a system consisting of at most two particles on a d-dimensional lattice Z^{d}, interacting via creation and annihilation operators. The spectrum of A_μ plays an important role in the study of the spectral properties of the Hamiltonians associated with the energy operator of a lattice system describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles on a lattice.

Usually the off-diagonal elements A_{01} and A_{01}^* are called annihilation and creation operators [10], respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

Let us recall the notion of the essential spectrum and the discrete spectrum of a linear bounded self-adjoint operator in Hilbert space. Let X be the Hilbert space and $A : X \to X$ be the linear, bounded and self-adjoint operator. The set of all isolated eigenvalues of A with finite multiplicity is called the discrete spectrum of A and we denote this set by $\sigma_{\mathrm{disc}}(A)$. The set, which is defined as $\sigma(A) \setminus \sigma_{\mathrm{disc}}(A)$ is called the essential spectrum of A and we denote this set by $\sigma_{\mathrm{ess}}(A)$.

It is easy to see that the perturbation $\mathrm{A}_\mu - \mathrm{A}_0$ of the operator A_0 is a self-adjoint operator of rank 2. Therefore, using Weyl's theorem [31] about the invariance of the essential spectrum under the finite rank perturbations, we obtain that the essential spectrum $\sigma_{\mathrm{ess}}(\mathrm{A}_\mu)$

of A_μ is coincide with the essential spectrum $\sigma_{\text{ess}}(\mathrm{A}_0)$ of A_0. By the construction for the $\sigma_{\text{ess}}(\mathrm{A}_0)$ we have

$$\sigma_{\text{ess}}(\mathrm{A}_0)=[a+m,a+M],$$

where the numbers m and M are defined by

$$m:=\min_{x\in T^{\mathrm{d}}} w(x), \quad M:=\max_{x\in T^{\mathrm{d}}} w(x).$$

Using the last facts we obtain the following equality

$$\sigma_{\text{ess}}(\mathrm{A}_\mu)=[a+m,a+M]$$

for the the essential spectrum $\sigma_{\text{ess}}(\mathrm{A}_\mu)$ of A_μ and it is does not depend on μ. Moreover, for the point spectrum $\sigma_{\text{pp}}(\mathrm{A}_0)$ of A_0 the equality $\sigma_{\text{pp}}(\mathrm{A}_0)=\{a\}$ holds and the corresponding eigenvector-function f has form $f=(f_0,0)$ with $f_0=\text{const}\neq 0$. If $a\in[a+m,a+M]$, then a is an embedded eigenvalue of A_0, that is, $a\in\sigma_{\text{ess}}(\mathrm{A}_0)$. For example, if $m\leq 0$ and $M>0$, then $a\in[a+m,a+M]$. In the case $a\notin[a+m,a+M]$, the number a is a simple isolated eigenvalue of A_0, and hence for the discrete spectrum $\sigma_{\text{disc}}(\mathrm{A}_0)$ of A_0 the equality $\sigma_{\text{disc}}(\mathrm{A}_0)=\{a\}$ holds.

Example. If $\mathrm{d}=1$ and $w(x)=1-\cos x$, then $\sigma_{\text{ess}}(\mathrm{A}_\mu)=[a,a+2]$. If $\mathrm{d}=2$ and $w(x)=2-\cos x_1-\cos x_2$, then $\sigma_{\text{ess}}(\mathrm{A}_\mu)=[a,a+4]$. If $\mathrm{d}=3$ and $w(x)=3-\cos x_1-\cos x_2-\cos x_3$, then $\sigma_{\text{ess}}(\mathrm{A}_\mu)=[a,a+6]$. If $\mathrm{d}\in N$ and $w(x)=\mathrm{d}-\cos x_1-...-\cos x_{\mathrm{d}}$, then $\sigma_{\text{ess}}(\mathrm{A}_\mu)=[a,a+2\mathrm{d}]$.

To define the discrete spectrum $\sigma_{\text{disc}}(\mathrm{A}_\mu)$ of A_μ we define an analytic function $\Delta_\mu(\cdot)$ in the domain $C\setminus\sigma_{\text{ess}}(\mathrm{A}_\mu)$ by

$$\Delta_\mu(z):=a-z-\mu^2\int_{T^{\mathrm{d}}}\frac{v^2(t)dt}{a+w(t)-z}.$$

This function is called the Fredholm determinant associated to the operator A_μ.

The following statement establishes connection between the eigenvalues of the operator matrix A_μ and zeros of the function $\Delta_\mu(\cdot)$.

Lemma 3.1. *For any $\mu\in R$ the block operator matrix A_μ has an eigenvalue $z_\mu\in C\setminus\sigma_{\text{ess}}(\mathrm{A}_\mu)$ if and only if $\Delta_\mu(z_\mu)=0$.*

Proof. Suppose $f=(f_0,f_1)\in\mathrm{H}$ is an eigenvector-function of the block operator matrix A_μ associated with the eigenvalue $z_\mu\in C\setminus[a+m;a+M]$. Then f_0 and f_1 satisfy the following system of equations

$$\begin{aligned}&(a-z_\mu)f_0+\mu\int_{T^{\mathrm{d}}}v(t)f_1(t)dt=0;\\&\mu v(x)f_0+(a+w(x)-z_\mu)f_1(x)=0.\end{aligned}\tag{2}$$

Since $z_\mu\in C\setminus[a+m;a+M]$, the relation $a+w(x)-z_\mu\neq 0$ holds for all $x\in T^{\mathrm{d}}$. Then from the second equation of the system (2) for f_1 we have

$$f_1(x)=-\frac{\mu v(x)f_0}{a+w(x)-z_\mu}.\tag{3}$$

Substituting the expression (3) for f_1 into the first equation of the system (2), we conclude that the system of equations (2) has a nontrivial solution if and only if the condition $\Delta_\mu(z)=0$ is satisfied. Lemma is proved.

From Lemma 3.1 it follows that

$$\sigma_{\text{disc}}(\mathrm{A}_\mu)=\{z\in C\setminus\sigma_{\text{ess}}(\mathrm{A}_\mu):\Delta_\mu(z)=0\}.$$

4 The number and location of the eigenvalues of A_μ

In this section first we investigate main properties of the function $\Delta_\mu(\cdot)$ and then we study the eigenvalues of A_μ.

We formulate the first property of $\Delta_\mu(\cdot)$ about the monotonicity.

Lemma 4.1. *For any $\mu\in R$ the function $\Delta_\mu(\cdot)$ is monotonically decreasing in the intervals $(-\infty;a+m)$ and $(a+M;+\infty)$.*

Proof. From the definition of the function $\Delta_\mu(\cdot)$ it follows that this function is a regular function in $C\setminus\sigma_{ess}(\mathrm{A}_\mu)$. Simple calculations show that

$$\frac{d}{dz}\Delta_\mu(z)=-1-\mu^2\int_{T^{\mathrm{d}}}\frac{v^2(t)}{(a+w(t)-z)^2},\quad z\in C\setminus\sigma_{\text{ess}}(\mathrm{A}_\mu).$$

It is clear that $\frac{d}{dz}\Delta_\mu(z)<0$ for any $z\in R\setminus\sigma_{\text{ess}}(\mathrm{A}_\mu)$. This means that the function $\Delta_\mu(\cdot)$ is monotonically decreasing in $R\setminus\sigma_{\text{ess}}(\mathrm{A}_\mu)$.

Since for any $\mu\in R$ the function $\Delta_\mu(\cdot)$ is monotonically decreasing in the intervals $(-\infty;a+m)$ and $(a+M;+\infty)$, from here and from the Lebesgue dominated theorem it follows the existence of the finite or infinite limits

$$\Delta_\mu(a+m)=\lim_{z\to a+m-0}\Delta_\mu(z);\qquad \Delta(a+M)=\lim_{z\to a+M+0}\Delta_\mu(z).$$

Lemma 4.2. *For any fixed $\mu\in R$ the operator A_μ has an unique eigenvalue in $(-\infty;z_\mu)$, $z_\mu\le a+m$ if and only if $\Delta_\mu(z_\mu)<0$.*

Proof. Let $e_\mu\in(-\infty;z_\mu)$, $z_\mu\le a+m$ be the eigenvalue of the operator A_μ. Then by Lemma 4.1 we have $\Delta_\mu(e_\mu)=0$. Since for any fixed $\mu\in R$ the function $\Delta_\mu(\cdot)$ is decreasing in $(-\infty;a+m)$, we obtain

$$\Delta_\mu(z_\mu)<\Delta_\mu(e_\mu)=0.$$

Conversely, let for some z_μ, $z_\mu\le a+m$ the condition $\Delta_\mu(z_\mu)<0$ be hold. Since $\lim_{z\to-\infty}\Delta_\mu(z)=+\infty$ and the function $\Delta_\mu(\cdot)$ is decreasing and continuous by z in the interval $(-\infty;z_\mu)$, there exists the number $e_\mu\in(-\infty;z_\mu)$ such that $\Delta_\mu(e_\mu)=0$. By Lemma 4.1 the number e_μ is an eigenvalue of the operator A_μ. Lemma 4.2 is completely proved.

The following Lemma can be proven analogously to Lemma 4.2.

Lemma 4.3. *For any $\mu\in R$ the operator A_μ has an unique eigenvalue in $(z_\mu;+\infty)$, $z_\mu\ge a+M$ if and only if $\Delta_\mu(z_\mu)>0$.*

From Lemmas 4.2 and 4.3 we obtain the following corollary.

Corollary 4.4. *If for some* $\mu_0 \in R$ *the relations* $\Delta_{\mu_0}(a+m) < 0$ *and* $\Delta_{\mu_0}(a+M) > 0$ *are hold, then the operator* A_{μ_0} *has two simple eigenvalues* $e^{(1)}_{\mu_0}$ *and* $e^{(2)}_{\mu_0}$ *such that* $e^{(1)}_{\mu_0} < a+m$ *and* $e^{(2)}_{\mu_0} > a+M$.

Let us prove one additional Lemma.

Lemma 4.5. *Let the functions* $v(\cdot)$ *and* $w(\cdot)$ *be defined as*

$$v(x) \equiv 1, \quad w(x) = \sum_{i=1}^{\mathrm{d}} (1-\cos x_i), x = (x_1, \dots, x_{\mathrm{d}}) \in T^{\mathrm{d}}.$$

If $a = 0$, $\mathrm{d} \geq 3$ and

$$\mu_0 = \left(2\mathrm{d} \int_{T^{\mathrm{d}}} \frac{dt}{2\mathrm{d} - \varepsilon(t)} \right)^{-1/2},$$

then for any $|\mu| > \mu_0$ we have $\Delta_\mu(0) < 0$ and $\Delta_\mu(2\mathrm{d}) > 0$. Here $m = 0$ and $M = 2\mathrm{d}$.

Proof. We set $\bar{0} := (0, \dots, 0), \bar{\pi} := (\pi, \dots, \pi) \in T^{\mathrm{d}}$. It is clear that the function $w(\cdot)$ has an unique non-degenerate zero minimum (i.e. $m = 0$) at the point $\bar{0} \in T^{\mathrm{d}}$ and an unique non-degenerate maximum, which is equal to $2\mathrm{d}$ (i.e. $M = 2\mathrm{d}$) at the point $\bar{\pi} \in T^{\mathrm{d}}$. Depending on d there exist finite or infinite integrals

$$\int_{T^{\mathrm{d}}} \frac{dt}{w(t)} > 0, \quad \int_{T^{\mathrm{d}}} \frac{dt}{2\mathrm{d} - w(t)} > 0.$$

In the case $\mathrm{d} \geq 3$ these integrals are finite. Indeed, let

$$|x| = \sqrt{x_1^2 + \dots + x_{\mathrm{d}}^2}, \quad x = (x_1, \dots, x_{\mathrm{d}}) \in T^{\mathrm{d}}$$

and

$$U_\delta(\bar{0}) = \{x \in T^{\mathrm{d}} : |x| < \delta\}, \quad \delta > 0.$$

There exist the numbers $C_1, C_2 > 0$ and $\delta > 0$ such that

$$C_1 |x|^2 \leq w(x) \leq C_2 |x|^2, \quad x \in U_\delta(\bar{0});$$
$$w(x) \geq C_1, \quad x \in T^{\mathrm{d}} \setminus U_\delta(\bar{0}).$$

Then taking into account last inequalities T^{d} we have

$$0 < \int_{T^{\mathrm{d}}} \frac{dt}{w(t)} \leq \int_{T^{\mathrm{d}}} \frac{dt}{w(t)} \leq C_1 \int_{U_\delta(\bar{0})} \frac{dt}{|x|^2} + C_2 < \infty.$$

There exist the numbers $C_3, C_4 > 0$ and $\gamma > 0$ such that

$$C_3 |x - \bar{\pi}|^2 \leq 2\mathrm{d} - w(x) \leq C_4 |x - \bar{\pi}|^2, \quad x \in U_\gamma(\bar{\pi});$$
$$2\mathrm{d} - w(x) \geq C_3, \quad x \in T^{\mathrm{d}} \setminus U_\gamma(\bar{\pi}).$$

Analogously using the latter estimates one can show the finiteness of the following integral

$$\int_{T^{\mathrm{d}}} \frac{dt}{2\mathrm{d} - w(t)}.$$

From the definition of the function $w(\cdot)$ we obtain

$$\Delta_\mu(0) = -\mu^2 \int_{T^{\mathrm{d}}} \frac{dt}{w(t)} < 0$$

for all $\mu \neq 0$.

Let now $|\mu| > \mu_0$. Then we have

$$\Delta_\mu(2\mathrm{d}) = -2\mathrm{d} + \mu^2 \int_{T^{\mathrm{d}}} \frac{dt}{2\mathrm{d} - w(t)} > 0.$$

Lemma is completely proved.

Similar properties of the Friedrichs model is studied in [32].

Particular special case: let $w(x) \equiv w_0 = const.$ Then for the essential spectrum of the block operator matrix A_μ we have the following equality $\sigma_{ess}(\mathrm{A}_\mu) = \{a + w_0\}$.

To find the discrete spectrum of the block operator matrix A_μ we solve the equation $\Delta_\mu(z) = 0$ with respect to z, that is, the equation

$$a - z - \mu^2 \int_{T^{\mathrm{d}}} \frac{v^2(t)dt}{a + w_0 - z} = 0.$$

The latter equation can be represented as

$$(a - z)(a + w_0 - z) - \mu^2 \int_{T^{\mathrm{d}}} v^2(t)dt = 0$$

or as quadratic equation

$$z^2 - (2a + w_0)z + a(a + w_0) - \mu^2 \| v \|^2 = 0.$$

Let us calculate the discriminant D_μ of the latter quadratic equation

$$D_\mu = (2a + w_0)^2 - 4a(a + w_0) + 4\mu^2 \| v \|^2 .$$

After simple calculations we have

$$D_\mu = 4a^2 + 4aw_0 + w_0^2 - 4a^2 - 4aw_0 + 4\mu^2 \| v \|^2 = w_0^2 + 4\mu^2 \| v \|^2 \ge 0.$$

Therefore, in this particular case for any $\mu \in R$ the block operator matrix A_μ has two eigenvalues $z_\mu^{(1)}$ and $z_\mu^{(2)}$, and they have the forms:

$$z_\mu^{(1)} = \frac{2a + w_0 - \sqrt{w_0^2 + 4\mu^2 \| v \|^2}}{2}, \quad z_\mu^{(2)} = \frac{2a + w_0 + \sqrt{w_0^2 + 4\mu^2 \| v \|^2}}{2}.$$

Therefore, for the discrete spectrum of the block operator matrix A_μ we have

$$\sigma_{\mathrm{disc}}(\mathrm{A}_\mu) = \left\{ \frac{2a + w_0 - \sqrt{w_0^2 + 4\mu^2 \| v \|^2}}{2}, \quad \frac{2a + w_0 + \sqrt{w_0^2 + 4\mu^2 \| v \|^2}}{2} \right\}.$$

It is clear that one of the eigenvalues $z_\mu^{(1)}$ and $z_\mu^{(2)}$ is located on the left hand side of $a + w_0$, other one is located on the right hand side of $a + w_0$. At the end we conclude that for the spectrum of the block operator matrix A_μ the equality

$$\sigma(\mathrm{A}_\mu) = \left\{ a + w_0, \quad \frac{2a + w_0 - \sqrt{w_0^2 + 4\mu^2 \| v \|^2}}{2}, \quad \frac{2a + w_0 + \sqrt{w_0^2 + 4\mu^2 \| v \|^2}}{2} \right\}$$

holds.

5 Resolvent operator

In this section we construct the resolvent operator corresponding to the block operator matrix A_μ.

For any fixed $z \in C \setminus \sigma(\mathrm{A}_\mu)$ we introduce the block operator matrix $R(\mu,z)$, acting in the Hilbert space H as

$$R(\mu,z) := \begin{pmatrix} R_{00}(\mu,z) & R_{01}(\mu,z) \\ R_{10}(\mu,z) & R_{11}(\mu,z) \end{pmatrix}, \tag{4}$$

where the matrix elements $R_{ij}(\mu,z) : \mathrm{H}_j \to \mathrm{H}_i$, $i,j = 0,1$ are defined by

$$R_{00}(\mu,z)g_0 = \frac{g_0}{\Delta_\mu(z)}, \quad R_{01}(\mu,z)g_1 = -\frac{\mu}{\Delta_\mu(z)} \int_{T^{\mathrm{d}}} \frac{v(t)g_1(t)dt}{a+w(t)-z},$$

$$(R_{10}(\mu,z)g_0)(x) = -\frac{\mu v(x) g_0}{\Delta_\mu(z)(a+w(x)-z)},$$

$$(R_{11}(\mu,z)g_1)(x) = \frac{g_1(x)}{a+w(x)-z} + \frac{\mu^2 v(x)}{\Delta_\mu(z)(a+w(x)-z)} \int_{T^{\mathrm{d}}} \frac{v(t)g_1(t)dt}{a+w(t)-z}.$$

In the following theorem we describe the resolvent operator for A_μ.

Theorem 5.1. *For any fixed* $z \in C \setminus \sigma(\mathrm{A}_\mu)$ *the operator* $R(\mu,z)$, *defined by* (4), *is the resolvent operator corresponding to* A_μ.

Proof. Let $z \in C \setminus \sigma(\mathrm{A}_\mu)$. For the construction of the resolvent operator $R(\mu,z)$ we have to consider the equation $\mathrm{A}_\mu f - zf = g$ for any $f = (f_0, f_1)$, $g = (g_0, g_1) \in \mathrm{H}$. For convenience we represent this operator equation as the following system of equations

$$(a-z)f_0 + \mu \int_{T^{\mathrm{d}}} v(t) f_1(t) dt = g_0;$$
$$\mu v(x) f_0 + (a+w(x)-z) f_1(x) = g_1(x). \tag{5}$$

Since $z \notin [a+m; a+M]$, for any $x \in T^{\mathrm{d}}$ we have $a+w(x)-z \neq 0$. From the second equation of the system (5) for f_1 we have

$$f_1(x) = \frac{g_1(x)}{a+w(x)-z} - \frac{\mu v(x) f_0}{a+w(x)-z}. \tag{6}$$

Substituting the expression (6) for f_1 to the first equation of the system (5), we obtain

$$\Delta_\mu(z) f_0 + \mu \int_{T^{\mathrm{d}}} \frac{v(t)g_1(t)dt}{a+w(t)-z} = g_0.$$

Taking into account the relation $z \notin \sigma_{\mathrm{disc}}(\mathrm{A}_\mu)$, from the last equality for f_0 we have

$$f_0 = \frac{g_0}{\Delta_\mu(z)} - \frac{\mu}{\Delta_\mu(z)} \int_{T^{\mathrm{d}}} \frac{v(t)g_1(t)dt}{a+w(t)-z}. \tag{7}$$

We rewrite the equality (7) as

$$f_0 = R_{00}(\mu,z)g_0 + R_{01}(\mu,z)g_1. \tag{8}$$

Next, substituting the expression (8) for f_0 to the equality (6), we obtain

$$f_1(x) = \frac{g_1(x)}{a+w(x)-z} - \frac{\mu v(x)}{a+w(x)-z}\left[\frac{g_0}{\Delta_\mu(z)} - \frac{\mu}{\Delta_\mu(z)} \int_{T^{\mathrm{d}}} \frac{v(t)g_1(t)dt}{a+w(t)-z}\right] =$$

$$= -\frac{\mu v(x) g_0}{\Delta_\mu(z)(a+w(x)-z)} + \frac{g_1(x)}{a+w(x)-z} + \frac{\mu^2 v(x)}{\Delta_\mu(z)(a+w(x)-z)} \int_{T^{\mathrm{d}}} \frac{v(t)g_1(t)dt}{a+w(t)-z}.$$

We rewrite the latter equality in the following form

$$f_1(x) = (R_{10}(\mu,z)g_0)(x) + (R_{11}(\mu,z)g_1)(x).$$

Comparing the obtained expressions for f_0 and f_1 via g_0 and g_1 we obtain the equality $f = R(\mu, z)g,\ z \in C \setminus \sigma(\mathrm{A}_\mu),\ \ f, g \in \mathrm{H}$. Theorem 5.1 is completely proved.

6 Conclusion

In the present paper the 2×2 operator matrix A_μ (so called generalized Friedrichs model) with real parameter $\mu \in R$ is considered. We notice that this operator matrix is associated with the Hamiltonian of a system consisting of at most two particles on a d -dimensional lattice Z^{d}, interacting via creation and annihilation operators. It is acting in the direct sum of zero-particle and one-particle subspaces of the Fock space. The essential and discrete spectra of the operator matrix A_μ are found. The Fredholm determinant and resolvent operator associated to A_μ are constructed.

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