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Inverse Coefficient Problem for a Fractional-Diffusion Equation with a Bessel Operator

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Abstract—The second initial-boundary value problem in a bounded domain for a fractional-diffusion equation with the Bessel operator and the Gerasimov–Caputo derivative is investigated. Theorems of existence and uniqueness of the solution to the inverse problem of determining the lowest coefficient in a one-dimensional fractional-diffusion equation under the condition of integral observation are obtained. The Schauder principle was used to prove the existence of the solution.

Keywords: inverse problem, Fourier–Bessel series, eigenvalue, eigenvalue function, uniqueness, Schauder fixed-point theorem

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INTRODUCTION AND PROBLEM STATEMENT

It is well known that partial differential equations play an important role in constructing mathematical models of many real processes. Equations of mathematical physics with the Bessel operator belong to the class of degenerate differential equations, for which the theory of boundary value problems is currently one of the important sections of the theory of partial differential equations (PDEs).

Differential equations with fractional derivatives serve as the basis for mathematical modeling of processes occurring in continua with a fractal structure. Especially, during the last few decades, many applications of various types of fractional differential equations have become the objects of attention of specialists for theoretical and practical reasons [1]. Many types of boundary value problems, including direct [2, 3] and inverse problems [4], were formulated for various types of PDEs of integer order and with several differential operators of fractional order.

In works [5, 6] the unique solvability of the direct and inverse source problem for a fractional time partial differential equation with the Gerasimov–Caputo and Bessel operators was studied. Solutions to these problems are constructed on the basis of corresponding expansions in terms of eigenfunctions, and the existence and uniqueness of the resulting solutions are proved. In works [7, 8] the main boundary value problems for differential equations with the Bessel operator acting on a spatial variable and the Riemann–Liouville and Caputo partial derivatives with respect to a time variable were studied.

In this work, the posed problem is equivalently reduced to a nonlinear integral equation of Volterra type. One of the best mathematical tools for studying a system of nonlinear integral equations is the principle of contraction mappings, and on its basis the existence and uniqueness of solutions to the problems posed is proved, see [9–16]. In this work, unlike the above works, the Schauder principle is used to prove the existence theorem for a solution to a nonlinear integral equation [17, 18].

In the region $\Omega := \{(x,t) : 0 < x < 1, 0 < t < T\}$, we consider the differential equation

$$\partial_{0t}^{\alpha} u(x,t) = u_{xx}(x,t) + \frac{1}{x}u_x(x,t) - \frac{1}{x^2}u(x,t) + q(t)u(x,t) + f(x,t) \tag{1}$$

with initial condition

$$u(x,0) = a(x), \quad 0 \le x \le 1, \tag{2}$$

and boundary conditions

$$\lim_{x\to 0} xu_x(x,t) = 0, \quad u(1,t) = 0, \tag{3}$$

where $0 < \alpha < 1$ is a given number and ∂_{0t}^{α} is the Gerasimov–Caputo derivative of fractional order α starting at point 0, which is defined as follows ([1], p. 90):

$$\partial_{0t}^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} v'(\tau) d\tau.$$

Relations (1)–(3) are the direct problem, i.e., if the functions $q(t)$, $f(x,t)$, and $a(x)$ are known and the constants α, then the solution $u(x,t)$ can be found from Eqs. (1)–(3).

Definition 1. Under the solution to the direct problem (1)–(3) we will understand the function $u(x,t)$ from the class

$$u(x,t) \in C_{x,t}^{2,\alpha}(\Omega) \cap C(\overline{\Omega}), \tag{4}$$

where $C_{x,t}^{2,\alpha}(\Omega) := \left\{ u(x,t) | u_{xx}(\cdot,t) \in C(0,1), t \in (0,T); \partial_{0t}^{\alpha} u(x,\cdot) \in C(0,T), x \in (0,1) \right\}$, which is a solution to Eq. (1) in the region Ω and satisfies conditions (1)–(3).

Inverse problem. Define a function $q(t)$ if the following additional information about solving a direct problem (1)–(3) is known:

$$\int_0^1 u(x,t) dx = b(t), \quad 0 \le t \le T, \tag{5}$$

where $b(t)$ is a given sufficiently smooth function.

Definition 2. The solution to the inverse problem (1)–(5) is the functions $u(x,t)$ and $q(t)$ from the class (4) and $C[0,T]$, respectively, satisfying relations (1)–(5).

1. SYMBOLS AND SUPPORTING INFORMATION

In this section we introduce the necessary notation and statements that will be needed below.

We use the following function spaces: by $C_{\gamma}[0,T]$ we denote the function class $f(t)$ for which the function $t^{\gamma} f(t) \in C[0,T]$, where $0 \le \gamma < 1$, and

$$\|f\|_{\gamma} = \left\| t^{\gamma} f(t) \right\|_{C[0,T]}, \quad C_0[0,T] = C[0,T],$$

$$\|f\|_0 = \|f\| := \max_{t \in [0,T]} |f(t)|,$$

and

$$C_{\gamma}^{\alpha}[0,T] := \left\{ f(t) \in C[0,T] : \partial_{0t}^{\alpha} f \in C_{\gamma}[0,T] \right\},$$

$$C^{\alpha,n-1}[0,T] := \left\{ f(t) \in C^{n-1}[0,T] : \partial_{0t}^{\alpha} f \in C[0,T] \right\}, \quad C^{\alpha,0}[0,T] = C^{\alpha}[0,T],$$

where $\alpha > 0$, $n = -[-\alpha]$ and $[\alpha]$ is the integer part of number α.

The *Mittag–Leffler function* is an entire function defined by the series

$$E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad z, \alpha \in \mathbb{C}, \quad \Re(\alpha) > 0.$$

The two-parameter Mittag–Leffler function is the sum of a more general series

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0;$$

the equality $E_{\alpha,1}(z) = E_{\alpha}(z)$ is obvious.

Statement ([19]). *For* $0 < \alpha < 1$, $\eta > 0$ *it is true that* $0 \le E_{\alpha,\alpha}(-\eta) \le \frac{1}{\Gamma(\alpha)}$. *Moreover,* $E_{\alpha,\alpha}(-\eta)$ *is a monotonically decreasing function with* $\eta > 0$.

Lemma 1 ([20], p. 188)**.** *Let $b \geq 0$, $\alpha > 0$, and $a(t)$ is a nonnegative function that is locally integrable on $0 \leq t < T$ (for some positive T) and suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\alpha-1} u(s)ds$$

on this interval. Then we have

$$u(t) \leq a(t) + b\Gamma(\alpha)\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(b\Gamma(\alpha)(t-s)^{\alpha}\right) a(s)ds.$$

2. STUDY OF THE DIRECT PROBLEM (1)–(4)

In this section we will study the direct problem and study its properties. First, let us study the initial boundary value problem for the equation

$$\partial_{0t}^{\alpha} u(x,t) - u_{xx}(x,t) - \frac{1}{x}u_x(x,t) + \frac{1}{x^2}u(x,t) = F(x,t), \quad (x,t) \in \Omega, \tag{6}$$

with conditions (2), (3).

In (1) let us introduce the notation $F(x,t) := q(t)u(x,t) + f(x,t)$. We will seek partial solutions to Eq. (6) with $F \equiv 0$, not equal to zero in the region Ω and satisfying zero boundary conditions (3), in the form of a product $u(x,t) = X(x)T(t)$. Substituting this product into Eq. (6), for $X(x)$ we obtain the following spectral problem:

$$X''(x) + \frac{1}{x}X'(x) + \left(\lambda^2 - \frac{1}{x^2}\right)X(x) = 0, \quad 0 < x < 1, \tag{7}$$

$$\lim_{x\to 0} xX'(x) = 0, \quad X(1) = 0, \tag{8}$$

where λ^2 is the separation constant.

Let us find a general solution to Eq. (7). Multiplying (7) by x^2, performing a change of variable, according to the desired function

$$X(x) = v(\xi), \quad \xi = \lambda x, \tag{9}$$

we reduce the original equation to the Bessel equation

$$\xi^2 v''(\xi) + \xi v'(\xi) + (\xi^2 - 1)v(\xi) = 0. \tag{10}$$

It is well-known ([21], p. 249) that the general solution to Eq. (10) has the form

$$v(\xi) = c_1 J_1(\xi) + c_2 Y_1(\xi), \tag{11}$$

where $J_1(\xi)$ is the Bessel function of the first kind, $Y_1(\xi)$ is the Bessel function of the second kind, and c_1, c_2 are arbitrary constants.

Passing in (11) to the original variable x and functions $X(x)$ according to formula (9), we obtain a general solution to Eq. (7) as

$$X(x) = c_1 J_1(\lambda x) + c_2 Y_1(\lambda x). \tag{12}$$

We find the constants c_1 and c_2 from the requirement that the general solution (12) satisfies conditions (8). Solution (12) satisfies the first boundary condition from (8) at $c_1 = 1$ and $c_2 = 0$. As a result, we have

$$X(x) = J_1(\lambda x). \tag{13}$$

Now, we require that solution (13) satisfied the second boundary condition from (8):

$$J_1(\lambda) = 0. \tag{14}$$

According to Lommel's theorem ([22], p. 530) Eq. (14) has a countable number of simple real roots $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, which determine the eigenvalues of the spectral problem (7), (8). Assuming $\lambda = \lambda_n$ in (13), we obtain the corresponding eigenfunctions of the problem (7) and (8):

$$\tilde{X}_n(x) = J_1(\lambda_n x), \quad n = 1, 2, \ldots.$$

For the convenience of further calculations, we orthonormalize this system of functions:

$$X_n(x) = \frac{1}{\|\tilde{X}_n\|_{L_{2,x}[0,1]}} \tilde{X}_n(x), \tag{15}$$

where

$$\|\tilde{X}_n\|^2_{L_{2,x}[0,1]} = \int_0^1 x J_1^2(\lambda_n x) dx = \frac{1}{2} J_2^2(\lambda_n).$$

It is well-known ([23], p. 633) that the system of eigenfunctions (15) is complete in space $L_2[0,1]$ with the weight x, i.e., in $L_{2,x}[0,1]$.

Note that for the eigenvalues of the problem (7), (8) at large n the asymptotic formula is valid ([23], p. 317)

$$\lambda_n = \pi n + \frac{\pi}{4} + O(1/n).$$

Let us now expand the required function and the right-hand side of Eq. (6) into the Fourier–Bessel series in terms of eigenfunctions $X_n(x)$:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{16}$$

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) X_n(x), \tag{17}$$

where

$$u_n(t) = \int_0^1 x u(x,t) X_n(x) dx, \quad F_n(t) = \int_0^1 x F(x,t) X_n(x) dx, \quad n = 1, 2, \ldots.$$

Substituting (16), (17) into (1), we get

$$\partial_{0t}^{\alpha} u_n(t) = -\lambda_n^2 u_n(t) + F_n(t), \quad 0 < t \le T. \tag{18}$$

As follows from (2), the initial conditions for $u_n(t)$ have the form

$$u_n(0) = a_n, \tag{19}$$

where $a_n = \int_0^1 x a(x) X_n(x) dx$, $n = 1, 2, \ldots$, are the Fourier–Bessel coefficients of the series

$$a(x) = \sum_{n=1}^{\infty} a_n X_n(x). \tag{20}$$

It is easy to see that the solution to Eq. (18) with the initial conditions (19) has the following form ([1], p. 231):

$$u_n(t) = a_n E_{\alpha}(-\lambda_n^2 t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^{\alpha}) F_n(\tau) d\tau.$$

Next, we put $F_n(t) = f_n(t) + q(t)u_n(t)$. Then we obtain the following integral equation for $u_n(t)$:

$$u_n(t) = a_n E_\alpha(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) f_n(\tau) d\tau$$
$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) q(\tau) u_n(\tau) d\tau, \tag{21}$$

where

$$f_n(t) = \int_0^1 x f(x,t) X_n(x) dx, \quad n = 1, 2, \ldots.$$

Equation (21) represents a weakly singular Volterra integral equation of the second kind. It is well-known ([1], p. 205) that, if the functions $q(t) \in C[0,T]$ and $f_n(t) \in C_\gamma[0,T]$ at each $n = 1,2,\ldots,$ then Eq. (21) has a unique solution belonging to the class $C^\alpha[0,T]$, where $0 \le \gamma < 1$, $\gamma \le \alpha$, and $\partial_{0t}^\alpha u_n(t) \in C[0,T]$.

Let $a(x) = 0$, $f(x,t) = 0$, Then $a_n = f_n(t) \equiv 0$. From Lemma 1 and (21) we get $u_n(t) = 0$ for all $n \in \mathbb{N}$. Then from (17) for any $t \in [0,T]$ we have $\int_0^1 u(x,t) x X_n(x) dx = 0$. Hence, the completeness of system (15) in the space $L_2[0,1]$ with weight x implies $u(x,t) = 0$ almost everywhere in the range [0, 1] for any $t \in [0,T]$. Because, according to (2), the function $u(x,t) \in C(\overline{\Omega})$, $u(x,t) \equiv 0$ in $\overline{\Omega}$. Thus, the uniqueness of the solution to the problem (1)–(4) is proved based on the completeness of the system of eigenfunctions of the one-dimensional spectral problem.

Together with series (16), consider the following series:

$$\partial_{0t}^\alpha u(x,t) = \sum_{n=1}^\infty \partial_{0t}^\alpha u_n(t) X_n(x), \quad u_x(x,t) = \sum_{n=1}^\infty u_n(t) X_n'(x), \tag{22}$$

$$u_{xx}(x,t) = \sum_{n=1}^\infty u_n(t) X_n''(x). \tag{23}$$

Let us prove the uniform convergence of series (16), (22), and (23) in the region $\overline{\Omega}$. To do this, we impose some additional conditions on the functions $a(x)$ and $f(x,t)$.

First, we prove the following statements.

Lemma 2. *For large $n \in \mathbb{N}$ the estimates hold*:

$$|u_n(t)| \le c_1 \left(1 + \|q\| E_{\alpha,\alpha+1}\left(T^\alpha \|q\|\right)\right)|a_n| + c_2 \left(1 + \|q\| E_{\alpha,2\alpha-\gamma+1}\left(T^\alpha \|q\|\right)\right)\|f_n\|_\gamma, \quad t \in [0,T],$$

$$\left|\partial_{0t}^\alpha u_n(t)\right| \le c_1 \left(\lambda_n^2 + \|q\|\right)\left(1 + \|q\| E_{\alpha,\alpha+1}\left(T^\alpha \|q\|\right)\right)|a_n|$$
$$+ c_2 \varepsilon^{-\gamma}\left(\lambda_n^2 + \|q\|\right)\left(1 + \|q\| E_{\alpha,2\alpha-\gamma+1}\left(T^\alpha \|q\|\right)\right)\|f_n\|_\gamma, \quad t \in [\varepsilon,T],$$

where $c_1 := \max\{1, T^\alpha\}$, $c_2 := \max\left\{\frac{T^{\alpha-\gamma}}{\Gamma(\alpha)} B(\alpha, 1-\gamma), \Gamma(1-\gamma) T^{2\alpha-\gamma}\right\}$, $B(\cdot,\cdot)$ *is the Euler integral of the first kind*, *and* $\varepsilon \in (0,T)$ *is a sufficiently small number*.

Proof. Let us write the integral equation (21) as

$$u_n(t) = a_n E_\alpha(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} \tau^{-\gamma} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \tau^\gamma f_n(\tau) d\tau$$
$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) q(\tau) u_n(\tau) d\tau.$$

Then, using Proposition 1, we have

$$|u_n(t)| \le |a_n| + \frac{\|f_n\|_\gamma}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\gamma}d\tau + \frac{\|q\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u_n(\tau)|\,d\tau$$

or

$$|u_n(t)| \le |a_n| + \frac{\|f_n\|_\gamma}{\Gamma(\alpha)} t^{\alpha-\gamma}B(\alpha,1-\gamma) + \frac{\|q\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u_n(\tau)|\,d\tau.$$

Further, according to Lemma 1

$$|u_n(t)| \le |a_n| + \frac{\|f_n\|_\gamma}{\Gamma(\alpha)} t^{\alpha-\gamma}B(\alpha,1-\gamma)$$

$$+ \|q\|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\|q\|(t-\tau)^\alpha\right)\left[|a_n| + \frac{\|f_n\|_\gamma}{\Gamma(\alpha)}\tau^{\alpha-\gamma}B(\alpha,1-\gamma)\right]d\tau$$

or

$$|u_n(t)| \le |a_n| + \frac{\|f_n\|_\gamma}{\Gamma(\alpha)} t^{\alpha-\gamma}B(\alpha,1-\gamma) + \|q\||a_n|\,t^\alpha E_{\alpha,\alpha+1}\left(\|q\|t^\alpha\right)$$

$$+ \Gamma(1-\gamma)t^{2\alpha-\gamma}\|q\|\,E_{\alpha,2\alpha-\gamma+1}\left(\|q\|t^\alpha\right)\|f_n\|_\gamma .$$

From the last inequality we immediately obtain the first estimate of Lemma 2 for any $t \in [0,T]$.

The second estimate follows from Eq. (18) and the first inequality of Lemma 2 for an arbitrary $t \in [\varepsilon,T]$.

□

Lemma 3. *For large n for all* $x \in [\delta,1]$, *where* δ *is a sufficiently small number*, *the estimates hold*:

$$|X_n(x)| \le c_3, \quad |X_n'(x)| \le c_4 n, \quad |X_n''(x)| \le c_5 n^2, \tag{24}$$

where c_i, $i = 3,4,5$, *are positive constants.*

Proof. It is well-known that at large ξ the estimate takes place ([23], p. 133):

$$J_\nu(\xi) = O(\xi^{-1/2}). \tag{25}$$

Because

$$\|\tilde{X}_n\|_{L_{2,x}[0,1]} = \frac{1}{\sqrt{2}}|J_2(\lambda_n)|,$$

the representation of λ_n due to (25) implies the validity of the first estimate (24). To justify the second estimate (24), we calculate

$$X_n'(x) = \frac{\lambda_n}{\sqrt{2}J_2(\lambda_n)}\left[J_0(\lambda_n x) - J_2(\lambda_n x)\right]. \tag{26}$$

Then formulas (26) and (25) imply the validity of the second estimate (24). Further, from Eq. (7) we get

$$X_n''(x) + \frac{1}{x}X_n'(x) + \left(\lambda_n^2 - \frac{1}{x^2}\right)X_n(x) = 0.$$

Hence, the proven first two inequalities in (24) result in the validity of the third estimate in (24).

□

According to Lemmas 2 and 3, for any $(x,t) \in \overline{\Omega}$ series (16) is majorized by the series

$$c_1c_3\left(1+\|q\|E_{\alpha,\alpha+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}|a_n| + c_2c_3\left(1+\|q\|E_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}\|f_n\|_{\gamma}, \tag{27}$$

series (22) and (23) are majorized accordingly by the series

$$c_1c_3\left(1+\|q\|E_{\alpha,\alpha+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}\left(\lambda_n^2+\|q\|\right)|a_n|$$

$$+\,c_2c_3\varepsilon^{-\gamma}\left(1+\|q\|E_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}\left(\lambda_n^2+\|q\|\right)\|f_n\|_{\gamma}, \tag{28}$$

$$c_1c_4\left(1+\|q\|E_{\alpha,\alpha+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}n|a_n| + c_2c_4\left(1+\|q\|E_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}n\|f_n\|_{\gamma},$$

$$c_1c_5\left(1+\|q\|E_{\alpha,\alpha+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}n^2|a_n| + c_2c_5\left(1+\|q\|E_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}\|q\|\right)\right)\sum_{n=1}^{\infty}n^2\|f_n\|_{\gamma}. \tag{29}$$

We explore number series (27)–(29) for convergence.

In accordance with the theory of Fourier–Bessel series, we need to obtain an estimate of the form

$$n^2|a_n| \le \frac{c_6}{(n^2)^{1+\epsilon}}, \quad n^2\|f_n\|_{\gamma} \le \frac{c_7}{(n^2)^{1+\epsilon}} \quad (c_6, c_7, \epsilon > 0 - \text{const}). \tag{30}$$

Let the functions $a(\cdot)$, $f(\cdot,t)$ be defined and continuously differentiable four times on the interval $x \in [0,1]$, and suppose that

(i) $a(0) = f(0,t) = a'(0) = \frac{\partial}{\partial x}f(0,t) = a''(0) = \frac{\partial^2}{\partial x^2}f(0,t) = a'''(0) = \frac{\partial^3}{\partial x^3}f(0,t) = 0.$

(ii) $a^{(4)}(x), \frac{\partial^4}{\partial x^4}f(x,t)$ is bounded (these derivatives may not exist at individual points).

(iii) $a(1) = f(1,t) = a'(1) = \frac{\partial}{\partial x}f(1,t) = a''(1) = \frac{\partial}{\partial x^2}f(1,t) = 0$.

Then for the Fourier–Bessel coefficients of the functions $a(\cdot)$, $f(\cdot,t)$ inequality (30) holds ([21], p. 283).

According to conditions (i)–(iii), series (27)–(29) are majorized by the number series

$$c_8\sum_{n=1}^{\infty}(n^{-2})^{1+\epsilon}, \tag{31}$$

and, therefore, series (16), (22), and (23) in a closed region $\overline{\Omega}$ converge uniformly, where c_8 are some constants. Thus, the constructed function $u(x,t)$, defined by series (16), satisfies all the conditions of the problem (1)–(4).

□

This proves

Theorem 1. *If the functions* $a(x)$ *and* $f(x,t)$ *meet conditions* (i)–(iii) *and* $f(x,\cdot) \in C_{\gamma}[0,T]$, *where* $0 \le \gamma \le \alpha < 1$, $q(t) \in C[0,T]$, *then there exists a unique solution* $u(x,t)$ *to the problem* (1)–(4), *which is determined by the sum of series* (16).

3. EXISTENCE OF A SOLUTION TO THE INVERSE PROBLEM (1)–(5)

Let us derive an operator equation for the unknown function $q(t)$. We rewrite series (16) taking into account (21) as

$$u(x,t)=\sum_{n=1}^{\infty}\Bigg[a_nE_\alpha(-\lambda_n^2t^\alpha)+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)f_n(\tau)d\tau$$

$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)q(\tau)u_n(\tau)d\tau\Bigg]X_n(x).$$

First, we integrate Eq. (1) with respect to x from 0 to 1. After this, integrating by parts with respect to x and taking into account conditions (3)–(5) and designation $\tilde{f}(t)=\int_0^1 f(x,t)dx$, we arrive at the relation

$$(\partial_{0t}^\alpha b)(t)=u_x(1,t)-u_x(0,t)-\lim_{x\to0}\frac{1}{x}u(x,t)+b(t)q(t)+\tilde{f}(t). \tag{32}$$

From here, using (15), (16), it is easy to see that

$$\lim_{x\to0}\frac{1}{x}u(x,t)=u_x(0,t)=\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{\lambda_n}{J_2(\lambda_n)}u_n(t).$$

As a result, dividing both sides of Eq. (32) by $b(t)$, we get

$$q(t)=p_0(t)-\frac{1}{b(t)}\int_0^t\left\{\sum_{n=1}^{\infty}\left(X'_n(1)-\frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)}\right)E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)u_n(\tau)\right\}(t-\tau)^{\alpha-1}q(\tau)d\tau, \tag{33}$$

where

$$p_0(t)=\frac{1}{b(t)}\Big[(\partial_{0t}^\alpha b)(t)-\tilde{f}(t)\Big]-\frac{1}{b(t)}\sum_{n=1}^{\infty}\Bigg[a_nE_\alpha(-\lambda_n^2t^\alpha)$$

$$+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)f_n(\tau)d\tau\Bigg]\left(X'_n(1)-\frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)}\right).$$

Let us introduce the nonlinear operator $\mathcal{F}:C[0,T]\to C[0,T]$ by the formula

$$\mathcal{F}[q]=p_0(t)-\frac{1}{b(t)}\int_0^t\left\{\sum_{n=1}^{\infty}\left(X'_n(1)-\frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)}\right)E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)u_n(\tau)\right\}(t-\tau)^{\alpha-1}q(\tau)d\tau,$$

where $u_n(t)$ is the solution to the problem (18), (19), defined by formula (21).

Let

$$0<b_0:=\min_{t\in[0,T]}|b(t)|<+\infty,\quad b_1:=\|b(t)\|_{C^1[0,T]}<+\infty,\quad b(0)=\int_0^1a(x)dx. \tag{A}$$

It is well-known that ([1], p. 94)

$$\left\|\partial_{0t}^\alpha b(t)\right\|_{C[0,T]}\le\frac{T^{1-\alpha}}{(2-\alpha)\Gamma(1-\alpha)}\|b\|_{C^1[0,T]}.$$

Due to conditions (i)–(iii) and (A), the operator $\mathcal{F}$ acts from $C[0,T]$ to $C[0,T]$, and relation (33) can be rewritten as

$$q=\mathcal{F}[q]. \tag{34}$$

Let us prove that the operator $\mathcal{F}$ maps a convex set

$$B_r[0] := \{q(t) \in C[0,T] : \|q\| \le r\} \subset C[0,T]$$

into itself.

Indeed, if we assume that conditions (i)–(iii), (A) of the statement are satisfied, as well as (31), then for any $q(t) \in C[0,T]$ we have

$$\begin{aligned}
\|\mathcal{F}[q]\| &= \left\| p_0(t) - \frac{1}{b(t)}\int_0^t \left\{ \sum_{n=1}^{\infty} \left(X_n'(1) - \frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)} \right) E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) u_n(\tau) \right\} (t-\tau)^{\alpha-1} q(\tau) d\tau \right\| \\
&\le \|p_0(t)\| + \left\| \frac{1}{b(t)}\int_0^t \left\{ \sum_{n=1}^{\infty} \left(X_n'(1) - \frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)} \right) E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) u_n(\tau) \right\} (t-\tau)^{\alpha-1} q(\tau) d\tau \right\| \\
&\le \left\| \frac{1}{b(t)} \left[(\partial_{0t}^{\alpha} b)(t) - \tilde{f}(t) \right] - \frac{1}{b(t)} \sum_{n=1}^{\infty} \left[a_n E_\alpha(-\lambda_n^2 t^\alpha) \right.\right. \\
&\quad \left.\left. + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) f_n(\tau) d\tau \right] \left(X_n'(1) - \frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)} \right) \right\| \\
&\quad + \frac{c_1 c_4}{b_0 \Gamma(\alpha+1)} T^\alpha \|q\| \left(1 + \|q\| E_{\alpha,\alpha+1}\left(T^\alpha \|q\|\right)\right) \sum_{n=1}^{\infty} n|a_n| \\
&\quad + \frac{c_2 c_4}{b_0 \Gamma(\alpha+1)} T^\alpha \|q\| \left(1 + \|q\| E_{\alpha,2\alpha-\gamma+1}\left(T^\alpha \|q\|\right)\right) \sum_{n=1}^{\infty} n\|f_n\|_\gamma \\
&\le \frac{1}{b_0} \left\{ \frac{T^{1-\alpha}}{(2-\alpha)\Gamma(1-\alpha)} b_1 + \tilde{f}_0 + \frac{\pi}{6} c_4 \left(c_6 + \frac{c_7}{\Gamma(1+\alpha)} T^\alpha \right) \right. \\
&\quad \left. + \frac{\pi}{6\Gamma(\alpha+1)} c_4 T^\alpha \|q\| \left(c_1 c_6 \left(1 + \|q\| E_{\alpha,\alpha+1}\left(T^\alpha \|q\|\right)\right) + c_2 c_7 \left(1 + \|q\| E_{\alpha,2\alpha-\gamma+1}\left(T^\alpha \|q\|\right)\right) \right) \right\},
\end{aligned} \tag{35}$$

where $\tilde{f}_0 := \max_{t\in[0,T]} \left|\tilde{f}(t)\right|$. The right-hand side of inequality (35) is a monotonically increasing function of T. Moreover, if we replace $\|q\|$ with r, then due to $\|q\| \le r$ this inequality only strengthens

$$\begin{aligned}
\|\mathcal{F}[q]\| &\le \frac{1}{b_0} \left\{ \frac{T^{1-\alpha}}{(2-\alpha)\Gamma(1-\alpha)} b_1 + \tilde{f}_0 + \frac{\pi}{6} c_4 \left(c_6 + \frac{c_7}{\Gamma(1+\alpha)} T^\alpha \right) \right. \\
&\quad \left. + \frac{\pi}{6\Gamma(\alpha+1)} c_4 T^\alpha r (c_1 c_6 (1 + r E_{\alpha,\alpha+1}(T^\alpha r)) + c_2 c_7 (1 + r E_{\alpha,2\alpha-\gamma+1}(T^\alpha r))) \right\}.
\end{aligned}$$

Let T_1 be a positive root of the equation

$$\begin{aligned}
m_1(T) &= \frac{1}{b_0} \left\{ \frac{T^{1-\alpha}}{(2-\alpha)\Gamma(1-\alpha)} b_1 + \tilde{f}_0 + \frac{\pi}{6} c_4 \left(c_6 + \frac{c_7}{\Gamma(1+\alpha)} T^\alpha \right) \right. \\
&\quad \left. + \frac{\pi}{6\Gamma(\alpha+1)} c_4 T^\alpha r (c_1 c_6 (1 + r E_{\alpha,\alpha+1}(T^\alpha r)) + c_2 c_7 (1 + r E_{\alpha,2\alpha-\gamma+1}(T^\alpha r))) \right\} = r.
\end{aligned}$$

Then for all $T \in [0, T_1]$ we have $\mathcal{F} B_r[0] \subset B_r[0]$.

Due to estimate (35) the set of functions

$$y(t) = \mathcal{F}[q(t)], \quad q \in B_r[0], \tag{36}$$

is uniformly bounded.

Let us now prove that $\mathcal{F}[q]$ is a completely continuous operator on $B_r[0]$. To do this, we use Arzel's theorem ([24], p. 207). Let us recall the definition of an equicontinuous operator: an operator $\mathcal{F}$ is called *equicontinuous* if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $q_1, q_2 \in B_r[0]$, for which $\|q_1 - q_2\| \le \delta$, the inequality holds

$$\|\mathcal{F}[q_1] - \mathcal{F}[q_2]\| \le \epsilon. \tag{37}$$

Let

$$c_9 := \frac{\pi T^\alpha}{6 b_0 \Gamma(\alpha+1)\Gamma(\alpha)} c_4 \left(c_1 c_6 \left(1 + r E_{\alpha,\alpha+1}(T^\alpha r)\right) + c_2 c_7 \left(1 + r E_{\alpha,2\alpha-\gamma+1}(T^\alpha r)\right)\right).$$

Next, for any $q_1, q_2 \in B_r[0]$ we have

$$\|\mathcal{F}[q_1] - \mathcal{F}[q_2]\| = \left\| \frac{1}{b(t)} \int_0^t \left\{ \sum_{n=1}^\infty E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) u_n(\tau) \left(X_n'(1) - \frac{\sqrt{2}\lambda_n}{J_2(\lambda_n)} \right) \right\} \right.$$

$$\left. \times (t-\tau)^{\alpha-1} [q_1(\tau) - q_2(\tau)] d\tau \right\| \le c_9 \|q_1 - q_2\| \le c_9 \delta_0.$$

Therefore, if we take $\delta_0 = \epsilon / c_9$, then inequality (37) holds for $\delta \in (0, \delta_0]$. The equicontinuity of the set of functions (36) is proved. According to Arzel's theorem, $\mathcal{F}[q]$ is completely continuous on $B_r[0]$. Besides, $\mathcal{F}$ reflects this ball into itself. Therefore, by Schauder's principle, the integral equation (33) has at least one solution in the ball $B_r[0]$.

Thus, the following theorem is proved.

Theorem 2. *Let the conditions of Theorems* 1 *and* (A) *hold. Then Eq.* (34) *has a fixed point on* $B_r[0]$.

4. UNIQUENESS OF SOLUTION TO THE INVERSE PROBLEM (1)–(5)

Let us assume that there are two different solutions to the inverse problem (1)–(5):

$$\{u^{(1)}(x,t),\ \ q^{(1)}(t)\},\ \ \{u^{(2)}(x,t),\ \ q^{(2)}(t)\}.$$

Let us put

$$z(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t), \quad \sigma(t) = q^{(1)}(t) - q^{(2)}(t).$$

Then this pair of functions satisfies the relations

$$\partial_{0t}^\alpha z(x,t) = z_{xx}(x,t) + \frac{1}{x} z_x(x,t) - \frac{1}{x^2} z(x,t) + \sigma(t) u^{(2)}(x,t) + q^{(1)}(t) z(x,t),$$

$$\lim_{x \to 0} x z_x(x,t) = 0, \quad z(1,t) = 0,$$

$$z(x,0) = 0, \quad 0 \le x \le 1,$$

$$\int_0^1 z(x,t) dx = 0.$$

Integrating over x ranging from 0 to 1, we arrive at the relation

$$0 = z_x(1,t) - z_x(0,t) - \lim_{x \to 0} \frac{z(x,t)}{x} + b(t)\sigma(t);$$

dividing both sides of the last equality by $b(t)$ and taking into account (33), we have

$$\begin{aligned}\sigma(t) &= \frac{1}{b(t)}\int_0^t \left\{\sum_{n=1}^{\infty} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})(2X'_n(0) - X'_n(1))u_n^{(2)}(\tau)\right\}(t-\tau)^{\alpha-1}\sigma(\tau)d\tau \\ &+ \frac{1}{b(t)}\int_0^t \left\{\sum_{n=1}^{\infty} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})(2X'_n(0) - X'_n(1))z_n(\tau)\right\}(t-\tau)^{\alpha-1}q^{(1)}(\tau)d\tau,\end{aligned} \tag{38}$$

where $z_n(t) = u_n^{(1)}(t) - u_n^{(2)}(t)$,

$$z_n(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\left[q^{(1)}(\tau)z_n(\tau) + \sigma(\tau)u_n^{(2)}(\tau)\right]d\tau.$$

Note that, by virtue of Theorems 1 and 2, for the functions $u_n^{(j)}(t)$ and $q^{(j)}(t)$, $j = 1, 2$, the estimates (31), (35) are valid. Then

$$\begin{aligned}|z_n(t)| &\le \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})q^{(1)}(\tau)z_n(\tau)d\tau\right| \\ &+ \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\sigma(\tau)u_n^{(2)}(\tau)d\tau\right|.\end{aligned}$$

Considering the first inequality written for $u_n^{(2)}(t)$ in Lemma 2, $\left\|q^{(1)}\right\| \le r$, and also the statement, we have

$$\begin{aligned}|z_n(t)| &\le \frac{1}{\Gamma(\alpha)}\left\{c_1\left(1 + rE_{\alpha,\alpha+1}\left(T^{\alpha}r\right)\right)|a_n| + c_2\left(1 + rE_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}r\right)\right)\|f_n\|_{\gamma}\right\}\int_0^t (t-\tau)^{\alpha-1}|\sigma(\tau)|\,d\tau \\ &+ \frac{r}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|z_n(\tau)|\,d\tau.\end{aligned}$$

Then, by Lemma 1, we get

$$\begin{aligned}|z_n(t)| &\le \frac{1}{\Gamma(\alpha)}\Big\{c_1(1 + rE_{\alpha,\alpha+1}(T^{\alpha}r))|a_n| \\ &+ c_2\left(1 + rE_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}r\right)\right)\|f_n\|_{\gamma}\Big\}\int_0^t (t-\tau)^{\alpha-1}|\sigma(\tau)|\,d\tau \\ &+ \frac{r}{(\Gamma(\alpha))^2}\left[c_1\left(1 + rE_{\alpha,\alpha+1}\left(T^{\alpha}r\right)\right)|a_n| + c_2\left(1 + rE_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}r\right)\right)\|f_n\|_{\gamma}\right] \\ &\times \int_0^t (t-\tau)^{\alpha-1}\int_0^{\tau} (\tau-s)^{\alpha-1}|\sigma(s)|\,ds d\tau.\end{aligned}$$

Changing the order of integration in the last integral using the Dirichlet formula, we arrive at the equality

$$\int_0^t |\sigma(s)|\,ds\int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1}d\tau = \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int_0^t (t-\tau)^{2\alpha-1}|\sigma(s)|\,ds.$$

Then from the inequality $(t-\tau)^{2\alpha-1} \le T^{\alpha}(t-\tau)^{\alpha-1}$ for $0 \le \tau \le t \le T$ we get

$$|z_n(t)| \le c_{10}\int_0^t (t-\tau)^{\alpha-1}|\sigma(\tau)|\,d\tau, \tag{39}$$

where

$$c_{10} := \left(c_1\left(1 + rE_{\alpha,\alpha+1}\left(T^{\alpha}r\right)\right)|a_n| + c_2\left(1 + rE_{\alpha,2\alpha-\gamma+1}\left(T^{\alpha}r\right)\right)\|f_n\|_{\gamma}\right)\left(\frac{1}{\Gamma(\alpha)} + \frac{rT^{\alpha}}{\Gamma(2\alpha)}E_{\alpha,\alpha}(rT^{\alpha})\right).$$

On the other hand, from (38), (39) we have

$$|\sigma(t)| \le c_{11}\int_0^t (t-\tau)^{\alpha-1}|\sigma(\tau)|\,d\tau. \tag{40}$$

Hence, by Lemma 1, for any $t \in [0,T]$ we conclude that $\sigma(t) \equiv 0$ or $q^{(1)}(t) \equiv q^{(1)}(t)$.

Remark. The integral inequality (40) has been obtained similarly to (39), where c_{11} is some constant depending only on α, T, r.

So, we have proven the following uniqueness theorem.

Theorem 3. *Let the conditions of Theorems* 1 *and* 2 *hold. Then the inverse problem* (1)–(5) *cannot have more than one solution.*

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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