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Determination of a Coefficient and Kernel in a Two-dimensional Fractional Integrodifferential Equation

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Abstract—This paper is devoted to obtaining a unique solution to an inverse problem for a two-dimensional time-fractional integrodifferential equation. In the case under additional data, we consider an inverse problem. The unknown coefficient and kernel are determined uniquely by the additional data. The existence and uniqueness result is based on the Fourier method, fractional calculus, properties of Mittag-Lefller function, and Banach fixed point theorem.

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1. INTRODUCTION

For studying objects or processes in the surrounding world, methods of mathematical modeling are extensively used. An efficient way to study processes by mathematical methods is modeling these processes in the form of differential equations.

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and engineering (see, e.g., other studies [1–8]). Other studies [9–14] demonstrate a number of interesting features of the fractional diffusion-wave equations, which represent a peculiar union of properties typical for second-order parabolic and wave differential equations.

According to the fractional order α, the diffusion process can be specified as sub-diffusion ($\alpha \in (0,1)$) and super-diffusion ($\alpha \in (1,2)$), respectively. There are abundant of literatures on the studies of fractional equations on various aspects, such as physical backgrounds, weak solution and maximum principle and numerical methods [15].

Practical needs often lead to problems of determining the coefficients, kernel or the right-hand side of a differential equation from a certain known information about its solution. Such problems have received the name inverse problems of mathematical physics. Inverse problems arise in various domains of human activity, such as seismology, prospecting for mineral deposits, biology, medical visualization, computer-aided tomography, remote sounding of Earth, spectral analysis, nondestructive control, etc. (see [1–4]).

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Let $Q_T := \bar{S} \times [0,T]$ for a given time $T > 0$, where S defined by the inequalities $0 < x < 1$, $0 < y < 1$. We consider a fractional integrodifferential equation with fractional derivative in time t

$$\partial_t^\alpha u = u_{xx} + u_{yy} + q(t)u_t + \int\limits_0^t k(t-\tau)u(x,y,\tau)d\tau + f(x,y,t), \quad (x,y,t) \in Q_T, \tag{1}$$

with the Gerasimov–Caputo time fractional derivative ∂_t^α of order $1 < \alpha < 2$, defined by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(2-\alpha)} \int\limits_0^t (t-\tau)^{1-\alpha} v''(\tau)d\tau,$$

where Γ is the Euler's Gamma function.

We supplement the above fractional wave equation with the following initial conditions

$$u(x,y,0) = a(x,y), \quad u_t(x,y,0) = b(x,y), \tag{2}$$

and the boundary conditions

$$u_x(0,y,t) = u(1,y,t) = 0, \quad 0 \le t \le T, \tag{3}$$

$$u(x,0,t) = u_y(x,1,t) = 0, \quad 0 \le t \le T, \tag{4}$$

In this paper, we take the following additional overdetermination and integral conditions

$$u(0,1,t) = h_0(t), \quad 0 \le t \le T, \tag{5}$$

$$\Lambda[u(\cdot,\cdot,t)] = h_1(t), \tag{6}$$

where Λ are defined by

$$\Lambda[h](t) := \iint\limits_S h(x,y,t)dxdy,$$

where $h_i(t)$ are given functions [21].

The inverse problem considered in this paper is stated as follows.

Definition. By the *classical solution of the inverse boundary value problem* (1)–(6), we mean the triple functions $u(x,y,t)$, $q(t)$, $k(t)$, if

$$u(x,y,t) \in Y_T, \tag{7}$$

$q(t) \in C[0,T]$, $k(t) \in C[0,T]$ and relations (1)–(6) hold in the usual sense, where

$$Y_T := \{u : \ u(\cdot,\cdot,t) \in C^2(\overline{S}),\ t \in [0,T] \quad \text{and} \quad u(x,y,\cdot) \in C_\gamma^{\alpha,1}[0,T], \quad (x,y) \in \overline{S}\},$$

$$C_\gamma^{\alpha,n}[0,T] := \Big\{v(t) : \ v(t) \in C^{(n)}[0,T], \quad \text{and} \quad \partial_t^\alpha v(t) \in C_\gamma[0,T]\Big\}, \quad C_\gamma^{\alpha,0}[0,T] = C_\gamma^\alpha[0,T],$$

where $\alpha > 0$, $n \in \mathbb{N}$, $0 \le \gamma < 1$ be such that $\gamma \le \alpha$ (see [5], p. 199) and here

$$C_\gamma[0,T] := \{f(t) : \ t^\gamma f(t) \in C[0,T]\}, \quad ||f||_{C_\gamma} := ||t^\gamma f(t)||_{C[0,T]}.$$

We make the following assumptions:

(C1) $\partial_t^\alpha h_i(t) \in C[0,T]$, $i = 1,2$;

(C2) $h_0(0) = a(0,1)$, $h_0'(0) = b(0,1)$;

(C3) $\iint_S a(x,y)dxdy = h_1(0)$, $\iint_S b(x,y)dxdy = h_1'(0)$;

(C4) $h(t) \equiv h_1(0)h_0'(t) - h_0(0)h_1'(t) \ne 0$.

The outline of the paper is as follows. In Section 2, an equivalent problem is presented. In Section 3, the existence and uniqueness of the solution of the direct problem (1)–(4) is established by using the Fourier method and the Gronwall–Bellman type integral inequality. In Section 4, an inverse problem is investigated.

2. EQUIVALENCE

Lemma 1. *Let* $l(t)=\int_0^t k(\tau)d\tau$. *Suppose that* $a(x,y),b(x,y)\in C(\overline{S})$, $f(x,y,t)\in C(\overline{S})\cap C[0,T]$, $q(t)\in C[0,T]$, $k(t)\in C[0,T]$, $h_i(t)\in C^2[0,T]$ $(i=0,1)$ *and the validity conditions*

$$a(0,1)=h_0(0),\quad b(0,1)=h_0'(0), \tag{8}$$

$$\iint\limits_S a(x,y)dxdy=h_1(0),\quad \iint\limits_S b(x,y)dxdy=h_1'(0) \tag{9}$$

hold. Then, the problem of finding a classical solution of (1)−(6) is equivalent to the problem of determining the functions $u(\cdot,\cdot,\cdot)\in Y_T$, $q(t)\in C^1[0,T]$ *and* $k(t)\in C[0,T]$ *satisfying (1)−(4), and the following conditions*

$$(\partial_t^\alpha h_0)(t)=(\triangle u)(0,1,t)+h_0'(t)q(t)+h_0(0)l(t)+\int\limits_0^t l(t-\tau)h_0'(\tau)d\tau+f_0(t),\quad 0\le t\le T; \tag{10}$$

$$(\partial_t^\alpha h_1)(t)=\Lambda[\triangle u](t)+h_1'(t)q(t)+h_1(0)l(t)+\int\limits_0^t l(t-\tau)h_1'(\tau)d\tau+f_1(t),\quad 0\le t\le T, \tag{11}$$

where $\triangle:=\partial_x^2+\partial_y^2$, $f_0(t):=f(x,y,t)\Big|_{(0,1)}$, $f_1(t):=\Lambda[f]$. *On the other hand, if (1)−(4), (10), and (11) has a solution and the compatibility conditions (8), (9) and the technical condition (C4) hold, then there exists a solution to the inverse problem (1)−(6).*

Remark 1. From Lemma 1, we know that (1)–(4), (10), and (11) is an equivalent form of the original inverse problem (1)–(6). So, in the following next sections, we turn to discuss (1)–(4), (10), and (11), other than the original one.

Proof. Let the three $\{u(x,y,t),q(t),k(t)\}$ functions be a classical solution of problem (1)–(6). Taking into account the conditions $h_i(t)\in C^2[0,T]$ $(i=0,1)$, and fractional differentiating twice both sides of (5) and (6) with respect to t gives

$$(\partial_t^\alpha u)(0,1,t)=(\partial_t^\alpha h_0)(t),\quad \Lambda[(\partial_t^\alpha u)](t)=(\partial_t^\alpha h_1)(t),\quad 0\le t\le T. \tag{12}$$

Setting $x=0$ and $y=1$ in Eq. (1), the procedure yields

$$(\partial_t^\alpha u)(0,1,t)=u_{xx}(0,1,t)+u_{yy}(0,1,t)$$
$$+q(t)u_t(0,1,t)+\int\limits_0^t k(t-\tau)u(0,1,\tau)d\tau+f(0,1,t),\quad 0\le t\le T. \tag{13}$$

From (13), taking into account (5) and the first equality of (12), we conclude that condition (10) is satisfied.

Moreover, integrating Eq. (1) with respect to x and y over the interval $[0,1]$ gives

$$(\partial_t^\alpha \Lambda[u])(t)=\Lambda[\triangle u](t)+q(t)\Lambda[u_t](t)+k(t)\Lambda[u](t)+\Lambda[f](t). \tag{14}$$

We note that $l(t)=\int_0^t k(\tau)d\tau$. Then, by integration by parts, we get the following equality

$$\int\limits_0^t k(\tau)h_1(t-\tau)d\tau=h_1(0)l(t)+\int\limits_0^t l(t-\tau)h_1'(\tau)d\tau. \tag{15}$$

From the equality (14), with the help of (5), (6), and (15), we obtain (11).

Now we assume that (u,q,k) satisfies (1)–(4), (10), and (11). In order to prove that (u,q,k) is the solution to the inverse problem (1)–(6), it suffices to show that (u,q,k) satisfies (5) and (6).

According to the definition $l(t)$, then from (10) and (13) we get

$$\partial_t^\alpha\left(u(0,1,t)-h_0(t)\right)=q(t)\frac{d}{dt}\left(u(0,1,t)-h_0(t)\right)+l(t)\left(u(0,1,0)-h_0(0)\right)$$
$$+\int_0^t l(t-\tau)\frac{d}{d\tau}\left(u(0,1,\tau)-h_0(\tau)\right)d\tau,\quad 0\le t\le T. \tag{16}$$

Using (2) and the matching conditions (8), we obtain the following relation

$$u(0,1,0)-h_0(0)=a(0,1)-h_0(0)=0,\quad u_t(0,1,0)-h_0'(0)=b(0,1)-h_0'(0)=0. \tag{17}$$

Since problem (16) and (17) has only a trivial solution, so from $u(0,1,t)-h_0(t)=0$, $0\le t\le T$, we get that the condition (5) is satisfied.

Now, from (11) and (14) we find

$$\partial_t^\alpha(\Lambda[u]-h_1)(t)=q(t)\frac{d}{dt}(\Lambda[u]-h_1)(t)+k(t)(\Lambda[u]-h_1)(t),\quad 0\le t\le T. \tag{18}$$

By using the initial conditions (2) and the matching conditions (9), we may write

$$\iint_S u(x,y,0)dxdy-h_1(0)=\iint_S a(x,y)dxdy-h_1(0)=0,$$

$$\iint_S u(x,y,0)dxdy-h_1'(0)=\iint_S b(x,y)dxdy-h_1'(0)=0. \tag{19}$$

Hence, relations (18) and (19) give us conclude that $\Lambda[u(\cdot,\cdot,t)]=h_1(t)$, $0\le t\le T$, i.e., the condition is satisfied. The proof is complete. □

3. CONSTRUCTION AND INVESTIGATION OF THE SOLUTION OF DIRECT PROBLEM

Let $L^2(S)$ be a usual L^2-space. Since $-\Delta:=\partial_x^2+\partial_y^2$ is a symmetric uniformly elliptic operator, the spectrum of $-\Delta$ is entirely composed of eigenvalues and counting to the multiplicities, we can set $0<\lambda_{11}\le\lambda_{12}\le\ldots$. Now, we will solve the following boundary-value problem

$$\begin{cases} v_{xx}+v_{yy}+\lambda^2 v=0;\\ v_x(0,y)=0,\quad v(1,y)=0,\quad y\in[0,1];\\ v(x,0)=0,\quad v_y(x,1)=0,\quad x\in[0,1].\end{cases} \tag{20}$$

Thus, the eigenvalue problem reduces to the solution of the homogeneous equation with homogeneous boundary conditions. We shall solve this problem by the method of separation of variables, assuming $v(x,y)=X(x)Y(y)$. Carrying out a separation of the variables, we obtain the following one-dimensional eigne-value problems

$$\begin{cases} X''+\mu^2X=0;\\ X'(0)=0,\quad X(1)=0;\end{cases} \tag{21}$$

$$\begin{cases} Y''+\nu^2Y=0;\\ Y(0)=0,\quad Y'(1)=0;\end{cases} \tag{22}$$

where μ and ν are constants of the separation of variables connected by the relationship $\mu^2+\nu^2=\lambda^2$. The boundary conditions for $X(x)$ and $Y(y)$ follows from the corresponding conditions for function v. For example, from $v(0,y)=X'(0)Y(y)=0$ it follows $X(0)=0$, since $Y(y)\ne 0$ (we have only non-trivial solutions).

The solutions of equations (21) and (22) have the form

$$X_m(x)=\cos(\mu_m x),\quad Y_m(y)=\sin(\nu_n y);$$

$$\mu_m = \frac{\pi}{2}\left(2m-1\right), \quad \nu_n = \frac{\pi}{2}\left(2n-1\right), \quad m,n \in \mathbb{N}.$$

The eigenvalues $\lambda_{mn}^2 = \mu_m^2 + \nu_n^2$ have corresponding eigenfunctions $v_{mn}(x,y) = A_{mn}\cos(\mu_m x) \times \sin(\nu_n y)$, where A_{mn} is some constant factor. We choose this so that the norm in $L^2(S)$ of function v_{mn} with weight 1 equals unity

$$\iint\limits_S v_{mn}^2 dxdy = A_{mn}^2 \int\limits_0^1 \cos^2(\mu_m x)dx \int\limits_0^1 \sin^2(\nu_n y)dy = 1.$$

Hence, $A_{mn} = 2$. So,

$$v_{mn}(x,y) = 2\cos(\mu_m x)\sin(\nu_n y) \tag{23}$$

is the orthonormal eigenfunction corresponding to $-\lambda_{mn}$. Then, the sequence $\{v_{mn}\}_{m,n\in\mathbb{N}}$ is orthonormal basis in $L^2(S)$.

3.1. Uniqueness of the Solution

The following theorem is valid.

Theorem 1. *If problem (1)–(4) has a solution such that*

$$\lim_{x\to 1-0} u_x \cos(\mu_m x) = 0, \quad 0 \le y \le 1, \quad 0 \le t \le T, \tag{24}$$

$$\lim_{y\to +0} u_y \sin(\nu_n y) = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T, \tag{25}$$

then this solution is unique.

Proof. Applying the method of separation of variables, we seek a solution of (1)–(4) with the form

$$u(x,y,t) = u(t)v(x,y). \tag{26}$$

Substituting (26) into (1) with $F(x,y,t;u,q,l,f) \equiv 0$, where

$$F(x,y,t;u,q,l,f) := q(t)u_t(x,y,t) + a(x,y)l(t) + \int\limits_0^t l(t-\tau)u_\tau(x,y,\tau)d\tau + f(x,y,t), \tag{27}$$

we require that $v(x,y)$ satisfies the spectral problem (20) and its non-trivial solution gives by (23).

Let a function $u(x,y,t)$ stand for a solution to problem (1)–(4) which satisfies conditions (24) and (25). We consider the integral

$$u_{mn}(t) = \iint\limits_S u(x,y,t)v_{mn}(x,y)dxdy. \tag{28}$$

Introduce an auxiliary integral, namely,

$$u_{mn}^{\varepsilon,\delta}(t) = \iint\limits_{S_{\varepsilon\delta}} u(x,y,t)v_{mn}(x,y)dxdy,$$

here ε, δ are given sufficiently small positive values,

$$S_{\varepsilon\delta} = \left\{(x,y) |\, \varepsilon \le x \le 1-\varepsilon,\, \delta \le y \le 1-\delta\right\}.$$

Fractional differentiating equality (28) and making use of Eq. (1), we get the correlation

$$\left(\partial_t^\alpha u_{mn}^{\varepsilon,\delta}\right)(t) = \iint\limits_{S_{\varepsilon\delta}} \partial_t^\alpha u(x,y,t)v_{mn}(x,y)dxdy = \iint\limits_{S_{\varepsilon\delta}} \left(a^2\triangle u + F(x,y,t;u,q,l,f)\right) v_{mn}(x,y)dxdy$$

$$= a^2 \left(\iint\limits_{S_{\varepsilon\delta}} u_{xx} v_{mn}(x,y)dxdy + \iint\limits_{S_{\varepsilon\delta}} u_{yy} v_{mn}(x,y)dxdy \right)$$

$$+ \iint\limits_{S_{\varepsilon\delta}} F(x,y,t;u,q,l,f) v_{mn}(x,y)dxdy = a^2 (I_1 + I_2) + F^{\varepsilon,\delta}_{mn}(t;u,q,l,f), \tag{29}$$

where

$$F^{\varepsilon,\delta}_{mn}(t;u,q,l,f) = \iint\limits_{S_{\varepsilon\delta}} F(x,y,t;u,q,l,f) v_{mn}(x,y)dxdy.$$

Calculating integrals I_1 and I_2 by parts, we conclude that

$$I_1 = \int\limits_{\delta}^{1-\delta} dy \int\limits_{\varepsilon}^{1-\varepsilon} u_{xx} v_{mn}(x,y)dx = \int\limits_{\delta}^{1-\delta} dy \left(u_x v_{mn}(x,y)|_{\varepsilon}^{1-\varepsilon} - \int\limits_{\varepsilon}^{1-\varepsilon} u_x v_{mnx}(x,y)dx \right)$$

$$= \int\limits_{\delta}^{1-\delta} dy \left(u_x v_{mn}(x,y)|_{\varepsilon}^{1-\varepsilon} - u v_{mnx}(x,y)|_{\varepsilon}^{1-\varepsilon} + \int\limits_{\varepsilon}^{1-\varepsilon} u v_{mnxx}(x,y)dx \right),$$

$$I_2 = \int\limits_{\delta}^{1-\varepsilon} dx \int\limits_{\delta}^{1-\delta} u_{yy} v_{mn}(x,y)dy = \int\limits_{\varepsilon}^{1-\varepsilon} dx \left(u_y v_{mn}(x,y)|_{\delta}^{1-\delta} - \int\limits_{\delta}^{1-\delta} u_y v_{mny}(x,y)dy \right)$$

$$= \int\limits_{\varepsilon}^{1-\varepsilon} dx \left(u_y v_{mn}(x,y)|_{\delta}^{1-\delta} - u v_{mny}(x,y)|_{\delta}^{1-\delta} + \int\limits_{\delta}^{1-\delta} u v_{mnyy}(x,y)dy \right).$$

Proceeding to the limit in integrals I_1 and I_2 as $\varepsilon \to 0$ and $\delta \to 0$, taking into account conditions (3), (4), (24), and (25), from formula (29) arrives

$$(\partial_t^\alpha u_{mn})(t) = -\lambda_{mn}^2 \iint\limits_{S_{\varepsilon\delta}} u(x,y,t) v_{mn}(x,y)dxdy + F_{mn}(t;u,q,l,f,p)$$

$$= -\lambda_{mn}^2 u_{mn}(t) + F_{mn}(t;u,q,l,f)$$

or

$$(\partial_t^\alpha u_{mn})(t) + \lambda_{mn}^2 u_{mn}(t) = F_{mn}(t;u,q,l,f), \tag{30}$$

where

$$F_{mn}(t;u,q,l,f) = \iint\limits_{S} F(x,y,t;u,q,l,f) v_{mn}(x,y)dxdy.$$

Taking into account (27), the last equality has a form

$$F_{mn}(t;u,q,l,f,p) = q(t)u'_{mn}(t) + a_{mn} l(t) + f_{mn}(t) + \int\limits_0^t l(t-\tau) u'_{mn}(\tau)d\tau,$$

where

$$f_{mn}(t) = \iint\limits_{S} f(x,y,t) v_{mn}(x,y)dxdy, \quad 0 \le t \le T. \tag{31}$$

The initial conditions (2) give

$$u_{mn}(0)=\iint\limits_S u(x,y,0)v_{mn}(x,y)dxdy=\iint\limits_S a(x,y)v_{mn}(x,y)dxdy=a_{mn}. \tag{32}$$

$$u'_{mn}(0)=\iint\limits_S u_t(x,y,0)v_{mn}(x,y)dxdy=\iint\limits_S b(x,y)v_{mn}(x,y)dxdy=b_{mn}. \tag{33}$$

According to the Lemma 3, the initial-value problem (30), (32), and (33) is equivalent in the space $C[0,T]$ to the Volterra integral equation of the second kind

$$u_{mn}(t)=a_{mn}E_\alpha(-\lambda_{mn}^2t^\alpha)+b_{mn}tE_{\alpha,2}(-\lambda_{mn}^2t^\alpha)$$
$$+\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)F_{mn}(\tau;u,q,l,f)d\tau \tag{34}$$

or

$$u_{mn}(t)=a_{mn}\Big[E_\alpha(-\lambda_{mn}^2t^\alpha)-q(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{mn}^2t^\alpha)\Big]+b_{mn}tE_{\alpha,2}(-\lambda_{mn}^2t^\alpha)$$
$$+\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)f_{mn}(\tau)d\tau$$
$$-\int\limits_0^t(t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)q(\tau)u_{mn}(\tau)d\tau$$
$$-\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)q'(\tau)u_{mn}(\tau)d\tau$$
$$+\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)\int\limits_0^\tau k(\tau-s)u_{mn}(s)dsd\tau. \tag{35}$$

This means that the solution to problem (1)–(4) is unique, because with $a(x,y)\equiv 0$, $b(x,y)\equiv 0$, and $f(x,y,t)\equiv 0$ we get identities $a_{mn}\equiv 0$, $b_{mn}\equiv 0$ and $f_{mn}(t)\equiv 0$, and then formula (34) implies that $u_{mn}(t)\equiv 0$ (this fact follows from Lemma 2). □

Lemma 2 (see [22], p. 188). *Suppose $b\geq 0$, $\beta>0$ and $a(t)$ is a nonnegative function locally integrable on $0\leq t<T$ (some $T\leq+\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0\leq t<T$ with*

$$u(t)\leq a(t)+b\int\limits_0^t(t-s)^{\beta-1}u(s)ds$$

on this interval. Then,

$$u(t)\leq a(t)+\theta\int\limits_0^t E'_\beta(\theta(t-s))a(s)ds,\quad 0\leq t<T,$$

where

$$\theta=(b\Gamma(\beta))^{1/\beta},\quad E_\beta(z)=\sum_{n=0}^\infty\frac{z^{n\beta}}{\Gamma(n\beta+1)},\quad E'_\beta(z)=\frac{d}{dz}E_\beta(z).$$

If $a(t)\equiv a$, constant, then $u(t)\leq aE_\beta(\theta t)$.

In view of formula (28) the latter equality is equivalent to that

$$\iint\limits_S u(x,y,t)v_{mn}(x,y)dxdy = 0.$$

Since the system $v_{mn}(x,y)$ is complete in the space $L^2(S)$, the function $u(x,y,t) = 0$ almost everywhere in S and with any $t \in [0,T]$. Since in view of condition (7) the function $u(x,y,t)$ is continuous on $\overline{Q}_T$, we conclude that $u(x,y,t) \equiv 0$ on $\overline{Q}_T$. Thus, we have proved the uniqueness of the solution to problem (1)–(4).

Remark 2. Note that the second condition of (25) means that first derivative u_y near the corresponding face of the parallelepiped $\overline{Q}_T$ can have singularities of order less than one.

Example 1. Here we can bring many functions of the fulfillment of the second condition of Theorem 1. For example, if we take $u(x,y,t) = y^{\alpha-1}f(x,t)$, where $f(x,t) \in C([0,1]\times[0,T])$, then the condition (25) is valid.

3.2. The Existence of a Solution

The following assertion is true.

Lemma 3. *Let $0 \le \gamma < 1$. If $q(t) \in C^1[0,T]$, $k(t) \in C[0,T]$, and $f_{mn}(t) \in C_\gamma[0,T]$, then there exists a unique solution $u_{mn}(t)$ to the Cauchy problem 30), (32), and (33) in the space $C_\gamma^{\alpha,1}[0,T]$.*

The proof is similar to that of Theorem 3.25 in [20].

Moreover, the following estimates are valid with large m and n:

$$|u_{mn}(t)| \le c_1 \left(|a_{mn}| + |b_{mn}| + ||f_{mn}||_{C_\gamma}\right), \quad t \in [0,T], \tag{36}$$

$$\left|u'_{mn}(t)\right| \le c_2 \left(\lambda_{mn}^2|a_{mn}| + |b_{mn}| + ||f_{mn}||_{C_\gamma}\right), \quad t \in [0,T], \tag{37}$$

$$|\partial_{0t}^\alpha u_{mn}(t)| \le \lambda_{mn}^2 c_3 \left(|a_{mn}| + |b_{mn}| + ||f_{mn}||_{C_\gamma}\right), \quad t \in [0,T], \tag{38}$$

hereinafter c_i are positive constant values independent of $a(x,y)$, $b(x,y)$ and $f(x,y,t)$ and dependent of α, T, besides continuous norm of $q(t)$, $k(t)$.

The validity of estimates (36)–(38) immediately follows from formula (30), (34) and Lemma 2.

Under certain requirements to functions $q(t)$, $k(t)$, $f(x,y,t)$, $a(x,y)$, and $b(x,y)$ we can prove that the function

$$u(x,y,t) = \sum_{m,n=1}^{\infty} u_{mn}(t)v_{mn}(x,y), \tag{39}$$

where $v_{mn}(x,y)$ obeys formula (23), while $u_{mn}(t)$ does (35) is a solution to problem (1)–(4).

Lemma 4. *If $a(x,y) \in C^{(1,1)}(\overline{S})$, $b(x,y) \in C^{(1,1)}(\overline{S})$, $f(x,y,t) \in C^{(1,1)}(\bar{S}) \cap C_\gamma^{\alpha,1}[0,T]$, and*

(i_1) $a(1,y) = 0$, $a'_x(x,0) = 0$, $(x,y) \in \overline{S}$;

(i_2) $b(1,y) = 0$, $b'_x(x,0) = 0$, $(x,y) \in \overline{S}$;

(i_3) $f(1,y,t) = 0$, $f'_x(x,0,t) = 0$, $(x,y,t) \in \overline{Q}_T$,

then the following representations are valid

$$a_{mn} = -\frac{a_{mn}^{(1,1)}}{\mu_m\nu_n}, \quad b_{mn} = -\frac{b_{mn}^{(1,1)}}{\mu_m\nu_n}, \quad f_{mn}(t) = -\frac{f_{mn}^{(1,1)}(t)}{\mu_m\nu_n}; \tag{40}$$

here $a_{mn}^{(1,1)}$, $b_{mn}^{(1,1)}$, and $f_{mn}^{(1,1)}(t)$ are coefficients of the expansion of functions $a_{xy}(x,y)$, $b_{xy}(x,y)$, and $f_{xy}(x,y,t)$ in series with respect to the function system $v_{mn}^{(1)}(x,y) = 2\sin(\mu_m x)\cos(\nu_n y)$, $m,n = 1,2,\ldots$ such that

$$\sum_{m,n=1}^{\infty} |a_{mn}^{(1,1)}|^2 \le \iint\limits_S \left(a''_{xy}(x,y)\right)^2 dxdy, \quad \sum_{m,n=1}^{\infty} |b_{mn}^{(1,1)}|^2 \le \iint\limits_S \left(b''_{xy}(x,y)\right)^2 dxdy, \tag{41}$$

$$\sum_{m,n=1}^{\infty} |f_{mn}^{(1,1)}(t)|^2 \le \iint\limits_S \left(f''_{xy}(x,y,t)\right)^2 dxdy, \quad 0 \le t \le T.$$

Proof. Calculating integrals (31)–(33) by parts (one time x and one in y), taking into account conditions (i_1)–(i_3), we get formula (40). Estimates (41) represent the Bessel inequalities with respect to the orthonormal system $v_{mn}^{(1)}(x,y)$. The Lemma 3 is proved. □

In view of Lemmas 3 and 4 series (39) with any $(x,y,t) \in \overline{Q}_T$ is majorized by the convergent series

$$c_4 \sum_{m,n=1}^{\infty} \frac{1}{mn}\left(|a_{mn}^{(1,1)}| + |b_{mn}^{(1,1)}| + ||f_{mn}^{(1,1)}(t)||_{C_\gamma}\right). \tag{42}$$

Consequently, the function $u(x,y,t)$ is continuous on $\overline{Q}_T$.

Formally termwise differentiating the series in formula (39), we get the following series

$$u_{xx}(x,y,t) = \sum_{m,n=1}^{\infty} (-\mu_m^2) u_{mn}(t) v_{mn}(x,y), \quad u_{yy}(x,y,t) = \sum_{m,n=1}^{\infty} (-\nu_n^2) u_{mn}(t) v_{mn}(x,y), \tag{43}$$

$$u_t(x,y,t) = \sum_{m,n=1}^{\infty} u'_{mn}(t) v_{mn}(x,y), \tag{44}$$

$$\left(\partial_{0t}^{\alpha} u\right)(x,y,t) = \sum_{m,n=1}^{\infty} \left(\partial_{0t}^{\alpha} u_{mn}\right)(t) v_{mn}(x,y), \tag{45}$$

Lemma 5. *If* $a(x,y) \in C^{(3,1)}(\overline{S})$, $b(x,y) \in C^{(3,1)}(\overline{S})$, $f(x,y,t) \in C^{(3,1)}(\bar{S}) \cap C_\gamma^{\alpha,1}[0,T]$, *and*

(i_4) $a(1,y) = 0$, $a'_x(0,y) = 0$, $a''_{xx}(1,y) = 0$, $a'''_{xxx}(x,0) = 0$, $(x,y) \in \overline{S}$;

(i_5) $b(1,y) = 0$, $b'_x(0,y) = 0$, $b''_{xx}(1,y) = 0$, $b'''_{xxx}(x,0) = 0$, $(x,y) \in \overline{S}$;

(i_6) $f(1,y,t) = 0$, $f'_x(0,y,t) = 0$, $f''_{xx}(1,y,t) = 0$, $f'''_{xxx}(x,0,t) = 0$, $(x,y,t) \in \overline{Q}_T$,

then the following representations are valid

$$a_{mn} = \frac{a_{mn}^{(3,1)}}{\mu_m^3 \nu_n}, \quad b_{mn} = \frac{b_{mn}^{(3,1)}}{\mu_m^3 \nu_n}, \quad f_{mn}(t) = \frac{f_{mn}^{(3,1)}(t)}{\mu_m^3 \nu_n}, \quad 0 \le t \le T, \tag{46}$$

here $a_{mn}^{(3,1)}$, $b_{mn}^{(3,1)}$, *and* $f_{mn}^{(3,1)}(t)$ *are coefficients of the expansion of functions* $a_{xxxy}(x,y)$, $b_{xxxy}(x,y)$, $f_{xxxy}(x,y,t)$ *in series with respect to the function system* $v_{mn}^{(1)}(x,y)$, $m,n = 1,2,\ldots$ *such that*

$$\sum_{m,n=1}^{\infty} |a_{mn}^{(3,1)}|^2 \le \iint\limits_S \left(a_{xxxy}^{(4)}(x,y)\right)^2 dxdy,$$

$$\sum_{m,n=1}^{\infty} |b_{mn}^{(3,1)}|^2 \le \iint\limits_S \left(b_{xxxy}^{(4)}(x,y)\right)^2 dxdy, \tag{47}$$

$$\sum_{m,n=1}^{\infty} |f_{mn}^{(3,1)}(t)|^2 \le \iint\limits_S \left(f_{xxxy}^{(4)}(x,y,t)\right)^2 dxdy, \quad 0 \le t \le T.$$

In view of Lemma 5, the first series (43) with any $(x,y,t) \in \overline{Q}_T$ is majorized by the convergent series

$$c_5 \sum_{m,n=1}^{\infty} \frac{1}{mn}\left(|a_{mn}^{(3,1)}| + |b_{mn}^{(3,1)}| + ||f_{mn}^{(3,1)}||_{C_\gamma}\right). \tag{48}$$

Lemma 6. *If* $a(x,y) \in C^{(1,3)}(\overline{S})$, $b(x,y) \in C^{(1,3)}(\overline{S})$, $f(x,y,t) \in C^{(1,3)}(\bar{S}) \cap C_{\gamma}^{\alpha,1}[0,T]$, *and*

(i_7) $a(x,0)=0,\ a'_y(x,1)=0,\ a''_{yy}(x,0)=0,\ a'''_{yyy}(1,y)=0, \quad (x,y)\in\overline{S}$;

(i_8) $b(x,0)=0,\ b'_y(x,1)=0,\ b''_{yy}(x,0)=0,\ b'''_{yyy}(1,y)=0, \quad (x,y)\in\overline{S}$;

(i_9) $f(x,0,t)=0,\ f'_y(x,1,t)=0,\ f''_{yy}(x,0,t)=0,\ f'''_{yyy}(1,y,t)=0, \quad (x,y,t)\in\overline{Q}_T$,

then the following representations are valid

$$a_{mn}=\frac{a_{mn}^{(1,3)}}{\mu_m\nu_n^3}, \quad b_{mn}=\frac{b_{mn}^{(1,3)}}{\mu_m\nu_n^3}, \quad f_{mn}(t)=\frac{f_{mn}^{(1,3)}(t)}{\mu_m\nu_n^3}, \quad 0\le t\le T, \tag{49}$$

here $a_{mn}^{(1,3)}$, $b_{mn}^{(1,3)}$, and $f_{mn}^{(1,3)}(t)$ are coefficients of the expansion of functions $a_{xyyy}(x,y)$, $b_{xyyy}(x,y)$, and $f_{xyyy}(x,y,t)$ in series with respect to the function system $v_{mn}^{(1)}(x,y),\ m,n=1,2,\ldots$ such that

$$\sum_{m,n=1}^{\infty}|a_{mn}^{(1,3)}|^2 \le \iint\limits_S \left(a_{xyyy}^{(4)}(x,y)\right)^2 dxdy, \quad \sum_{m,n=1}^{\infty}|b_{mn}^{(1,3)}|^2 \le \iint\limits_S \left(b_{xyyy}^{(4)}(x,y)\right)^2 dxdy,$$

$$\sum_{m,n=1}^{\infty}|f_{mn}^{(1,3)}(t)|^2 \le \iint\limits_S \left(f_{xyyy}^{(4)}(x,y,t)\right)^2 dxdy, \quad 0\le t\le T. \tag{50}$$

In view of Lemma 6, the second series (43) with any $(x,y,t)\in\overline{Q}_T$ is majorized by the convergent series

$$c_6\sum_{m,n=1}^{\infty}\frac{1}{mn}\left(|a_{mn}^{(1,3)}|+|b_{mn}^{(1,3)}|+||f_{mn}^{(1,3)}||_{C_\gamma}\right). \tag{51}$$

Now, we set

$$\mathcal{H}_1 := \{\varphi(x,y)\in C^{(4,4)}(\bar{S}):\ \varphi(x,0)=\varphi(1,y)=\varphi_x(0,y)=\varphi_y(x,1)=\varphi_x(x,0)$$
$$=\varphi''_{xx}(1,y)=\varphi''_{xx}(x,0)=\varphi''_{xy}(x,1)=\varphi''_{yy}(x,0)=\varphi'''_{xxx}(0,y)=\varphi'''_{xxx}(x,0)$$
$$=\varphi'''_{xyy}(x,0)=\varphi'''_{xxy}(x,1)=\varphi'''_{yyy}(x,1)=0,\ (x,y)\in\bar{S}\}.$$

Lemma 7. *Let* $a(x,y)\in\mathcal{H}_1$, *then the following representations are valid*

$$|a_{mn}|=\frac{1}{\lambda_{mn}^4}\sum_{i+j=4}\binom{4}{j}|a_{mn}^{(i,j)}|, \tag{52}$$

where

$$a_{mn}^{(4,0)}=-\iint\limits_{\bar{S}} a_x^{(4)}(x,y)v_{mn}(x,y)dxdy, \quad a_{mn}^{(3,1)}=-\iint\limits_{\bar{S}} a_{xxxy}^{(3,1)}(x,y)v_{mn}^{(1)}(x,y)dxdy,$$

$$a_{mn}^{(2,2)}=\iint\limits_{\bar{S}} a_{xxyy}^{(2,2)}(x,y)v_{mn}(x,y)dxdy, \quad a_{mn}^{(1,3)}=\iint\limits_{\bar{S}} a_{xyyy}^{(1,3)}(x,y)v_{mn}^{(1)}(x,y)dxdy,$$

$$a_{mn}^{(0,4)}=\iint\limits_{\bar{S}} a_y^{(4)}(x,y)v_{mn}(x,y)dxdy, \tag{53}$$

while

$$\sum_{m,n=1}|a_{m,n}^{(i,j)}|^2 \le \left|\left|\frac{\partial^4 a(x,y)}{\partial x^i\partial y^j}\right|\right|_{L_2(S)}^2, \quad i+j=4. \tag{54}$$

Proof. Suppose $a(x,y) \in \mathcal{H}_1$. Then, taking into account the condition (3.13) and four-times integration by parts in x, we get $a_{mn} = a_{mn}^{(4,0)}/\mu_m^4$. In a similar way, we can easily get the following representations

$$a_{mn} = \frac{a_{mn}^{(3,1)}}{\mu_m^3 \nu_n}, \quad a_{mn} = \frac{a_{mn}^{(2,2)}}{\mu_m^2 \nu_n^2}, \quad a_{mn} = \frac{a_{mn}^{(1,3)}}{\mu_m \nu_n^3}, \, a_{mn} = \frac{a_{mn}^{(0,4)}}{\nu_n^4},$$

where $a_{mn}^{(i,j)}$ are defined by (53). According to the binomial formula $(a+b)^4 = \sum\limits_{j=0}^{4} \binom{4}{j} a^{4-j} b^j$, we have

$$\lambda_{mn}^4 |a_{mn}| = |a_{mn}^{(4,0)}| + 4|a_{mn}^{(3,1)}| + 6|a_{mn}^{(2,2)}| + 4|a_{mn}^{(1,3)}| + |a_{mn}^{(0,4)}|$$

and this equality is (52).

Estimate (54) represent the Bessel inequalities with respect to the orthonormal systems $v_{mn}(x,y)$ and $v_{mn}^{(1)}(x,y)$. The proof is complete. □

Furthermore, the above Lemma 5 is valid for the functions $b(x,y) \in \mathcal{H}_1$ and $f(x,y,t) \in \mathcal{H}_1 \cap C_\gamma^{\alpha,1}[0,T]$. From this assertion and in view of Lemma 5 with any $(x,y,t) \in \overline{Q}_T$ the series (39) and (43)–(45) are estimated by the convergent series

$$c_7 \sum_{m,n=1}^{\infty} \frac{1}{mn} \left(|a_{mn}^{(i,j)}| + |b_{mn}^{(i,j)}| + ||f_{mn}^{(i,j)}||_{C_\gamma} \right). \tag{55}$$

The proved assertion and Lemma 3 implies the next theorem.

Theorem 2. *Let $q(t) \in C^1[0,T]$ and $k(t) \in C[0,T]$. If functions $a(x,y)$, $b(x,y)$, and $f(x,y,t)$ satisfy conditions Lemma 7, then problem (1)–(4) has a unique solution, which is defined by series (39) and belongs to class (7).*

4. INVESTIGATION OF THE INVERSE PROBLEM (1)–(4), (10), AND (11)

Substituting the expression of $u_{mn}(t)$, $m,n \in \mathbb{N}$, into (39), we find

$$u(x,y,t) = 2 \sum_{m,n=1}^{\infty} \Bigg\{ a_{mn} E_\alpha(-\lambda_{mn}^2 t^\alpha) + b_{mn} t E_{\alpha,2}(-\lambda_{mn}^2 t^\alpha)$$

$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{mn}^2 (t-\tau)^\alpha\right) F_{mn}(\tau; u, q, k, f) d\tau \Bigg\} \cos(\mu_m x) \sin(\nu_n y). \tag{56}$$

Now from (10) and (11), we get

$$h_0'(t) q(t) + h_0(0) l(t) = (\partial_t^\alpha h_0)(t) - f_0(t) - l(t) h_0'(t)$$
$$+ 2 \sum_{m,n=1}^{\infty} (-1)^{n+1} \lambda_{mn}^2 u_{mn}(t), \quad 0 \le t \le T, \tag{57}$$

$$h_1'(t) q(t) + h_1(0) l(t) = (\partial_t^\alpha h_1)(t) - f_1(t) - l(t) h_1'(t)$$
$$+ 2 \sum_{m,n=1}^{\infty} (-1)^{m+1} \frac{\lambda_{mn}^2}{\mu_m \nu_n} u_{mn}(t), \quad 0 \le t \le T. \tag{58}$$

Due to (C4), we can solve the system (57), (58) and we find

$$q(t) = \frac{1}{h(t)} \Bigg\{ \partial_t^\alpha \Big(h_1(0) h_0(t) - h_0(0) h_1(t) \Big) - \Big(h_1(0) f_0(t) - h_0(0) f_1(t) \Big) - l(t) h(t)$$

$$+2\sum_{m,n=1}^{\infty}\left((-1)^{n+1}h_1(0)-(-1)^{m+1}\frac{h_0(0)}{\mu_m\nu_n}\right)\lambda_{mn}^2u_{mn}(t)\Bigg\},\quad 0\le t\le T, \tag{59}$$

$$l(t)=\frac{1}{h(t)}\Bigg\{h_1'(t)(\partial_t^\alpha h_0)(t)-h_0'(t)(\partial_t^\alpha h_1)(t)-h_1'(t)f_0(t)+h_0'(t)f_1(t)$$
$$-h_1'(t)\left(l(t)h_0'(t)\right)+h_0'(t)\left(l(t)h_1'(t)\right)$$
$$+2\sum_{m,n=1}^{\infty}\left((-1)^{n+1}h_1'(t)-(-1)^{m+1}\frac{h_0'(t)}{\mu_m\nu_n}\right)\lambda_{mn}^2u_{mn}(t)\Bigg\},\quad 0\le t\le T. \tag{60}$$

The following expressions for the second and third components of the solution $\{u(x,y,t),q(t),l(t)\}$ to problem (1)–(4), (10), and (11)

$$q(t)=\frac{1}{h(t)}\Bigg\{\partial_t^\alpha\Big(h_1(0)h_0(t)-h_0(0)h_1(t)\Big)-\Big(h_1(0)f_0(t)-h_0(0)f_1(t)\Big)-l(t)h(t)$$
$$+2\sum_{m,n=1}^{\infty}\lambda_{mn}^2\left((-1)^{n+1}h_1(0)-(-1)^{m+1}\frac{h_0(0)}{\mu_m\nu_n}\right)\Big(a_{mn}E_\alpha(-\lambda_{mn}^2t^\alpha)$$
$$+b_{mn}tE_{\alpha,2}(-\lambda_{mn}^2t^\alpha)+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)F_{mn}(\tau;u,q,k,f)d\tau\Big)\Bigg\}, \tag{61}$$

$$l(t)=\frac{1}{h(t)}\Bigg\{h_1'(t)(\partial_t^\alpha h_0)(t)-h_0'(t)(\partial_t^\alpha h_1)(t)-h_1'(t)f_0(t)+h_0'(t)f_1(t)-h_1'(t)\Big(l(t)*h_0'(t)\Big)$$
$$h_0'(t)\Big(l(t)*h_1'(t)\Big)+2\sum_{m,n=1}^{\infty}\lambda_{mn}^2\left((-1)^{n+1}h_1'(t)-(-1)^{m+1}\frac{h_0'(t)}{\mu_m\nu_n}\right)\Big(a_{mn}E_\alpha(-\lambda_{mn}^2t^\alpha)$$
$$+b_{mn}tE_{\alpha,2}(-\lambda_{mn}^2t^\alpha)+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{mn}^2(t-\tau)^\alpha\right)F_{mn}(\tau;u,q,k,f)d\tau\Big)\Bigg\}, \tag{62}$$

respectively, were obtained by substituting (34) into (59) and (60). Thus, the solution of problem (1)–(4), (10), and (11) was reduced to the solution of systems (56), (61), and (62) with respect to unknown functions $u(x,y,t)$, $q(t)$, and $l(t)$.

The following lemma an important role in studying the uniqueness of the solution to problem (1)–(4), (10), and (11).

Lemma 8. *If $\{u(x,y,t),q(t),l(t)\}$ is any solution to problem (1)–(4), (10), and (11), then the functions*

$$u_{mn}(t)=\iint_S u(x,y,t)v_{mn}(x,y)dxdy\quad(m,n=1,2,...)$$

satisfy the system (34) on the segment $[0,T]$.

Proof. Let $\{u(x,y,t),q(t),l(t)\}$ be any solution of the problem (1)–(4), (10), and (11). Multiplying both sides of the Eq. (1) by the eigenfunctions $v_{mn}(x,y)$, $(m,n=1,2,...)$, integrating with respect to x and y over the rectangular $\bar{S}$ and using the relations

$$\iint_S\partial_{0t}^\alpha u(x,y,t)v_{mn}(x,y)dxdy=\partial_{0t}^\alpha\left(\iint_S u(x,y,t)v_{mn}(x,y)dxdy\right)$$
$$=(\partial_{0t}^\alpha u_{mn})(t),\quad m,n=1,2,...,$$

$$\iint\limits_S \triangle u(x,y,t)v_{mn}(x,y)dxdy = -\lambda_{mn}^2 \iint\limits_S u(x,y,t)v_{mn}(x,y)dxdy = -\lambda_{mn}^2 u_{mn}(t), \quad m,n=1,2,...,$$

we obtain that the Eq. (30) is satisfied.

In like manner, it follows from (2) that condition (31) is also satisfied. Thus, the system of functions $u_{mn}(t)$, $(m,n=1,2,...)$ is a solution of problem (30) and (31). Hence, it follows directly that the functions $u_{mn}(t)$, $(m,n=1,2,...)$ are also satisfy the system (34) on $[0,T]$. The lemma is proved. □

Obviously, if $u_{mn}(t)$, $(m,n=1,2,...)$ is a solution to system (34), then the triple $\{u(x,y,t), q(t), l(t)\}$ of functions $u(x,y,t)=\sum_{m,n=1}^{\infty} u_{mn}(t)v_{mn}(x,y)$, $q(t)$, and $l(t)$ is also a solution to system (56), (61), and (62). It follows from the Lemma 8 that

Corollary 1. *Assume that the system (56), (61), and (62) has a unique solution. Then the problem (1)–(4), (10), and (11) has at most one solution, i.e., if the problem (1)–(4), (10), and (11) has a solution, then it is unique.*

Let us consider the functional space $B_{2,2,T}^{3,2}$ that is introduced in the study of [23], where $B_{2,2,T}^{3,2}$ denotes a set of all functions of the form

$$u(x,y,t) = \sum_{m,n=1}^{\infty} u_{mn}(t)v_{mn}(x,y)$$

considered in $\overline{Q}_T$. Moreover, the functions $u_{mn}(t)$, $(m,n=1,2,...)$ contained in last sum are continuously differentiable on $[0,T]$ and

$$J(u) \equiv \left\{ \sum_{m,n=1}^{\infty} \left(\lambda_{mn}^3 ||u_{mn}(t)||_{C[0,T]}\right)^2 \right\}^{1/2} + \left\{ \sum_{m,n=1}^{\infty} \left(\lambda_{mn}^2 ||u'_{mn}(t)||_{C[0,T]}\right)^2 \right\}^{1/2} < \infty.$$

The norm on the set $J(u)$ is established as follows: $||u(x,y,t)||_{B_{2,2,T}^{3,2}} = J(u)$.

Let $E_T^{3,2}$ denote the space consisting of the topological product $B_{2,2,T}^{3,2} \times C[0,T] \times C[0,T]$, which is the norm of the element $z=\{u,q,l\}$ defined by the formula

$$||z||_{E_T^{3,2}} = ||u(x,y,t)||_{B_{2,2,T}^{3,2}} + ||q(t)||_{C[0,T]} + ||l(t)||_{C[0,T]}.$$

It is known that the spaces $B_{2,2,T}^{3,2}$ and $E_T^{3,2}$ are Banach spaces.

Let us now consider the operator

$$A(u,q,l) = A_1(u,q,l), A_2(u,q,l), A_3(u,q,l),$$

in the space $E_T^{3,2}$, where

$$A_1(u,q,l) = \tilde{u}(x,y,t) \equiv \sum_{m,n=1}^{\infty} \tilde{u}_{mn}(t)v_{mn}(x,y), \quad A_2(u,q,l) = \tilde{q}(t), \quad A_3(u,q,l) = \tilde{l}(t),$$

and the functions $\tilde{u}_{mn}(t)$, $(m,n=1,2,...)$, $\tilde{q}(t)$, and $\tilde{l}(t)$ are equal to the right-hand sides of (56), (61), and (62), respectively. The derivative of $u_{mn}(t)$ follows from (34), i.e.,

$$\begin{aligned} u'_{mn}(t) &= -\lambda_{mn}^2 a_{mn} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn}^2 t^{\alpha}) + b_{mn} E_{\alpha}(-\lambda_{mn}^2 t^{\alpha}) \\ &+ \int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\lambda_{mn}^2 (t-\tau)^{\alpha}\right) F_{mn}(\tau;u,q,l,f)d\tau, \end{aligned} \tag{63}$$

here we have used

$$\frac{d}{dt}(E_{\alpha}(-\lambda t^{\alpha})) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad \text{and} \quad \frac{d}{dt}(t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha})) = t^{\beta-2} E_{\alpha,\beta-1}(-\lambda t^{\alpha})$$

for $\alpha > 0$, $\beta > 0$, and $\lambda > 0$.

Obviously, $\tilde{u}_{mn}(t)$ $(m, n = 1, 2, ...)$ are determined by right-hand side of (63).

The following proposition we will use to get a uniqueness and stability estimate for our inverse problem.

Proposition 1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then, there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}, \quad \kappa \le |\arg(z)| \le \pi.$$

For the proof, we refer to [24] for example.

It is easy too see that

$$\lambda_{mn}^3 \le (\mu_m^2 + \nu_n^2)(\mu_m + \nu_n) = \mu_m^3 + \mu_m^2\nu_n + \mu_m\nu_n^2 + \nu_n^3.$$

Taking into account Proposition 1 and this relation, for all $t \in [0, T]$ and sufficiently large $m, n \in \mathbb{N}$, we obtain

$$\left\{\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^3||\tilde{u}_{mn}(t)||_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le c_8\left\{\left(\sum_{m,n=1}^{\infty}\left(\mu_m^3|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|a_{mn}|\right)^2\right)^{\frac{1}{2}}\right.$$
$$+\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\nu_n^3|a_{mn}|\right)^2\right)^{\frac{1}{2}} + T\left(\sum_{m,n=1}^{\infty}\left(\mu_m^3|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$
$$+T\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|b_{mn}|\right)^2\right)^{\frac{1}{2}} + T\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|b_{mn}|\right)^2\right)^{\frac{1}{2}} + T\left(\sum_{m,n=1}^{\infty}\left(\nu_m^3|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$
$$+T||q(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u'_{mn}(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + T||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|a_{mn}|\right)^2\right)^{\frac{1}{2}}$$
$$+T||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \sqrt{T}\left(\int_0^T\sum_{m,n=1}^{\infty}\left(\mu_m^2|f_{mn}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$
$$\left.+\sqrt{T}\left(\int_0^T\sum_{m,n=1}^{\infty}\left(\nu_n^2|f_{mn}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}} + T^{3/2}||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u'_{mn}(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right\}, \quad (64)$$

$$\left\{\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||\tilde{u}'_{mn}(t)||_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \le c_9\left\{\left(\sum_{m,n=1}^{\infty}\left(\mu_m^3|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|a_{mn}|\right)^2\right)^{\frac{1}{2}}\right.$$
$$+\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\nu_n^3|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$
$$+\left(\sum_{m,n=1}^{\infty}\left(\nu_n^2\nu_n|b_{mn}|\right)^2\right)^{\frac{1}{2}} + T^{\alpha-1}||q(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u'_{mn}(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$
$$+T^{(\alpha-1)/2}||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|a_{mn}|\right)^2\right)^{\frac{1}{2}} + T^{(\alpha-1)/2}||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T^{(\alpha-1)/2}\left(\int\limits_0^T\sum_{m,n=1}^{\infty}\left(\mu_m^2|f_{mn}(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}+\sqrt{T}\left(\int\limits_0^T\sum_{m,n=1}^{\infty}\left(\nu_n^2|f_{mn}(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$+T^{\alpha/2}||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u'_{mn}(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\Bigg\},\tag{65}$$

$$||\tilde{q}(t)||_{C[0,T]}$$

$$\leq c_9||(h(t))^{-1}||_{C[0,T]}\Bigg\{\Big|\Big|\partial_t^\alpha\Big(h_1(0)h_0(t)-h_0(0)h_1(t)\Big)-\Big(h_1(0)f_0(t)-h_0(0)f_1(t)\Big)\Big|\Big|_{C[0,T]}$$

$$+T||l(t)||_{C[0,T]}||h(t)||_{C[0,T]}+2\left(\sum_{m,n=1}^{\infty}\lambda_{mn}^{-2}\right)^{\frac{1}{2}}(|h_1(0)|+|h_0(0)|)$$

$$\times\left[\left(\sum_{m,n=1}^{\infty}\left(\mu_m^3|a_{mn}|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|a_{mn}|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{m,n=1}^{\infty}\left(\nu_n^3|a_{mn}|\right)^2\right)^{\frac{1}{2}}+T\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|b_{mn}|\right)^2\right)^{\frac{1}{2}}+T\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|b_{mn}|\right)^2\right)^{\frac{1}{2}}+T\left(\sum_{m,n=1}^{\infty}\left(\nu_m^3|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T||q(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u'_{mn}(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+T||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|a_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}}+\sqrt{T}\left(\int\limits_0^T\sum_{m,n=1}^{\infty}\left(\mu_m^2|f_{mn}(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left(\int\limits_0^T\sum_{m,n=1}^{\infty}\left(\nu_n^2|f_{mn}(\tau)|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$\left.+T^{3/2}||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u'_{mn}(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\Bigg\},\tag{66}$$

$$||\tilde{l}(t)||_{C[0,T]}$$

$$\leq c_{10}||(h(t))^{-1}||_{C[0,T]}\Bigg\{\Big|\Big|h'_1(t)(\partial_t^\alpha h_0)(t)-h'_0(t)(\partial_t^\alpha h_1)(t)-h'_1(t)f_0(t)+h'_0(t)f_1(t)\Big|\Big|_{C[0,T]}$$

$$+ 2T||l(t)||_{C[0,T]}||h_1'(t)||_{C[0,T]}||h(t)||_{C[0,T]} + 2\left(\sum_{m,n=1}^{\infty}\lambda_{mn}^{-2}\right)^{\frac{1}{2}} |||h_1'(t)| + |h_0'(t)|||_{C[0,T]}$$

$$\times\left[\left(\sum_{m,n=1}^{\infty}\left(\mu_m^3|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|a_{mn}|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \left(\sum_{m,n=1}^{\infty}\left(\nu_n^3|a_{mn}|\right)^2\right)^{\frac{1}{2}} + T\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2\nu_n|b_{mn}|\right)^2\right)^{\frac{1}{2}} + T\left(\sum_{m,n=1}^{\infty}\left(\mu_m\nu_n^2|b_{mn}|\right)^2\right)^{\frac{1}{2}} + T\left(\sum_{m,n=1}^{\infty}\left(\nu_m^3|b_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T||q(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u_{mn}'(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + T||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\mu_m^2|a_{mn}|\right)^2\right)^{\frac{1}{2}}$$

$$+T||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\nu_n^2|a_{mn}|\right)^2\right)^{\frac{1}{2}} + \sqrt{T}\left(\int_0^T\sum_{m,n=1}^{\infty}\left(\mu_m^2|f_{mn}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left(\int_0^T\sum_{m,n=1}^{\infty}\left(\nu_n^2|f_{mn}(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.+T^{3/2}||l(t)||_{C[0,T]}\left(\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u_{mn}'(t)||_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\}, \tag{67}$$

Here, we have used

$$\max_{0\le y<+\infty}\frac{y^{\frac{\alpha-1}{\alpha}}}{1+y} = \frac{(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{1+(\alpha-1)}$$

and $c_i,\ i = 8, 9$, and 10 are depend only α. Then, from (64)–(67), respectively, we obtain

$$||u(x,y,t)||_{B^{3,2}_{2,2,T}} = \left\{\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^3||u_{mn}(t)||_{C[0,T]}\right)^2\right\}^{1/2} + \left\{\sum_{m,n=1}^{\infty}\left(\lambda_{mn}^2||u_{mn}'(t)||_{C[0,T]}\right)^2\right\}^{1/2}$$

$$\le \gamma_1(T) + \beta_1(T)||l(t)||_{C[0,T]} + \kappa_1(T)\Big(||q(t)||_{C[0,T]} + ||l(t)||_{C[0,T]}\Big)||u(x,y,t)||_{B^{3,2}_{2,2,T}}, \tag{68}$$

$$||\tilde{q}(t)||_{C[0,T]} \le \gamma_2(T) + \beta_2(T)||l(t)||_{C[0,T]}$$
$$+ \beta_2(T)\Big(||q(t)||_{C[0,T]} + ||l(t)||_{C[0,T]}\Big)||u(x,y,t)||_{B^{3,2}_{2,2,T}}, \tag{69}$$

$$||\tilde{l}(t)||_{C[0,T]} \le \gamma_3(T) + \beta_3(T)||l(t)||_{C[0,T]}$$
$$+ \beta_3(T)\Big(||q(t)||_{C[0,T]} + ||l(t)||_{C[0,T]}\Big)||u(x,y,t)||_{B^{3,2}_{2,2,T}}, \tag{70}$$

where

$$\gamma_1(T) := c_8\Big\{2||a^{(3,0)}(x,y)||_{L_2(S)} + 2||a^{(2,1)}(x,y)||_{L_2(S)} + 2||a^{(1,2)}(x,y)||_{L_2(S)} + 2||a^{(0,3)}(x,y)||_{L_2(S)}$$

$$+T||b^{(3,0)}(x,y)||_{L_2(S)}+T||b^{(2,1)}(x,y)||_{L_2(S)}+T||b^{(1,2)}(x,y)||_{L_2(S)}+T||b^{(0,3)}(x,y)||_{L_2(S)}$$
$$+2\sqrt{T}||f^{(2,0)}(x,y)||_{L_2(S)\times C_\gamma[0,T]}+2\sqrt{T}||f^{(0,2)}(x,y)||_{L_2(S)\times C_\gamma[0,T]}\Big\},$$

$$\beta_1(T):=c_8(T^{(\alpha-1)/2}+T)\Big(||a^{(2,0)}(x,y)||_{L_2(S)}+||a^{(0,2)}(x,y)||_{L_2(S)}\Big),$$

$$\kappa_1(T):=2c_8T(1+T^{1/2}),$$

$$\gamma_2(T)=c_9||(h(t))^{-1}||_{C[0,T]}\left\{\Big|\Big|\partial_t^\alpha\Big(h_1(0)h_0(t)-h_0(0)h_1(t)\Big)-\Big(h_1(0)f_0(t)-h_0(0)f_1(t)\Big)\Big|\Big|_{C[0,T]}\right.$$
$$+2\left(\sum_{m,n=1}^{\infty}\lambda_{mn}^{-2}\right)^{\frac{1}{2}}(|h_1(0)|+|h_0(0)|)\Big[||a^{(3,0)}(x,y)||_{L_2(S)}+||a^{(2,1)}(x,y)||_{L_2(S)}+||a^{(1,2)}(x,y)||_{L_2(S)}$$
$$+||a^{(0,3)}(x,y)||_{L_2(S)}+T||b^{(3,0)}(x,y)||_{L_2(S)}+T||b^{(2,1)}(x,y)||_{L_2(S)}+T||b^{(1,2)}(x,y)||_{L_2(S)}$$
$$\left.+T||b^{(0,3)}(x,y)||_{L_2(S)}+\sqrt{T}||f^{(2,0)}(x,y)||_{L_2(S)\times C_\gamma[0,T]}+\sqrt{T}||f^{(0,2)}(x,y)||_{L_2(S)\times C_\gamma[0,T]}\Big]\right\},$$

$$\beta_2(T):=c_9T||h(t)||_{C[0,T]}+2T\left(\sum_{m,n=1}^{\infty}\lambda_{mn}^{-2}\right)^{\frac{1}{2}}(|h_1(0)|+|h_0(0)|)$$
$$\times\Big(||a^{(2,0)}(x,y)||_{L_2(S)}+||a^{(0,2)}(x,y)||_{L_2(S)}\Big),$$

$$\kappa_2(T):=\frac{c_9}{2}\kappa_1(T),$$

$$\gamma_3(T):=c_{10}||(h(t))^{-1}||_{C[0,T]}\Big\{\Big|\Big|h_1'(t)(\partial_t^\alpha h_0)(t)-h_0'(t)(\partial_t^\alpha h_1)(t)-h_1'(t)f_0(t)+h_0'(t)f_1(t)\Big|\Big|_{C[0,T]}$$
$$+2\left(\sum_{m,n=1}^{\infty}\lambda_{mn}^{-2}\right)^{\frac{1}{2}}\Big|\Big||h_1'(t)|+|h_0'(t)|\Big|\Big|_{C[0,T]}\Big[||a^{(3,0)}(x,y)||_{L_2(S)}+||a^{(2,1)}(x,y)||_{L_2(S)}$$
$$+||a^{(1,2)}(x,y)||_{L_2(S)}+||a^{(0,3)}(x,y)||_{L_2(S)}+T||b^{(3,0)}(x,y)||_{L_2(S)}$$
$$+T||b^{(2,1)}(x,y)||_{L_2(S)}+T||b^{(1,2)}(x,y)||_{L_2(S)}$$
$$+T||b^{(0,3)}(x,y)||_{L_2(S)}+\sqrt{T}||f^{(2,0)}(x,y)||_{L_2(S)\times C_\gamma[0,T]}+\sqrt{T}||f^{(0,2)}(x,y)||_{L_2(S)\times C_\gamma[0,T]},$$

$$\beta_3(T):=2c_{10}T||h_1'(t)||_{C[0,T]}||h(t)||_{C[0,T]}$$
$$+2T\left(\sum_{m,n=1}^{\infty}\lambda_{mn}^{-2}\right)^{\frac{1}{2}}|||h_1'(t)|+|h_0'(t)|||_{C[0,T]}\left(||a^{(2,0)}(x,y)||_{L_2(S)}+||a^{(0,2)}(x,y)||_{L_2(S)}\right),$$

$$\kappa_3(T):=c_{10}\kappa_1(T).$$

From inequalities (68)–(70), we conclude

$$||\tilde{u}(x,y,t)||_{B^{3,2}_{2,2,T}}+||\tilde{q}(T)||+||\tilde{l}(T)||$$

$$\leq \gamma(T) + \beta(T)||l(t)||_{C[0,T]} + \kappa(T)(||q(t)||_{C[0,T]} + ||l(t)||_{C[0,T]})||u(x,y,t)||_{B^{3,2}_{2,2,T}}, \tag{71}$$

where

$$\gamma(T) := \gamma_1(T) + \gamma_2(T) + \gamma_3(T), \quad \beta(T) := \beta_1(T) + \beta_2(T) + \beta_3(T),$$
$$\kappa(T) := \kappa_1(T) + \kappa_2(T) + \kappa_3(T).$$

Theorem 3. *Let the conditions Lemma 7 and the condition*

$$\kappa(T)(\gamma(T)+2)^2 < 2 \quad \text{and} \quad \beta(T) < 1/2 \tag{72}$$

be fulfilled. Then, problem (1)–(4), (10), and (11) has a unique solution in the ball $B = B_r(||z||_{E^{3,2}_T} \leq r = \frac{1}{2}(\gamma(T)+2)$ *of the space.*

Proof. Let us denote $z = (u(x,y,t), q(t), l(t))^*$ (where $*$ is the symbol of transposition) and rewrite the system of equations (56), (61), and (62) in the following operator equation

$$z = Az, \tag{73}$$

where $A = (A_1, A_2, A_3)^*$, $A_1(z)$, $A_2(z)$, and $A_3(z)$ defined by the right sides of (56), (61), and (62), respectively.

Analogously to (71) we obtain that for any $z, z_1, z_2 \in B_r$ the following estimates hold

$$||Az||_{E^{3,2}_T} \leq \gamma(T) + \beta(T)||l(t)||_{C[0,T]} + \kappa(T)(||q(t)||_{C[0,T]} + ||l(t)||_{C[0,T]})||u(x,y,t)||_{B^{3,2}_{2,2,T}}, \tag{74}$$

$$||Az_1 - Az_2||_{E^{3,2}_T} \leq r(\beta(T) + \kappa(T))||l_1(t) - l_2(t)||_{C[0,T]}$$
$$+ r\kappa(T)(||q_1(t) - q_2(t)||_{C[0,T]} + ||u_1(x,y,t) - u_2(x,y,t)||_{B^{3,2}_{2,2,T}}). \tag{75}$$

Then, taking into account (72) in (74) and (75), it follows that the operator A acts in the ball $B = B_r$ and is contractive. Therefore, in the ball $B = B_r$ the operator A has a unique fixed point $\{u, q, l\}$ that is a unique solution of (73), i.e., it is unique solution of system (56), (61), and (62), in the ball B_r. The Theorem 3 is proved. □

Remark 2. Inequality (72) is satisfied for sufficiently small values of T.

Theorem 4. *Suppose that all the conditions of Theorem 3 are satisfied and (C2)–(C4). Then, problem (1)–(6) has a unique classical solution in the ball* B_r *of the space* $E^{3,2}_T$.

5. CONCLUSIONS

In this paper, we investigate time-dependent coefficient and kernel for a two-dimensional fractional diffusion-wave equation. The existence and uniqueness for the direct problem and the uniqueness for the inverse problems are proved. The method of separation of variables are used to solve the problems. The conditions of unique solvability of a solution to the direct and inverse problem are obtained.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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