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MUNDARIJA *** СОДЕРЖАНИЕ *** CONTENTS

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ESTIMATES FOR CONVOLUTION OPERATORS RELATED TO A_∞ TYPE SINGULARITIES

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Annotatsiya. *Ushbu maqolada biz giperbolik tenglamalar bilan bog'liq bo'lgan o'rama operatori amplituda funksiyasining bir jinsli tartibi uchun baholarni olamiz. Bu natija M.Sugimoto tomonidan $n = 3$ holat uchun berilgan baholarni yaxshilaydi. Aniqrog'i, biz M.Sugimoto tomonidan A_∞ tipidagi o'ziga xosliklarga ega bo'lgan ixtiyoriy analitik faza funksiyasi uchun berilgan baholarni kengaytiramiz. Bu M.Sugimoto tomonidan qo'yilgan masalaning yechimini beradi.*

Kalit so'zlar: *O'rama operatori, Furye almashtirishi, Bessel potensiali, amplitude, faza, cho'zish, Hausdor-Yung tengsizligi, tebranuvchan integral.*

Аннотация. *В настоящей работе получены оценки однородного порядка амплитудной функции оператора свёртки, связанного с гиперболическими уравнениями. Этот результат улучшает оценки М.Сугимото для случая n=3. Точнее, мы распространяем оценки, данные М.Сугимото, для произвольных аналитических фазовых функций, имеющих особенности типа A_∞. Это даёт решение проблемы, предложенное М.Сугимото.*

Ключевые слова: *Свёрточный оператор, преобразование Фурье, Бесселев потенциал, амплитуда, фаза, локализация, неравенство Хаусдорфа-Юнга, осциллирующий интеграл.*

Abstract. *In this paper, we obtain estimates for the homogeneous order of the amplitude function of the convolution operator related to hyperbolic equations. This result improves the estimates given by M.Sugimoto for the case $n = 3$. More precisely, we extend the estimates given by M.Sugimoto for arbitrary analytic phase functions having A_∞ type singularities. This gives a solution of the problem proposed by M.Sugimoto.*

Keywors: *Convolution operator, Fourier transform, Bessel potential, amplitude, phase, localization, Hausdor-Young inequality, oscillatory integral.*

Introduction. In this paper we consider the convolution operators of type:

$$M_k = F^{-1}[e^{i\varphi(\xi)}a_k]F, \tag{1}$$

where F is the Fourier transform operator, $\varphi \in C^\infty(\mathbb{R}^\nu \setminus \{0\})$ is homogeneous of order one, so-called phase function, $a_k \in C^\infty(\mathbb{R}^\nu_\xi)$ is a homogeneous function of order $-k$ for large ξ. a_k is called to be an amplitude function.

Let $1 \le p \le 2$ be a fixed number: We consider the problem: *find the minimal number $k(p)$ such that the operator $M_k: L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu)$ is bounded for any $k > k(p)$.*

In the case $\nu = 1$, the problem is easy to solve. Since in this case due to homogeneity of the function φ we have

$$\varphi(\xi) = \begin{cases} c_1\xi, & for \quad \xi > 0, \\ c_2\xi, & for \quad \xi < 0, \end{cases}$$

where c_1, c_2 are non-zero constants and hence $F^{-1}e^{i\varphi}F$ is nothing but a sum of translations up to the $L^p(\mathbb{R}) \to L^{p'}(\mathbb{R})$ bounded operator for any $1 < p \le 2$, consequently M_k is essentially a Bessel potential, the boundedness of which is well known due to Hardy-Littlewood-Sobolev's inequality [5].

Similar results hold true for the case $\nu > 1$ and $\varphi \equiv 0$.

Further, we will assume that $\nu > 1$. Also, it is assumed that $\varphi(\xi) \neq 0$ for any $\xi \in \mathbb{R}^\nu \setminus \{0\}$. Since φ is a smooth function on $\mathbb{R}^\nu \setminus \{0\}$ and its domain is a connected set, whenever $\nu \ge 2$, then without loss of generality we may and will assume that $\varphi(\xi) > 0$ for any $\xi \neq 0$. Moreover, φ is a homogeneous function of order one, then, due to the Euler's homogeneity relation we have the identity:

$$\sum_{j=1}^{n} \xi_j \frac{\partial \varphi(\xi)}{\partial \xi_j} = \varphi(\xi),$$

for any $\xi \in \mathbb{R}^\nu \setminus \{0\}$ and hence the set Σ defined by the following

$$\Sigma = \{\xi \in \mathbb{R}^\nu : \varphi(\xi) = 1\}$$

is a smooth or an analytic hypersurface provided φ is a smooth or a real analytic function respectively. Next, we will assume that φ is a real analytic function defined on $\mathbb{R}^\nu \setminus \{0\}$.

Further, we use notation:

$$k_p := k_p(\Sigma) := \inf_{k>0}\{k > 0: M_k \ is \ L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu) \text{ bounded for any } a_k\} \tag{2}$$

It turns out that the number $k_p(\Sigma)$ depends on geometric properties of the hypersurface Σ. The number $k_p(\Sigma)$ is called to be a critical exponent.

In the case $\Sigma = S^{\nu-1}$, which is related to the classical wave equation, the $L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu)$ boundedness of M_k is obtained by Strichartz ([6] and [7]). It should be noted that the sphere has a positive, constant Gaussian curvature, which is essential in the estimates for the convolution operator.

This result has been extended to the case when Σ has non-vanishing Gaussian curvature by Brenner [3], and the case when Σ is convex by Sugimoto [8]. Note that the estimates, in the paper [8], are sharp for partial class of convex hypersurfaces.

In this article, we shall consider the case when $\Sigma \subset \mathbb{R}^3$ is not necessarily convex and at least one of the principal curvatures does not vanish at any point. Moreover, we obtain the sharp value of the number k_p for arbitrary analytic hypersurface having A_∞ type singularity improving results of the paper [10].

It is noteworthy that, in the case $\nu = 2$, by M. Sugimoto have been obtained the sharp bound for k_p (see [9]). In this case the sharp bound for k_p can be defined by the index $\gamma(\Sigma)$, which is the maximal order of contact of the curve Σ to its tangent line.

Similarly, one can define the index $\gamma_0(\Sigma)$ for the case $\nu \geq 3$ by the following:

$$\gamma_0(\Sigma) = \sup_p \inf_H \gamma(\Sigma; x, H).$$

For a point $x \in \Sigma$ and for a plane H (of dimension 2) which contains the normal line of Σ at x, we have defined the index $\gamma(\Sigma; x, H)$ to be the order of contact of the curve $\Sigma \cap H$ to the line $T \cap H$ at x, where T denotes the tangent hyperplane of Σ at x.

Then Sugimoto obtained the upper bound for k_p:

$$k_p(\Sigma) \leq \left(2n - \frac{2}{\gamma_0(\Sigma)}\right)\left(\frac{1}{p} - \frac{1}{2}\right).$$

Actually, the upper bound for $k_p(\Sigma)$ is closely related to estimates for the Fourier transform of measures supported on the hypersurfaces Σ. It should be noted that except the case $\nu = 2$ in general the quantity $\gamma_0(\Sigma)$ does not define the sharp uniform (with respect to the direction of the frequency vector) bound for the Fourier transform of measures supported on the hypersurface Σ. On the other hand, in general even the sharp uniform bound for the Fourier transform of surface-carried measures does not give the sharp bound for the number $k_p(\Sigma)$ (compare with [4]). Actually, this phenomenon can be observed in the case when Σ is related to A_∞ type singularities.

We obtain an upper and lower bounds for the number $k_p(\Sigma)$ extending the results proved by M. Sugimoto [10]. It is noteworthy that the upper bound agree with the lower bound. Thus, we obtain the sharp value of the number $k_p(\Sigma)$ for arbitrary analytic surfaces having A_∞ type singularity improving the result proved by Sugimoto [10] (compare with Theorem 2 page 306).

The paper organized as follows, in the next section 2 we give preliminary results on localization of the problem. Then we formulate relation oscillatory integrals in the next section 3. We formulate our main results in the section 4. Also, we give the upper bound for the critical exponent. In the concluding section 5 we obtain a lower bound for the critical exponent finishing a proof of the main Theorem.

Conventions: Throughout this article, we shall use the variable constant notation, i.e., many constants appearing in the course of our arguments, often denoted by $c, C, \varepsilon, \delta$; will typically have different values at different lines. Moreover, we shall use symbols such as $\sim, \lesssim$; or $\ll$ in order to avoid writing down constants, as explained in [4] (Chapter 1). By χ_0 we shall denote a non-negative smooth, spherical symmetric cut-off function on $\mathbb{R}^\nu$ with typically small compact support which is identically 1 on a small neighborhood of the origin, and also $\chi_1(x) := \chi_0(x) - \chi_0(2x)$.

Localization. Since $\boldsymbol{\Sigma}$ is a compact hypersurface, then following M. Sugimoto it is enough to consider the local version of the problem. More precisely, we may assume that the amplitude function $\boldsymbol{a_k(\xi)}$ is concentrated in a sufficiently small conic neighborhood $\boldsymbol{\Gamma}$ of a fixed point $\boldsymbol{v} \in \boldsymbol{S^2}$ (where $\boldsymbol{S^2}$ is the unit sphere centered at the origin of the space $\mathbb{R}^{\boldsymbol{3}}$) and $\boldsymbol{\varphi(\xi)} \in \boldsymbol{C}^\infty(\boldsymbol{\Gamma})$. Fixing such a point $\boldsymbol{v} \in \mathbb{R}^{\boldsymbol{3}}$, let us define the following local exponent $\boldsymbol{k_p(v)}$ associated to this point:

$$k_p(v) := \inf_{k>0}\{k: \exists \Gamma, M_k: L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3) \text{ is bounded, whenever } supp(a_k) \subset \Gamma\}. \tag{3}$$

Further, we use the standard notation assuming F being a sufficiently smooth function:

$$\partial^\mu F(x) := \partial_1^{\mu_1} \dots \partial_\nu^{\mu_\nu} F(x) := \frac{\partial^{|\mu|} F(x)}{\partial x_1^{\mu_1} \dots \partial x_\nu^{\mu_\nu}},$$

where $\mu = (\mu_1, \dots, \mu_\nu) \in \mathbb{Z}_+^\nu$ is a multi-index, with $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, and $|\mu| := \mu_1 + \dots + \mu_\nu$.

Also, for the sake of being definite we will assume that $v = (0,0,1)$ and $\varphi(0,0,1) = 1$. Then after possible a linear transform in the space $\mathbb{R}_\xi^3$, which preserves the point v, we may assume $\partial_1 \varphi(0,0,1) = 0$ as well as $\partial_2 \varphi(0,0,1) = 0$. Thus, in a neighborhood of the point v the hypersurface Σ is given as the graph of a smooth function:

$$\Sigma \cap \Gamma = \{\xi \in \Gamma : \varphi(\xi) = 1\} = \{(\xi_1, \xi_2, 1 + \phi(\xi_1, \xi_2)) \in \mathbb{R}^3 : (\xi_1, \xi_2) \in U\},$$

where $U \subset \mathbb{R}^2$ is a sufficiently small neighborhood of the origin and, $\phi \in C^\infty(U)$ is a smooth function satisfying the conditions: $\phi(0,0) = 0, \nabla\phi(0,0) = 0$ (compare with [10]).

Singularity of the function ϕ is called to be an A_∞ type singularity at the point $(0,0)$ if there exists a diffeomorphic map $x = G(y)$ defined in a neighborhood of the origin such that $G(0,0) = (0,0)$ and the relation

$$\phi(G(y)) = \pm y_1^2$$

holds true (see [2]).

Further, we assume that ϕ has A_∞ type singularity at the origin.

Surely, similarly one can define Σ in a neighborhood of the point $v = (0, \dots, 0, 1) \in \mathbb{R}^\nu$ as the graph of a smooth function ϕ defined in a sufficiently small neighborhood U of the origin of $\mathbb{R}^{\nu-1}$.

Preliminaries (related oscillatory integrals). Note that boundedness of the convolution operators is related to behaviour of the following convolution kernel:

$$K_k(x) := F^{-1}(e^{i\varphi(\xi)} a_k(\xi))(x),$$

where we define the Fourier operator and its inverse by the followings [11]:

$$F(u)(\xi) := \frac{1}{(\sqrt{2\pi})^\nu} \int_{\mathbb{R}^\nu} e^{-i\xi x} u(x) dx,$$

and

$$F^{-1}(u)(\xi) := \frac{1}{(\sqrt{2\pi})^\nu} \int_{\mathbb{R}^\nu} e^{i\xi x} u(x) dx$$

respectively for Schwartz functions u, where ξx is the usual inner product of the vectors ξ and x. Then the Fourier transform and its inverse are defined by standard arguments for distributions.

Proposition 3.1. *Let $\varphi \in C^\infty(\mathbb{R}^\nu \setminus \{0\})$ be a positive, homogeneous function of order one and $k > 0$ be a fixed positive number. Then there exists a positive numbers $\varepsilon > 0, M > 0$ such that the following relation*

$$\chi_{\{|x| \le \varepsilon\} \cup \{|x| > M\}}(\cdot) K_k(\cdot) \in L^{1+0}(\mathbb{R}^\nu)$$

holds, where we used the standard notation: $L^{1+0}(\mathbb{R}^\nu) := \cap_{p>1} L^p(\mathbb{R}^\nu)$ and χ_A is the indicator function of the set A.

Proof. A proof of the Proposition 3.1 is straightforward. Indeed, we write

$$K_k(x) = K_k^0(x) + K_k^\infty(x),$$

where

$$K_k^0(x) := \int_{\mathbb{R}^\nu} e^{i(\varphi(\xi) + x\xi)} a_k(\xi) \chi_0(\xi) d\xi$$

and

$$K_k^\infty(x) := \int_{\mathbb{R}^\nu} e^{i(\varphi(\xi) + x\xi)} a_k(\xi)(1 - \chi_0(\xi)) d\xi,$$

where χ_0 is a smooth cutoff function such that $\chi_0(\xi) = 1$ in a neighborhood of the origin of $\mathbb{R}^\nu$.

Note that $K_k^0(x)$ is a smooth function of x. Let $\chi_1 \in C_0^\infty(\mathbb{R}^\nu)$ be a smooth function supported near the point 1 satisfying the conditions $\chi_1(x) \equiv 0$ in a neighborhood of the origin and also:

$$\sum_{j \in \mathbb{Z}} \chi_1(2^j |\xi|) = 1 \quad \text{for any} \quad x \neq 0.$$

So, we can write

$$K_k^0(x) = \sum_{j=1}^\infty \int_{\mathbb{R}^\nu} e^{i(\varphi(\xi) + x\xi)} a_k(\xi) \chi_0(\xi) \chi_1(2^j |\xi|) d\xi = \sum_{j=1}^\infty K_{kj}^0(x),$$

where

$$K_{kj}^0(x) := \int_{\mathbb{R}^\nu} e^{i(\varphi(\xi)+x\xi)} a_k(\xi)\chi_0(\xi)\chi_1(2^j|\xi|)d\xi.$$

By using scaling arguments we obtain:

$$K_{kj}^0(x) = 2^{-\nu j}\int_{\mathbb{R}^\nu} e^{i2^{-j}(\varphi(\xi)+x\xi)} a_k(2^{-j}\xi)\chi_0(2^{-j}\xi)\chi_1(|\xi|)d\xi.$$

If $|x2^{-j}| \lesssim 1$ then we use the trivial estimate $|K_{kj}^0(x)| \lesssim 2^{-\nu j}$. Otherwise, e.g. if $|2^{-j}x| >> 1$ then we can use integration by parts arguments and obtain:

$$|K_{kj}^0(x)| \lesssim_N \frac{2^{-\nu j}}{(1+|x2^{-j}|)^N},$$

where N is any fixed number. We can choose $N > \nu$ and have that

$$\| K_{kj}^0(x) \|_{L^p(\mathbb{R}^\nu)} \lesssim_p 2^{-\frac{j\nu}{p'}}.$$

Thus, the series $\sum_j K_{kj}^0$ converge in L^p for any p with $1 < p < \infty$, hence $K_k^0 \in L^{1+0}(\mathbb{R}^\nu)$.

Now, we will estimate K_k^∞. If $|x| \gg 1$ then we can use integration by parts arguments and obtain:

$$|K_k^\infty(x)| \lesssim_N \frac{1}{|x|^N}.$$

From now, on we assume that $|x| \ll 1$. Then, we can write

$$K_k^\infty(x) = \sum_{j=1}^{\infty}\int_{\mathbb{R}^\nu} e^{i(\varphi(\xi)+x\xi)} a_k(\xi)(1-\chi_0(\xi))\chi_1(2^{-j}|\xi|)d\xi =: \sum_{j=1}^{\infty} K_{kj}^\infty(x).$$

By using the scaling arguments we obtain:

$$K_{kj}^\infty(x) = 2^{-jk+\nu j}\int_{\mathbb{R}^\nu} e^{i2^j(\varphi(\xi)+x\xi)} a_0(\xi,2^j)(1-\chi_0(2^j\xi))\chi_1(|\xi|)d\xi,$$

where $a_0(\xi,2^j) := 2^{jk}a_k(2^j\xi) \in C_0^\infty(D)\otimes C^\infty(\mathbb{R})$ is a classical symbol of order zero.

Note that if $j \gg 1$ then we have $\chi_0(2^j\xi) = 0$ because $|\xi| \sim 1$. Thus for $j \gg 1$ we get

$$K_{kj}^\infty(x) = 2^{-jk+\nu j}\int_{\mathbb{R}^\nu} e^{i2^j(\varphi(\xi)+x\xi)} a_0(\xi,2^j)\chi_1(|\xi|)d\xi,$$

Since the phase function $\varphi(\xi)+x\xi$ has no critical point on the annulus D provided x is sufficiently small. Therefore we can use integration by parts arguments and get

$$|K_{kj}^\infty(x)| \lesssim_N 2^{-Nj},$$

where N is as large as we wish.

Consequently, there exist positive numbers $\varepsilon > 0, M > 0$ such that the following inclusion $\chi_{\{|x|\le\varepsilon\}\cup\{|x|>M\}}(\cdot)K_k^\infty(\cdot) \in L^p(\mathbb{R}^\nu)$ holds true for any $p \ge 1$.

Thus, we came to the following conclusion:

Corollary 3.2. *The main contribution to K_k gives points x which belongs to a sufficiently small neighborhood of the set $-\nabla\varphi(supp(a_k)\backslash\{0\})$.*

Proof. Indeed, if $x^0 \notin -\nabla\varphi(supp(a_k)\backslash\{0\})$ is a point, then there exist a positive number $\delta > 0$ and a neighborhood $V(x^0)$ of the point x^0 such that the inequality

$$|\nabla\varphi(\xi) - x| \ge \delta$$

holds for any $x \in V(x^0)$ and $\xi \in supp(a_k)\backslash\{0\}$. Thus, we can repeat all arguments of the proof of the Proposition 3.1 and came to a proof of the Corollary 3.2.

In the paper [10] have been shown relation between the boundedness of the convolution operator M_k and behaviour of the following oscillatory integral:

$$I(\lambda,z) = \int_{\mathbb{R}^{\nu-1}} e^{i\lambda(z\cdot x+\phi(x))} g(x)dx, (\lambda > 0, z \in \mathbb{R}^{\nu-1}),$$

where $g \in C_0^\infty(U)$ and U is a sufficiently small neighborhood of the origin.

More precisely there are proved the following statements [10]:

Proposition 3.3. *Let $q \ge 2$ and $\alpha \ge 0$. Suppose for all $g \in C_0^\infty(U)$ and $\lambda > 1$,*

$$\| I(\lambda,\cdot) \|_{L^q(\mathbb{R}_z^{\nu-1})} \le C_g\lambda^{-\alpha}, \tag{4}$$

where C_g is independent of λ. Then $K_k(\cdot) := F^{-1}[e^{i\varphi(\xi)}a_k(\xi)](\cdot) \in L^q(\mathbb{R}^\nu)$ and $M_k: L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu)$ bounded for $p = \frac{2q}{2q-1}$, if $k > \nu - \alpha - \frac{1}{q}$.

The main results. **The main Theorem**. *Suppose that* $\boldsymbol{\phi(x)}$ *has* $\boldsymbol{A_\infty}$ *type singularly at the origin with* $\boldsymbol{\partial_1^2\phi(0,0) \neq 0, \partial_1\partial_2\phi(0,0) = 0, \partial_2^2\phi(0,0) = 0}$ *and the equation* $\boldsymbol{\frac{\partial\phi(x_1,x_2)}{\partial x_1} = 0}$ *has the solution* $\boldsymbol{x_1 = x_2^m\omega(x_2)}$ *with a real analytic function* $\boldsymbol{\omega}$ *satisfying the conditions* $\boldsymbol{\omega(0) \neq 0}$ *and* $\boldsymbol{m \geq 3}$*. Then* $\boldsymbol{M_k}$ *is* $\boldsymbol{L^p}(\mathbb{R}^3) \to \boldsymbol{L^{p'}}(\mathbb{R}^3)$ *bounded if*

$$k > k_p(v) := \max\{6(1/p - 1/2) - 1/2, (5 - 1/m)(1/p - 1/2)\}.$$

Moreover if $k < k_p(v)$ then there exists an amplitude function a_k such that the M_k is not $L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ bounded.

Proof. First, we prove the upper bound for the number $k_p(v)$. Note that $\phi(0,0) = 0$ and $\nabla\phi(0,0) = 0$. Also, by assumptions of the Theorem 2 we may suppose that $\frac{\partial^2\phi(0,0)}{\partial x_1} \neq 0$ and $\frac{\partial^2\phi(0,0)}{\partial x_2^2} = 0$ and also $\frac{\partial^2\phi(0,0)}{\partial x_1\partial x_2} = 0$. Hence by implicit function theorem the equation

$$\frac{\partial\phi(x_1,x_2)}{\partial x_1} = 0$$

has a smooth solution $x_1 = \Psi(x_2)$ satisfying the conditions $\Psi(0) = \Psi'(0) = \Psi''(0) = \cdots = \Psi^{(m-1)} = 0$ and $\Psi^{(m)}(0) \neq 0$. Thus, there exists an analytic function ω such that $\omega(0) \neq 0$ and $\Psi(x_2) = x_2^m\omega(x_2)$. Therefore by the division theorem $\phi(x_1,x_2)$ can be written in the form: $\phi(x_1,x_2) = b(x_1,x_2)(x_1 - \Psi(x_2))^2$ and also by the assumptions of the main Theorem we have $\Psi(x_2) = x_2^m\omega(x_2)$. Now, we consider the oscillatory integral given by the following

$$I(\lambda; z) = \int_{R^2} e^{i\lambda\Phi(x,z)} g(x)dx,$$

where

$$\Phi(x,z) = b(x_1,x_2)(x_1 - x_2^m\omega(x_2))^2 + z_1x_1 + z_2x_2.$$

If $|z| > \varepsilon$ then the phase function has no critical points provided g is supported in a sufficiently small neighborhood of the origin. Hence as before we consider behavior of the oscillatory integral when $z \in U$, where U is a sufficiently small neighborhood of the origin. We prove the following inequality which was proved in the paper [10] in a particular case:

$$\| I(\lambda;\cdot) \|_{L^{m+1}(U)} \leq C_{g,\varepsilon}\lambda^{-(1/2+2/(m+1))+\varepsilon} \tag{5}$$

for any $\varepsilon > 0$. From the estimate (5) it follows a proof of the following upper bound:

$$k_p \leq \max\{6(1/p - 1/2) - 1/2, (5 - 1/m)(1/p - 1/2)\}.$$

The rest of the section is devoted to prove of the inequality (5).

First, we use change variables $x_1 - x_2^m\omega(x_2) = y_1$ $x_2 = y_2$ in the integral $I(\lambda, z)$ and obtain:

$$I(\lambda, z) = \int e^{i\lambda(b(y_1+y_2^m\omega(y_2),y_2)y_1^2+z_1y_1+z_1y_2^m\omega(y_2)+z_2y_2)} g(y_1 + y_2^m\omega(y_2), y_2)dy.$$

Consider the following interior integral:

$$I_{in}(\lambda, z_1, y_2) := \int e^{i\lambda\Phi_1(y_1,y_2,z_1)} g(y_1 + y_2^m\omega(y_2), y_2)dy_1,$$

where $\Phi_1(y_1,y_2,z_1) := b(y_1 + y_2^m\omega(y_2), y_2)y_1^2 + z_1y_1$ is the new phase function. Now, the variables y_2, z_1 can be considered as parameters. The phase function $\Phi_1(y_1,y_2,z_1)$ has a unique non-degenerate critical point with respect to y_1. It is a real analytic function of the parameters. The critical point can be written in the form: $y_1^c = z_1B(y_2,z_1)$, where B is a real analytic function with $B(0,0) = -\frac{1}{2b(0,0)}$. Thus by using stationary phase method in the variable y_1 we get:

$$I_{in} = C\frac{e^{i\lambda z_1^2B_1(y_2,z_1)}g(y_1^c(y_2,z_1) + y_2^m\omega(y_2), y_2)}{\lambda^{\frac{1}{2}}} + O\left(\frac{1}{|\lambda|^{\frac{3}{2}}}\right),$$

where C is a constant. Moreover, asymptotic relation in the remainder term holds uniformly with respect to parameters. Therefore it is enough to investigate contribution of the principal part of the interior integral. So, we consider the following integral:

$$I_1(\lambda, z) = \frac{C}{\lambda^{\frac{1}{2}}}\int e^{i\lambda(z_1(y_2^m\omega(y_2)+z_1B_1(y_2,z_1))+z_2y_2)} g(y_1^c + y_2^m\omega(y_2), y_2)dy_2.$$

Note that

$$|I(\lambda, z) - I_1(\lambda, z)| \leq \frac{C}{|\lambda|^{\frac{3}{2}}}.$$

By using the classical van der Corpute Lemma [11] we have the estimate:

$$|I_1(\lambda,z)| \le \frac{C}{\lambda^{\frac{1}{2}+\frac{1}{m}}|z_1|^{\frac{1}{m}}} \tag{6}$$

Suppose $|z_1| \le M|z_2|$ (where M is a fixed positive real number) then the phase function of the integral $I_1(\lambda,z)$ has no critical point provided that the amplitude function is concentrated in a sufficiently small neighborhood of the origin. Then we can use integration by parts arguments and obtain:

$$|I_1(\lambda,z)| \le \frac{C}{|\lambda||z_2|^{\frac{1}{2}}}. \tag{7}$$

Interpolating the inequalities (6) and (7) with the interpolation parameter

$$\theta := \frac{m^2-3m}{m^2-m-2} + \frac{2\varepsilon m}{m-2}, \tag{8}$$

where ε is a sufficiently small positive real number, we get:

$$|I_1(\lambda,z)| \le \frac{C}{|\lambda|^{\frac{1}{2}+\frac{2}{m+1}-\varepsilon}|z_2|^{\frac{1}{2}(1-\theta)}|z_1|^{\frac{\theta}{m}}}$$

It is easy to show that

$$\frac{\chi_{\{|z_1|\le M|z_2|\}}(z)}{|z_2|^{\frac{1}{2}(1-\theta)}|z_1|^{\frac{\theta}{m}}} \in L^{m+1}(U),$$

where U is any bounded neighborhood of the origin.

Now, we assume $|z_2| \le \delta|z_1|$, where $\delta = \frac{1}{M}$ is a fixed positive real number. Then from the results of the paper [1] there exists a function $\Omega(z_1,\eta)$ satisfying the conditions:

1) for any number $1 \le p < \frac{2(m-1)}{m-2}$ and for any z_1 the inequality

$$\int_{|\eta|<\delta} (\Omega(z_1,\eta))^p d\eta \le C,$$

holds, with constant C not depending on z_1;

2) the following estimate holds true:

$$(1-\chi_{\{|z_1|\le M|z_2|\}}(z))|I_1(\lambda,z)| \le \frac{\Omega\left(z_1,\frac{z_2}{z_1}\right)}{|\lambda||z_1|^{\frac{1}{2}}}$$

Finally, interpolating the last inequality with (6) by using interpolating parameter (8) we get:

$$(1-\chi_{\{|z_1|\le M|z_2|\}}(z))|I_1(\lambda,z)| \le \frac{\left(\Omega\left(z_1,\frac{z_2}{z_1}\right)\right)^{1-\theta}}{|\lambda|^{\frac{1}{2}+\frac{2}{m+1}-\varepsilon}|z_1|^{\frac{2}{m+1}-\varepsilon}}.$$

It is easy to show that the following inclusion holds:

$$(1-\chi_{\{|z_1|\le M|z_2|\}}(z))\frac{\left(\Omega\left(z_1,\frac{z_2}{z_1}\right)\right)^{1-\theta}}{|z_1|^{\frac{2}{m+1}-\varepsilon}} \in L^{m+1}(U).$$

The last inclusion completes a proof of the upper bound for $k_p(v)$. Indeed, for $p_0 = \frac{2m+2}{2m+1}$ we have $k > \frac{5}{2} - \frac{3}{m+1} + \varepsilon$ for and $\varepsilon > 0$. Hence $k_{p_0}(v) \le \frac{5}{2} - \frac{3}{m+1}$. The required estimate for other values of p can be obtained by using interpolation arguments (more detailed arguments one can seen from [10]).

Actually, we prove the following more better result.

Corollary 1. Assume the function ϕ satisfies conditions of the main Theorem. Then for any positive real number ε there exists a function $\Omega \in L^{m+1}_{loc}(\mathbb{R}^2)$ such that the following inequality

$$|I(\lambda,z)| \le \frac{\Omega(z)}{|\lambda|^{\frac{1}{2}+\frac{2}{m+1}-\varepsilon}}$$

holds.

Lower bounds. Assume $\boldsymbol{k < 6(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}$. We show that $\boldsymbol{M_k}$ is not $\boldsymbol{L^p}(\mathbb{R}^3) \mapsto \boldsymbol{L^p}(\mathbb{R}^3)$ bounded. We a little modified the M. Sugimoto sequence and consider the sequence

$$u_j = 2^{-\frac{3j}{p'}} F^{-1}(v(2^{-j}\cdot))(x),$$

where

$$v(\xi)=f\left(\frac{\xi_2}{\varphi(\xi)}-\left(\frac{\xi_1}{\varphi(\xi)}\right)^m\omega\left(\frac{\xi_1}{\varphi(\xi)}\right)\right)\frac{g\left(\frac{\xi_1}{\varphi(\xi)}\right)\psi(\varphi(\xi))|\xi|^k}{\varphi^2(\xi)G\left(\frac{\xi_1}{\varphi(\xi)},\frac{\xi_2}{\varphi(\xi)}\right)},$$

where $f,g,\psi\in C_0^\infty(\mathbb{R})$ are non-negative smooth functions satisfying the conditions: $f(0)=g(0)=1,\psi(1)=1$, and supports of f,g lie in a sufficiently small neighborhood of the origin of $\mathbb{R}$ and support of ψ belongs to a sufficiently small neighborhood of one (cf. [10]). Also, following, M. Sugimoto we introduce the function: $G(y)=1+\phi(y_1,y_2)-y\nabla\phi(y)$.

Obviously $v\in C_0^\infty(\mathbb{R}^3)$ and $\parallel v\parallel_{L^{p'}(\mathbb{R}^3)}\neq 0$. Thus for large j we have

$$\parallel u_j\parallel_{L^p(\mathbb{R}^3)}\sim 1.$$

Now, we consider estimate for $\parallel M_k u_j\parallel_{L^p(\mathbb{R}^3)}$.

We have:

$$M_k u_j=F^{-1}e^{i\varphi(\xi)}a_k(\xi)Fu_j=2^{-\frac{3j}{p'}}F^{-1}(e^{i\varphi(\xi)}a_k(\xi)v(2^{-j}\xi))(x).$$

We perform change of variables given by the scaling $2^j\xi\to\xi$ and obtain:

$$M_k u_j(x)=\frac{2^{\frac{3j}{p}-kj}}{\sqrt{(2\pi)^3}}\int_{\mathbb{R}^3}e^{i2^j(\varphi(\xi)-\xi x)}f\left(\frac{\xi_2}{\varphi(\xi)}-\left(\frac{\xi_1}{\varphi(\xi)}\right)^m\omega\left(\frac{\xi_1}{\varphi(\xi)}\right)\right)$$
$$\times\frac{g\left(\frac{\xi_1}{\varphi(\xi)}\right)\psi(\varphi(\xi))}{\varphi^2(\xi)G\left(\frac{\xi_1}{\varphi(\xi)},\frac{\xi_2}{\varphi(\xi)}\right)}d\xi.$$

Finally, we use change of variables $\xi\to\lambda(y_1,y_2,1+\phi(y_1,y_2))$. Then we have:

$$M_k u_j(x)=\frac{2^{\frac{3j}{p}+\frac{j(m+1)}{p'}-kj}}{\sqrt{(2\pi)^3}}\int_{\mathbb{R}^3}e^{i2^j\lambda(1-x_3-(y_1x_1+y_2x_2+x_3\phi(y_1,y_2)))}\times$$
$$\times f(y_2-y_1^m\omega(y_1))g(y_1)\psi(\lambda)d\lambda dy_1dy_2.$$

Now, we perform the change of variables

$$y_1=z_1,y_2=y_1^m\omega(y_1)+z_2.$$

Thus

$$M_k u_j(x)=2^{\frac{3j}{p}-kj}\int e^{i2^j\lambda\Phi_3(z,x,j)}f(z_2)g(z_1)\psi(\lambda)d\lambda dz_1dz_2,$$

where

$$\Phi_3(z,x,j):=1-x_3-(x_1z_1+x_2z_1^m\omega(z_1)+z_2x_2+x_3z_2^2b(z_1,(z_1^m\omega(z_1)+z_2))).$$

We use stationary phase method assuming, $|1-x_3|\ll 2^{-j},|x_1|\ll 2^{-j},|x_2|\ll 2^{-j}$ and obtain:

$$M_k u_j(x)=2^{j(\frac{3}{p}-\frac{1}{2}-k)}\left(\int_{\mathbb{R}^2}e^{i2^j\lambda\Phi_4}f(z_2^c(z_1,x_2))g(z_1)\psi(\lambda)d\lambda dz_1+O(2^{-j}\right),$$

where

$$\Phi_4:=\Phi_4(z_1,x,j):=1-x_3-x_1z_1+x_2z_1^m\omega(z_1)+x_2^2B(z_1,x_2).$$

From here we obtain the lower bound:

$$\parallel M_k u_j\parallel_{L^{p'}(\mathbb{R}^3)}\geq 2^{j(\frac{3}{p}-\frac{1}{2}-\frac{3}{p'}-k)}c,$$

where $c>0$ is a constant which does not depend on j. Thus if $k<6\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}$ then the operator M_k is not $L^p(\mathbb{R}^3)\mapsto L^{p'}(\mathbb{R}^3)$ bounded.

Now, we show that if

$$k<(5-1/m)(1/p-1/2)\}$$

then the corresponding operator M_k is not $L^p(\mathbb{R}^3)\mapsto L^{p'}(\mathbb{R}^3)$ bounded. Indeed, we can write ϕ in the form

$$\phi(x_1,x_2)=b(0,0)(x_1-x_2^m\omega(0))^2+R(x_1,x_2),$$

Take $\kappa_1=1/2,\kappa_2=1/(2m)$. Then $\phi_\pi(x_1,x_2):=b(0,0)(x_1-x_2^m\omega(0))^2$ is a weighted homogeneous polynomial function of degree one with weights $\kappa:=(\kappa_1,\kappa_2)$ and R is a real analytic functions consisting of sum of terms with degree strictly bigger that one with the weight κ.

We show that if $k<(6-2(\kappa_1+\kappa_2))(1/p-1/2)=(5-1/m)(1/p-1/2)$ then the operator M_k is not $L^p(\mathbb{R}^3)\mapsto L^{p'}(\mathbb{R}^3)$ bounded.

Actually, we show that the modified sequence of functions suggested by M. Sugimoto in the paper [10] can be used to prove sharpness of the upper for $k_p(v)$. Let us take a smooth function in $\mathbb{R}^3$ such that $a_k(\xi) = |\xi|^{-k}$ for large ξ. Define non-negative functions $f(0) = g(0) = 1$ concentrated in a sufficiently small neighborhood of the origin, and a smooth function with $\psi(1) = 1$ and with support in a sufficiently small neighborhood of the point 1.

We set

$$u_j(x) = 2^{j(3-|\kappa|)\left(-\frac{1}{p'}\right)} F^{-1}(v_j(2^{-j}\xi))(x),$$

where

$$v_j(\xi) = \frac{f\left(2^{\kappa_1 j}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\kappa_2 j}\frac{\xi_2}{\varphi(\xi)}\right)\psi(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2 G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)} \in C_0^\infty(\mathbb{R}^3).$$

The sequence $\{F^{-1}(v_j(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3))\}_{j=1}^\infty$ is bounded in $L^p(\mathbb{R}^3)$. Indeed, the classical Hausdorff-Young inequality yields:

$$\| F^{-1}(v_j(2^{-j\kappa_1}\cdot, 2^{-j\kappa_2}\cdot,\cdot)) \|_{L^p} \lesssim \| v_j(2^{-j\kappa_1}\cdot, 2^{-j\kappa_2}\cdot,\cdot) \|_{L^{p'}}.$$

On the other hand

$$\| v_j(2^{-j\kappa_1}\cdot, 2^{-j\kappa_2}\cdot,\cdot) \|_{L^{p'}}^{p'} = \int f^{p'}\left(\frac{\xi_1}{\varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3)}\right) g^{p'}\left(\frac{\xi_2}{\varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3)}\right)$$
$$\frac{\psi^{p'}(\varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3))((2^{-j\kappa_1}\xi_1)^2 + (2^{-j\kappa_2}\xi_2)^2 + \xi_3^2)^{k/2}}{\varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3)^{2p'} G^{p'}(\xi_1/\varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3), \xi_2/\varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3))} d\xi.$$

Since ψ is supported in a sufficiently small neighborhood of one, then we have: $\frac{1}{2} \le \varphi(2^{-j\kappa_1}\xi_1, 2^{-j\kappa_2}\xi_2, \xi_3) \le 2$. On the other hand supports of the functions f and g are concentrated in a sufficiently small neighborhood of the origin. Hence, $|\xi_1| < 1$ and $|\xi_2| < 1$ and also $|\xi_3| \sim 1$, because $\varphi(0,0,1) = 1$. This yields:

$$\| v_j(2^{-j\kappa_1}\cdot, 2^{-j\kappa_2}\cdot,\cdot) \|_{L^{p'}} \lesssim 1.$$

Consequently,

$$\| F^{-1}(v_j(2^{-j}\cdot)) \|_{L^p} \lesssim 2^{j(3-|\kappa|)\frac{1}{p'}}.$$

Hence the sequence $\{u_j\}_{j=1}^\infty$ is bounded in the space $L^p(\mathbb{R}^3)$.

On the other hand we have the relation:

$$M_k u_j(x) = 2^{j(3-|\kappa|)(-\frac{1}{p'})-kj+2j} F^{-1}\left(e^{i\varphi(\xi)} \frac{f\left(2^{j\kappa_1}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{j\kappa_2}\frac{\xi_2}{\varphi(\xi)}\right)\psi(2^{-j}\varphi(\xi))}{\varphi(\xi)^2 G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)}\right).$$

We perform the change of variables given by the scaling $2^{-j}\xi \mapsto \xi$ and obtain:

$$M_k u_j(x) = \frac{2^{j((3-|\kappa|)(-\frac{1}{p'})-k+3)}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{2^j i(\varphi(\xi)-x\xi)} \frac{f\left(2^{j\kappa_1}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{j\kappa_2}\frac{\xi_2}{\varphi(\xi)}\right)\psi(\varphi(\xi))}{\varphi^2(\xi) G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)} d\xi.$$

Then following M. Sugimoto we use change of variables $\xi = (\lambda y, \lambda(1 + \phi(y)))$ and get:

$$M_k u_j(x) = \frac{2^{j((3-|\kappa|)(-\frac{1}{p'})-k+3)}}{\sqrt{(2\pi)^3}} \int e^{i2^j\lambda(1-(x_1y_1+x_2y_2+x_3(1+\phi(y))))} f(2^{j\kappa_1}y_1) g(2^{j\kappa_2}y_2)\psi(\lambda) d\lambda dy.$$

Finally, we use change of variables $2^{j\kappa_1}y_1 \mapsto y_1, 2^{j\kappa_2}y_2 \mapsto y_2$ and obtain:

$$M_k u_j(x) = 2^{j((3-|\kappa|)(-\frac{1}{p'})-k-|\kappa|+3)} \int_{\mathbb{R}^3} e^{2^j i\lambda((x_3-1)-2^{-j\kappa_1}y_1x_1-2^{-j\kappa_2}y_2x_2-x_3\phi(2^{-j\kappa_1}y_1, 2^{-j\kappa_2}y_2))}$$
$$\times f(y_1) g(y_2)\psi(\lambda) d\lambda dy.$$

If $|x_3 - 1| \ll 2^{-j}, |x_1| \ll 2^{-j(1-\kappa_1)}, |x_2| \ll 2^{-j(1-\kappa_2)}$, then the phase is the non-oscillating function, because $\phi(2^{-j\kappa_1}y_1, 2^{-j\kappa_2}y_2) = o(2^{-j})$ provided the supports of f, g are small enough.

Consequently, we have the following lower bound:

$$\| M_k u_j \|_{L^{p'}} \gtrsim 2^{j(6-2(\kappa_1+\kappa_2)|)(\frac{1}{p}-\frac{1}{2})-k)}.$$

Therefore, if $k < 2\big(3-(\kappa_1+\kappa_2)\big)\left(\frac{1}{p}-\frac{1}{2}\right) = \left(5-\frac{2}{m}\right)\left(\frac{1}{p}-\frac{1}{2}\right)$, then $\| M_k u_j \|_{L^{p'}} \to \infty (as\ j \to +\infty)$. Thus, the operator $M_k: L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ is unbounded. Thus, we conclude that if $k < k_p(v)$ then the operator M_k is not bounded. This finishes a proof of the main Theorem.

Remark. Actually, the result of the main Theorem holds true for the case $m = \infty$. Thus, in terms of the Sugimoto for all surfaces of classes II and III, except the case $m = 2$ we have the sharp value of the number $k_p(v)$.

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