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Askar Rahmonov ✉; Durdimurod Durdiev; Dilshoda Akramova

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A 2D Inverse Problem for a Fractional-Wave Equation

Askar Rahmonov,[1, a)] Durdimurod Durdiev,[1, b)] and Dilshoda Akramova[2, c)]

[1)] *V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, University str., Tashkent 100174, Uzbekistan.*
[2)] *Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan.*

[a)] *Corresponding author:araxmonov@mail.ru*
[b)] *durdiev65@mail.ru*
[c)] *akramova.shoda@mail.ru*

Abstract. In this paper, we consider two dimensional inverse problem for a fractional-wave equation with variable coefficient. The inverse problem is reduced to the equivalent integral equation. For solving this equation, the contracted mapping principle is applied. The local existence and global uniqueness results are proven. Also, the stability estimate is obtained.

INTRODUCTION

Denoting as usual x, t the space and time variables and $u = u(x,t)$ the response variable, this equation read:

$$\left({}^C\mathscr{D}_t^\alpha u\right)(x,t) - \triangle_x u(x,t) + q(x_1,t)u(x,t) = f(x,t), \quad x \in \mathbb{R}^2,\ 0 < t \leq T, \tag{1}$$

where $1 < \alpha < 2$, ${}^C\mathscr{D}_t^\alpha$ is the Caputo fractional derivative, that is

$$\left({}^C\mathscr{D}_t^\alpha u\right)(x,t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} u_{\tau\tau}(x,\tau)d\tau, \tag{2}$$

and $\triangle_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, $f(x,t)$ is the known source term.

For the Cauchy problem corresponding to the initial conditions

$$u(x,0) = u_1(x), \quad u_t(x,0) = u_2(x), \quad x \in \mathbb{R}^2, \tag{3}$$

where $u_j(x)$, $j = 1,2$ are given functions.

We assume that the functions f, u_1, u_2 are bounded; f is a continuous in $(x,t) \in \mathbb{R}^2 \times [0,T]$, and locally Hölder continuous in x, uniformly respect to t, u_1 and its first derivatives are bounded and Hölder continuous with the exponent $\frac{1}{\alpha} < \gamma \leq 1$; u_2 is Hölder continuous. Under this assumption, the problem (1), (3) possesses a classical solution $u(x,t)$ as [1]. This means that $u(x,t)$ belongs to C^2 in x for each $t > 0$; for each $x \in \mathbb{R}^2$ $u(x,t)$ belongs to C^1 in (x,t) on $\mathbb{R}^2 \times [0,T]$, and for any $x \in \mathbb{R}^2$ the Caputo derivative $\left({}^C\mathscr{D}_t^\alpha u\right)(x,t)$ is continuously in t for $t > 0$; $u(x,t)$ satisfies the equation (1) and initial conditions (3).

For the given functions $q(x_1,t)$, $f(x,t)$, $u_1(x)$, $u_2(x)$ and a number $\alpha \in (1,2)$, the problem of determining the solution to Cauchy problem (1) and (3) we call as the direct problem.

Many modern science and engineering technology areas can be described very successfully by models using fractional differential equations (see [2, 3]). When $q(x_1,t) \equiv 0$, this equation was studied by many authors (mostly [1, 4, 5] and the references therein). The equation (1) describes the propagation of stress pulses in a viscoelastic medium [1]. In practical situations, the function q represents some physical property, which is very hard to be measured directly in advance. So we consider an inverse problem of determining source term function q from some additional measurement on u.

For classical integro-differential and time-fractional wave equations, sometimes the initial value, part of boundary value, wave coefficient, kernel or source term are not known. If we recover some of them by additional measured data, we can deduce many inverse problems for integro-differential and time-fractional wave equations. There have been some results for integro-differential and time-fractional wave equations, for instance the reconstruction of the kernel [6, 7, 8, 9, 10, 11, 12, 13, 14], the reconstruction of the time-dependent source term [15, 16, 17, 18, 19, 20, 21], the Cauchy problem [21, 22, 23, 24, 25], the reconstruction of order of fractional derivative and source term [26, 27]. However, as we know, the inverse problems of fractional wave equations have only a few papers such as inverse problems on an unbounded domain [28, 29].

Inverse problem consists of determining the time and horizontal variable dependent unknown coefficient of the source term $q(x_1,t)$ and the wave distribution $u(x,t)$, from the initial condition (3) and

$$u(x_1,0,t) = g(x_1,t), \quad x_1 \in \mathbb{R},\ t \in [0,T], \tag{4}$$

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where $g(x_1,t)$ is given.

The remainder of this paper is composed of three sections. Section 2 is devoted to the study of the properties of the solution of the direct problem (1), (3). In section 3, the existence and uniqueness of the solution of the inverse problem (1)-(4) is established by using the Banach fixed point theorem and the continuous dependence of the solution of the inverse problem upon the data of $\{f,u_1,u_2,g\}$ is shown.

By $D_T := \{(x,t) : x\in\mathbb{R}^2, 0\le t\le T\}$ we denote a strip with the thickness T, where $T>0$ is any fixed number.

Let $C^{\alpha,m}(D_T)$ be the class of the m times continuously differentiable, bounded with all derivatives of order up to m with respect to $x\in\mathbb{R}^2$ and its fractional derivative ${}^{C}\mathscr{D}_t^{\alpha}$ is continuous in t on $[0,T]$.

Everywhere in this paper we will denote by $H^l(\mathbb{R}^2)$ locally Hölder continuous functions with exponent $l\in(0,1)$. The norms in $H^l(\mathbb{R}^2)$ are determined in [30].

By $C(H^l(\mathbb{R}^2),[0,T])$ we denote the class of continuous with respect to t variable on the segment $[0,T]$ with values in $H^l(\mathbb{R}^2)$ functions. For a fixed t, the norm of the function $\phi(x,t)$ in $H^l(\mathbb{R}^2)$ will be denoted by $|\phi|^l(t)$. The norm of a function $\phi(x,t)$ in $C(H^l(\mathbb{R}^2),[0,T])$ is defined by the equality

$$\|\phi\|^l := \max_{t\in[0,T]} \big||\phi|^l(t)\big|.$$

INVESTIGATION OF DIRECT PROBLEM (1)-(3)

The functions q,f and u_i, $(i=1,2)$ satisfy the following assumptions:

(A1) $q(x_1,t)$ is bounded, uniformly Hölder continuous in x_1, uniformly with respect to t;

(A2) $f(x,t)$ is a bounded function, jointly continuous in $(x,t)\in D_T$, and locally Hölder continuous in x, uniformly with respect to t;

(A3) $u_1(x)$ is bounded, continuously differentiable, and its first derivatives are bounded and Hölder continuous with the exponent $\frac{1}{\alpha}<\gamma\le 1$;

(A4) $u_2(x)$ is Hölder continuous.

Let us transpose the last term on the left in (1), the fractional differential equation becomes

$$\left({}^{C}\mathscr{D}_t^{\alpha}u\right)(x,t) - \triangle u(x,t) = F(x,t), \quad x\in\mathbb{R}^2,\ 0<t\le T, \tag{5}$$

where $F(x,t) = -q(x_1,t)u(x,t) + f(x,t)$.

To solve problem (1) and (3), we apply the Laplace transform with respect to t:

$$(\mathscr{L}_t u)(x,s) = \int_0^\infty e^{-ts}u(x,t)dt, \quad x\in\mathbb{R}^2,\ s\in\mathbb{C}, \tag{6}$$

and the Fourier transforms with respect to $x\in\mathbb{R}^2$:

$$(\mathscr{F}_x u)(\xi,t) = \int_{\mathbb{R}^2} e^{-i(x,\xi)}u(x,t)dx, \quad \xi\in\mathbb{R}^2,\ t>0, \tag{7}$$

here $x=(x_1,x_2)\in\mathbb{R}^2$, $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, $(x,\xi)=x_1\cdot\xi_1+x_2\cdot\xi_2$, $dx=dx_1dx_2$.

Applying the Fourier transform (7) to (5) and using a formula of the form

$$(\mathscr{F}_x\triangle_x u)(\xi,t) = -|\xi|^2(\mathscr{F}_x u)(\xi,t),$$

we arrive at the following relation:

$$({}^{C}\mathscr{D}_t^{\alpha}\mathscr{F}_x u)(\xi,t) + |\xi|^2(\mathscr{F}_x u)(\xi,t) = (\mathscr{F}_x F)(\xi,t). \tag{8}$$

Applying the Laplace transform (6) to (8), and taking into account the formula of the form

$$(\mathscr{L}_t{}^{C}\mathscr{D}_t^{\alpha}\mathscr{F}_x)(\xi,s) = s^{\alpha}(\mathscr{L}_t\mathscr{F}_x)(\xi,s) - s^{\alpha-1}(\mathscr{F}_x u)(\xi,0) - s^{\alpha-2}(\mathscr{F}_x u_t)(\xi,0)$$

with $\xi\in\mathbb{R}^2$ and initial conditions in (3), we have

$$(\mathscr{L}_t\mathscr{F}_x u)(\xi,s) = \frac{s^{\alpha-1}}{s^{\alpha}+|\xi|^2}(\mathscr{F}_x u_1)(\xi) + \frac{s^{\alpha-2}}{s^{\alpha}+|\xi|^2}(\mathscr{F}_x u_2)(\xi) + \frac{1}{s^{\alpha}+|\xi|^2}(\mathscr{L}_t\mathscr{F}_x F)(\xi,s)$$

$$= \Phi_1(\xi,s)+\Phi_2(\xi,s)+\Phi_3(\xi,s), \quad \xi\in\mathbb{R}^2, s\in\mathbb{C}. \tag{9}$$

Now we obtain the explicit solution $u(x,t)$ by using the inverse Laplace transform with respect to s:

$$(\mathscr{L}_s^{-1}u)(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{ts}u(x,s)ds, \qquad x\in\mathbb{R}^2, t>0$$

and the inverse Fourier transform with respect to ξ:

$$(\mathscr{F}_\xi^{-1}u)(x,t)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{i(x,\xi)}u(\xi,t)d\xi, \qquad \xi\in\mathbb{R}^2, t>0,$$

where $d\xi=d\xi_1 d\xi_2$ and $\gamma=\Re(s)>\inf|s|$. First, these operations we carry out for $\Phi_1(\xi,s)$. It may be performed by using the equality

$$\frac{s^{\alpha-1}}{s^\alpha+|\xi|^2}=s^{-1}\cdot\frac{1}{1+\frac{|\xi|^2}{s^\alpha}} \tag{10}$$

and expanding the second factor on the right side of this expression into an infinitely decreasing geometric series:

$$\frac{1}{1+\frac{|\xi|^2}{s^\alpha}}=\sum_{n=0}^{\infty}\Big(-\frac{|\xi|^2}{s^\alpha}\Big)^n$$

for $|\xi|<s^{\alpha/2}$. On bases of (10) from last equality we have

$$\frac{s^{\alpha-1}}{s^\alpha+|\xi|^2}=\sum_{n=0}^{\infty}\Big(-|\xi|^2\Big)^n s^{-n\alpha-1}. \tag{11}$$

Then, according to the following relation

$$(\mathscr{L}_s^{-1}s^{-\nu})(t)=\frac{t^{\nu-1}}{\Gamma(\nu)}$$

for $\Re(\nu)>0$ and $\Re(s)>0$, we have

$$(\mathscr{L}_s^{-1}\Phi_1)(\xi,t)=E_\alpha(-|\xi|^2t^\alpha)(\mathscr{F}_x u_1)(\xi).$$

Similarly, we can calculate the inverse Laplace transform for Φ_j, $j=2,3$ like that and generally after applying this transform to (10), we get

$$(\mathscr{F}_x u)(\xi,t)=E_\alpha(-|\xi|^2t^\alpha)(\mathscr{F}_x u_1)(\xi)+tE_{\alpha,2}(-|\xi|^2t^\alpha)(\mathscr{F}_x u_2)(\xi)$$

$$+t^{\alpha-1}E_{\alpha,\alpha}(-|\xi|^2t^\alpha)*_t(\mathscr{F}_x F)(\xi,t), \quad \xi\in\mathbb{R}^2, t>0, \tag{12}$$

where $*_t$ and $E_{\alpha,\beta}(z):=\sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$, $\alpha,\beta,z\in\mathbb{C}, \Re(\alpha)>0, \Re(\beta)>0$ are the Laplace convolution operator and Mittag-Leffler functions respectively (see [31], p. 19 and p. 42, [32]). Before applying the inverse Fourier transform to (12), we bring a formula which is relation between Mittag-Leffler and Fox functions, i.e.,

$$E_{\alpha,\beta}(z)=H_{1,2}^{1,1}\Big[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\Big],$$

(more information about Fox's $H-$function see [33].) Besides the following formula useful for the Fourier transform of a radial function, that is, if $\psi(x)\in L^1(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ is radial with $\psi(x)=\varphi(|x|)$, then its Fourier transform is also radial and is given by (see [32])

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x,\xi)}\varphi(|\xi|)d\xi=\frac{1}{(2\pi)^{n/2}}|x|^{(2-n)/2}\int_0^\infty r^{n/2}\varphi(r)J_{n/2-1}(|x|r)dr. \tag{13}$$

In particular,

$$\int_{\mathbb{R}^2} e^{i(x,\xi)} E_{\alpha,\beta}(-a|\xi|^2)d\xi = 2\pi \int_0^\infty r E_{\alpha,\beta}(-ar^2) J_0(|x|r)dr. \tag{14}$$

The right hand of (13) is the Hankel transform of $\varphi(r)$ and we will use it later. Firstly, we do change the last equality by Fox function and so (14) becomes

$$\int_{\mathbb{R}^2} e^{i(x,\xi)} E_{\alpha,\beta}(-a|\xi|^2)d\xi = 2\pi \int_0^\infty r J_0(|x|r) H^{1,1}_{1,2}\left[ar^2 \Big|^{(0,1)}_{(0,1),(1-\beta,\alpha)}\right]dr. \tag{15}$$

Furthermore, using the Hankel transform of the $H-$function [33], i.e.,

$$\int_0^\infty x^{\rho-1} J_\nu(ax) H^{m,n}_{p,q}\left[bx^\sigma \Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]dx = \frac{2^{\rho-1}}{a^\rho} H^{m,n+1}_{p+2,q}\left[b\left(\frac{2}{a}\right)^\sigma \Big|^{(1-\frac{\rho+\nu}{2},\frac{\sigma}{2}),(a_p,A_p),(1-\frac{\rho-\nu}{2},\frac{\sigma}{2})}_{(b_q,B_q)}\right].$$

If, we apply the last formula to (15), then we get the inverse Fourier transform for two parameter Mittag-Leffler function:

$$\int_{\mathbb{R}^2} e^{i(x,\xi)} E_{\alpha,\beta}(-a|\xi|^2)d\xi = \frac{4\pi}{|x|^2} H^{2,0}_{1,2}\left[\frac{|x|^2}{4a} \Big|^{(\beta,\alpha)}_{(1,1),(1,1)}\right], \tag{16}$$

here we used the reduction formula of the $H-$function (see [31], p. 11, formula (1.57)). We can represent the last function by definite integral and Wright function (see, [34]). Firstly, we bring the following useful formula [5]:

$$f_\beta(z;\mu,\delta) = \frac{1}{2\varepsilon\sqrt{\pi}}\left(\frac{z}{2}\right)^{-\gamma/\varepsilon} H^{2,0}_{1,2}\left[\left(\frac{z}{2}\right)^{1/\varepsilon} \Big|^{\left(\delta+\frac{\gamma\beta}{\varepsilon},\frac{\beta}{\varepsilon}\right)}_{\left(\frac{\gamma}{2\varepsilon},\frac{1}{2\varepsilon}\right),\left(\frac{1-\mu}{2}+\frac{\gamma}{2\varepsilon},\frac{1}{2\varepsilon}\right)}\right], \tag{17}$$

where

$$f_\beta(z;\mu,\delta) = \begin{cases} \frac{2}{\Gamma(\frac{\mu}{2})} \int\limits_1^\infty W(-\beta,\delta;-zy)(y^2-1)^{\frac{\mu}{2}-1}dy, & \mu > 0, \\ W(-\beta,\delta;-z), & \mu = 0, \end{cases}$$

$$W(-\beta,\delta;z) = \sum_{n=0}^\infty \frac{z^n}{n!\Gamma(\delta-\beta n)}, \quad \beta \in (0,1).$$

Then from (16) for $a = t^\alpha$ follows

$$\int_{\mathbb{R}^2} e^{i(x,\xi)} E_{\alpha,\beta}(-a|\xi|^2)d\xi = \pi^{3/2} f_{\alpha/2}\left(\frac{|x|}{t^{\alpha/2}};1,\beta-\alpha\right).$$

If taking into account $\beta \in \{1,2,\alpha\}$ and using the definition of the Fourier convolution operator, then after applying the inverse Fourier transform to (12), we have

$$u(x,t) = u_0(x,t) - \int_0^t d\tau \int_{\mathbb{R}^2} Y(x-\xi,t-\tau) q(\xi_1,\tau) u(\xi,\tau) d\xi, \tag{18}$$

where $d\xi = d\xi_1 d\xi_2$,

$$u_0(x,t) := \int_{\mathbb{R}^2} Z_1(x-\xi,t)u_1(\xi)d\xi + \int_{\mathbb{R}^2} Z_2(x-\xi,t)u_2(\xi)d\xi$$

$$+ \int_0^t d\tau \int_{\mathbb{R}^2} Y(x-\xi,t-\tau) f(\xi,\tau) d\xi, \tag{19}$$

and the Green kernels Z_j, $(j = 1,2)$, Y have the following form

$$Z_j(x,t) = \pi^{3/2} f_{\alpha/2}\left(\frac{|x|}{t^{\alpha/2}};1,j-\alpha\right), \quad Y(x,t) = \pi^{3/2} f_{\alpha/2}\left(\frac{|x|}{t^{\alpha/2}};1,0\right).$$

Lemma 1. *Suppose (A1)-(A4) be satisfied. Then there exists a unique solution of the integral equation (18) with* $u(x,t) \in C^{\alpha,2}(D_T)$ *where* $\alpha \in (1,2)$.

Proof. Since the unknown function u appears in the integral, this is not a solution formula, but an integral equation for u. It can be shown that the only continuous solution of the integral equation (18) is the solution of the problem (1), (3). To solve the integral equation (18), we use the method of successive approximations and consider the sequence of functions defined recursively by the formulas:

$$u_{n+1}(x,t) = u_0(x,t) - \int_0^t d\tau \int_{\mathbb{R}^2} Y(x-\xi, t-\tau) q(\xi_1, \tau) u_n(\xi, \tau) d\xi, \quad n = 1, 2, ..., \tag{20}$$

where $u_0(x,t)$ is determined by the equality (19). We wish to show that the sequence u_n converges to the solution u. Further we use the following estimates [1]. Let $\rho_\sigma(x,t) = \exp\{-\sigma(t^{-\alpha/2}|x|)^{\frac{2}{2-\alpha}}\}$, $\sigma > 0$. Then

$$|Z_1(x,t)| \leq Ct^{-\alpha}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \tag{21}$$

$$|Z_2(x,t)| \leq Ct^{-\alpha+1}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \tag{22}$$

$$|Y(x,t)| \leq Ct^{-1}\rho_\sigma(x,t), \tag{23}$$

$$|D_x^m Y(x,t)| \leq Ct^{-\alpha-1}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \quad |m| = 1, \tag{24}$$

$$|\partial_t Z_1(x,t)| \leq Ct^{-\alpha-1}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \tag{25}$$

$$|\partial_t Z_2(x,t)| \leq Ct^{-\alpha}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \tag{26}$$

$$|\partial_t Y(x,t)| \leq Ct^{-2}\rho_\sigma(x,t), \tag{27}$$

$$\left|\left({}^C\mathscr{D}_t^\alpha Z_1\right)(x,t)\right| \leq Ct^{-2\alpha}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \tag{28}$$

$$\left|\left({}^C\mathscr{D}_t^\alpha Z_2\right)(x,t)\right| \leq Ct^{-2\alpha+1}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t), \tag{29}$$

$$\left|\left({}^C\mathscr{D}_t^\alpha Y\right)(x,t)\right| \leq Ct^{-\alpha-1}\left[1 + |\ln(t^{-\alpha/2}|x|)|\right]\rho_\sigma(x,t). \tag{30}$$

The estimates

$$|D_x^m Z_1(x,t)| \leq Ct^{-\alpha}|x|^{-m}\rho_\sigma(x,t), \tag{31}$$

$$|D_x^m Z_2(x,t)| \leq Ct^{-\alpha+1}|x|^{-m}\rho_\sigma(x,t) \tag{32}$$

with $1 \leq m \leq 3$ and the estimate

$$|D_x^m Y(x,t)| \leq Ct^{-1}|x|^{-m}\rho_\sigma(x,t) \tag{33}$$

with $2 \leq m \leq 3$, valid. In (21)-(33) the letter C denotes various positive constants. These estimates for $Z_j, j = 1,2, Y$ are based on the following inequality (see, [5])

$$|f_\beta(z;\mu,\delta)| \leq C\exp(-\sigma z^{\frac{1}{1-\beta}})\ln z,$$

where C and σ are positive constants, and related only the parameters β, μ, δ. We also note that the following integral formulas hold:

$$\int_{\mathbb{R}^2} Z_1(x,t)dx = 1, \quad \int_{\mathbb{R}^2} Z_2(x,t)dx = t, \tag{34}$$

$$\int_{\mathbb{R}^2} Y(x,t)dx = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{35}$$

Note that $u_0(x,t)$ is the solution to the problem (1), (3) for $q(x_1,t) \equiv 0$. For a comparison, note that for the wave equation corresponding formally to $\alpha = 2$, with the initial conditions (3), a counterpart of the kernel Z_j, $j = 1,2$, Y from (19) are a distribution supported on light cone $\{|x-\xi| \le t\}$.

Thus in the fractional order case, though the fundamental solution of the Cauchy problem is not concentrated on the set $\{|x-\xi| \le t^{\alpha/2}\}$, it decays exponentially outside it, possessing a kind of weak hyperbolicity property. Of course, the equation (1) "interpolates" between the heat and wave equations, and possesses some "parabolic" properties too, as it is clear from (19).

Under the assumptions of Lemma 1 it is true inclusion $u_0(x,t) \in C^{\alpha,2}(D_T)$. The last assertion we will show step-by-step. Indeed, in accordance with the estimates (21)-(33), the first derivatives in t of function u_0, given by formula (19) is exists i.e., the following differentiation formula is valid:

$$\frac{\partial}{\partial t}u_0(x,t) = \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_1(x-\xi,t)u_0(\xi)d\xi$$

$$+\int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_2(x-\xi,t)u_1(\xi)d\xi + \int_0^t d\tau \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Y(x-\xi,t-\tau)f(\xi,\tau)d\xi, \tag{36}$$

that is the last improper integral converges in D_T. Indeed,

$$u_{0h}(x,t) = \int_{\mathbb{R}^2} Z_1(x-\xi,t-h)u_1(\xi)d\xi$$

$$+\int_{\mathbb{R}^2} Z_2(x-\xi,t-h)u_2(\xi)d\xi + \int_0^{t-h} d\tau \int_{\mathbb{R}^2} Y(x-\xi,t-\tau)f(\xi,\tau)d\xi.$$

Then

$$\frac{\partial}{\partial t}u_{0h}(x,t) = \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_1(x-\xi,t-h)u_1(\xi)d\xi + \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_2(x-\xi,t-h)u_2(\xi)d\xi$$

$$+\int_{\mathbb{R}^2} Y(x-\xi,h)f(\xi,t-h)d\xi + \int_0^{t-h} d\tau \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Y(x-\xi,t-\tau)f(\xi,\tau)d\xi$$

$$= I_1 + I_2 + I_3 + I_4.$$

By (26) and first equality in (34),

$$|I_1| = \left| \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_1(x-\xi,t-h)[u_1(\xi)-u_1(x)]d\xi \right| \le C(t-h)^{-\alpha-1} \int_{\mathbb{R}^2} \left(1 + \left| \ln \frac{|x-\xi|}{(t-h)^{\alpha/2}} \right| \right)$$

$$\times \rho_\sigma(x-\xi,t-h)|x-\xi|^\gamma d\xi \le C_\alpha (t-h)^{\alpha\gamma-1} < +\infty,$$

as $h \to 0$, here we used the fact that

$$\int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_1(x-\xi,t-h)d\xi = 0.$$

Using (26) and second equality in (34), we have

$$|I_2| = \left| \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Z_2(x-\xi,t-h)[u_2(\xi)-u_2(x)]d\xi \right| + |u_2(x)| \le C(t-h)^{-\alpha} \int_{\mathbb{R}^2} \left(1 + \left| \ln \frac{|x-\xi|}{(t-h)^{\alpha/2}} \right| \right)$$

$$\times \rho_\sigma(x-\xi,t-h)|x-\xi|^{\gamma_1}d\xi \le C_\alpha (t-h)^{\alpha\gamma} < +\infty.$$

as $h \to 0$. Similarly, by using (23), we find

$$|I_3| = \left| \int_{\mathbb{R}^2} Y(x-\xi,h) f(\xi,t-h) d\xi \right| \le C_\alpha h^{\alpha-1} \to 0, \tag{37}$$

as $h \to 0$.

Let

$$I_{40} = \int_{\mathbb{R}^2} \frac{\partial}{\partial t} Y(x-\xi,t-\tau) f(\xi,\tau) d\xi.$$

Using (23), then

$$|I_{40}| \le C(t-h)^{-1} \int_{\mathbb{R}^2} \rho_\sigma(x-\xi,t-h) d\xi \le C_\alpha (t-h)^{\alpha-1}. \tag{38}$$

Since $\alpha > 1$, that means the convergence of I_4, as $h \to 0$, which implies (36).

It follows from (37) and (38) that

$$\lim_{t\to 0} I_3 = \lim_{t\to 0} \frac{\partial}{\partial t} I_4 = 0. \tag{39}$$

It is straightforward to check, using the estimates for $Z_j(x,t)$, $j = 1,2$ and $Y(x,t)$, that the first derivative in x of $u_0(x,t)$ can be obtained by differentiating under the sign of integral in (19). Other derivatives from $u_0(x,t)$ repeats the reasoning in [4] (pp. 342-347).

Now, we continue of the proof of Lemma 1.

Set $d_1 := |u_1|^\gamma$, $d_2 := |u_2|^{\gamma_1}$, $q_0 := \|q\|^\alpha$ and $d_3 := \|f\|^{\gamma_2}$, where $0 < \gamma_1, \gamma_2 \le 1$. We wish to show that the sequence u_n converges to the solution u. We consider the difference

$$\vartheta_n \equiv u_n - u_{n-1}.$$

We subtract (20) with n replaced by $n-1$ from (20) to find that

$$\vartheta_{n+1}(x,t) = -\int_0^t d\tau \int_{\mathbb{R}^2} Y(x-\xi,t-\tau) q(\xi_1,\tau) \vartheta_n(\xi,\tau) d\xi. \tag{40}$$

From (19) we can easily obtain upper estimate for ϑ_1, i.e.,

$$|\vartheta_1(x,t)| \le \frac{d_0 q_0}{\Gamma(1+\alpha)} t^\alpha, \quad \text{for} \quad t \le T,$$

where $d_0 := d_1 + d_2 T + \frac{d_3}{\Gamma(1+\alpha)} T^\alpha$.

Then by (40) with $n = 1$

$$|\vartheta_2(x,t)| \le \frac{d_0 q_0^2}{\Gamma(1+2\alpha)} t^{2\alpha}.$$

Again by (20), but with $n = 2$

$$|\vartheta_3(x,t)| \le \frac{d_0 q_0^3}{\Gamma(1+3\alpha)} t^{3\alpha}.$$

Continuing in this fashion, we see that

$$|u_n(x,t) - u_{n-1}(x,t)| = |\vartheta_n(x,t)| \le \frac{d_0 q_0^n}{\Gamma(1+n\alpha)} t^{n\alpha}, \quad \text{for} \quad t \le T. \tag{41}$$

Thus for any $m > n$ and $t \le T$

$$|u_m(x,t) - u_n(x,t)| = |\vartheta_m + \vartheta_{m-1} + \cdots + \vartheta_{n+1}| \le |\vartheta_m| + |\vartheta_{m-1}| + \ldots |\vartheta_{n+1}|$$

$$\leq d_0 \sum_{k=n}^{m-1} \frac{q_0^k}{\Gamma(1+k\alpha)} t^{k\alpha}. \tag{42}$$

The series $\sum\limits_{k=0}^{\infty} \frac{q_0^k}{\Gamma(1+k\alpha)} t^{k\alpha}$ converges uniformly for $t \leq T$ to $E_\alpha(q_0 T^\alpha)$. Hence $u_m - u_n \to 0$ uniformly in t. This is the Cauchy criterion. Hence the sequence $u_n(x,t)$ converges uniformly in x and t (for $t \leq T$) to a function. Letting $n \to \infty$ in the recursion relation (20) gives the integral equation (18). The limiting function $u(x,t)$ solves (18) and hence the problem (1), (3).

The uniqueness of the solution follows the same considerations. For if ϑ is the difference of two solution of (18), we have

$$\vartheta(x,t) = -\int_0^t d\tau \int_{\mathbb{R}^2} Y(x-\xi, t-\tau) q(\xi_1, \tau) \vartheta(\xi, \tau) d\xi.$$

The derivation of (37) leads to the inequality

$$|\vartheta(x,t)| \leq \frac{d_0 q_0^n}{\Gamma(1+n\alpha)} t^{n\alpha} \sup_{(x,t)\in D_T} |\vartheta(x,t)|$$

for all n. Letting $n \to \infty$, we find that $\vartheta \equiv 0$.

Letting $m \to \infty$ in (38) leads to the inequality

$$|u(x,t) - u_n(x,t)| \leq d_0 \sum_{k=n}^{\infty} \frac{q_0^k}{\Gamma(1+k\alpha)} t^{k\alpha} = d_0 \Big(E_\alpha(q_0 t^\alpha) - \sum_{k=0}^{n-1} \frac{q_0^k}{\Gamma(1+k\alpha)} t^{k\alpha} \Big).$$

This is a bound for the error $|u - u_n|$ in terms of the maximum d_0 of the difference $|u_1 - u_0|$ between the first and second approximations. For fixed t the error bound approaches zero quite rapidly as $n \to \infty$.

AUXILIARY PROBLEM AND INVESTIGATION OF INVERSE PROBLEM (1)-(4)

Let $u(x,t)$ be a classical solution to Cauchy problem (1), (3) and f, u_0, u_1, g be enough smooth functions. We carry out the next converting of the inverse problem (1)-(4). Denote for this purpose the second derivative of $u(x,t)$ with respect to x_2, by $v(x,t)$ i.e. $v(x,t) := u_{x_2x_2}(x,t)$. Differentiating (1) and (3) twice in x_2, we get

$$\left({}^C\mathscr{D}_t^\alpha v\right)(x,t) - \triangle v(x,t) + q(x_1,t) v(x,t) = \partial_{x_2}^2 f(x,t), \quad x \in \mathbb{R}^2,\ 0 < t \leq T, \tag{43}$$

$$v(x,0) = (\partial_{x_2}^2 u_1)(x), \quad v_t(x,0) = (\partial_{x_2}^2 u_2)(x), \quad x \in \mathbb{R}^2, \tag{44}$$

To obtain an additional condition for the function $v(x,t)$, we note that the second term of Laplacian in (1) is $v(x,t)$. Setting $x_2 = 0$ in (1) and using equality (4), we obtain

$$v(x_1,0,t) = \left({}^C\mathscr{D}_t^\alpha g\right)(x_1,t) - (\partial_{x_1}^2 g)(x_1,t)$$

$$+q(x_1,t) g(x_1,t) - f(x_1,0,t), \quad x_1 \in \mathbb{R}, \quad 0 < t \leq T. \tag{45}$$

When the matching conditions $u_1(x_1,0) = g(x_1,0)$ and $u_2(x_1,0) = g_t(x_1,0)$ are fulfilled, it is easy to derive from (43)-(45) the equations (1)-(4).

In (43), introducing the notation $\partial_{x_2}^2 f(x,t) - q(x_1,t) v(x,t) =: F(x_1,x_2,t)$ and applying the formula (18) to direct problem (43), (44), we obtain the integral equation for determining $v(x,t)$:

$$v(x,t) = v_0(x,t) - \int_0^t d\tau \int_{\mathbb{R}^2} Y(x-\xi, t-\tau) q(\xi_1, \tau) v(\xi, \tau) d\xi, \tag{46}$$

where

$$v_0(x,t) := \int_{\mathbb{R}^2} Z_1(x-\xi, t)(\partial_{\xi_2}^2 u_1)(\xi) d\xi$$

$$+\int_{\mathbb{R}^2} Z_2(x-\xi,t)(\partial^2_{\xi_2}u_2)(\xi)d\xi+\int_0^t d\tau\int_{\mathbb{R}^2} Y(x-\xi,t-\tau)(\partial^2_{\xi_2}f)(\xi,\tau)d\xi. \tag{47}$$

Set

$$d_4:=|u_1|^{\gamma+2},\quad d_5:=|u_2|^{\gamma+2},\quad d_6:=\|f\|^{l+2}.$$

The following lemma is valid.

Lemma 2. *Let $q(x_1,t)\in C(H^{\alpha}(\mathbb{R});[0,T])$, $f(x,t)\in C(H^{\gamma_2+2}(\mathbb{R}^2);[0,T])$, $u_1(x)\in H^{\gamma+2}(\mathbb{R}^2)$, $u_2(x)\in H^{\gamma_1+2}(\mathbb{R}^2)$, where γ, γ_i, $i=1,2$ are defined in Lemma 1. Then, there exists a unique solution of the integral equation (8) with $v(x,t)\in C^{2-\alpha,2}(D_T)$ where $\alpha\in(1,2)$ and*

$$|v(x,t)|\le d_7E_{\alpha}(q_0T^{\alpha})=:d_{00} \tag{48}$$

estimate holds, where $d_7:=d_4+d_5T+\frac{d_6}{\Gamma(1+\alpha)}T^{\alpha}$.

The proof of Lemma 2 is similar to the proof of Lemma 1 and so we omitted it.

Let $M(d_{00})$ be the set of functions $v(x,t)\in C^{2-\alpha,2}(D_T)$ satisfying the inequality $|v(x,t)|\le d_{00}$ with a fixed positive constant d_{00} for $(x,t)\in D_T$. This constant is determined by (48).

Lemma 3. *Let $v(x,t)\in M(d_{00})$ and $\bar{v}(x,t)\in M(d_{00})$ be solutions of integral equation (47) with respective data sets $\{q,f_{x_2x_2},(u_1)_{x_2x_2},(u_2)_{x_2x_2}\}$ and $\{\bar{q},\bar{f}_{x_2x_2},(\bar{u}_1)_{x_2x_2},(\bar{u}_2)_{x_2x_2}\}$. Then the stability estimate*

$$|v-\bar{v}|\le d_8\left[|u_1-\bar{u}_1|^{\gamma+2}+|u_2-\bar{u}_2|^{\gamma_1+2}+\|f-\bar{f}\|^{\gamma_2+2}+\|q-\bar{q}\|^{\alpha}\right]$$

is hold, where d_8 will be defined bellow.

Proof. Let v, $\bar{v}$ denote the solutions to (46) corresponding to the functions q, $\bar{q}$. If the difference between two functions, whose only difference in notation is the overbar, is denoted by the same letter with a tilde ($\sim$), for instance $\tilde{v}=v-\bar{v}$, $\tilde{q}=q-\bar{q}$, etc., then equation (46) give the following equality

$$\tilde{v}(x,t)=\tilde{v}_0(x,t)-\int_0^t d\tau\int_{\mathbb{R}^2} Y(x-\xi,t-\tau)[\tilde{q}(\xi_1,\tau)v(\xi,\tau)+\bar{q}(\xi_1,\tau)\tilde{v}(\xi,\tau)]d\xi,$$

where

$$\tilde{v}_0(x,t):=\int_{\mathbb{R}^2} Z_1(x-\xi,t)(\partial^2_{\xi_2}\tilde{u}_1)(\xi)d\xi$$

$$+\int_{\mathbb{R}^2} Z_2(x-\xi,t)(\partial^2_{\xi_2}\tilde{u}_2)(\xi)d\xi+\int_0^t d\tau\int_{\mathbb{R}^2} Y(x-\xi,t-\tau)(\partial^2_{\xi_2}\tilde{f})(\xi,\tau)d\xi.$$

Note that the functions $\tilde{v}$ and $\tilde{v}_0$ included in it can be estimated on the basis of the a priori information on the problem data. Indeed, there is the obvious estimate

$$|\tilde{v}_0(x,t)|\le|\tilde{u}_1|^{\gamma+2}+|\tilde{u}_2|^{\gamma_1+2}+\frac{T^{\alpha}}{\Gamma(1+\alpha)}\|\tilde{f}\|^{\gamma_2+2}$$

and

$$|\tilde{v}(x,t)|\le|\tilde{v}_0(x,t)|+\frac{d_7}{\Gamma(1+\alpha)}T^{\alpha}E_{\alpha}(q_0T^{\alpha})\|\tilde{q}\|^{\alpha}+\bar{q}_0\int_0^t d\tau\int_{\mathbb{R}^2} Y(x-\xi,t-\tau)|\tilde{v}(\xi,\tau)|d\xi, \tag{49}$$

where $\bar{q}_0:=\|\bar{q}\|^{\alpha}$.

Let $d_9:=\max\left\{1,\frac{T^{\alpha}}{\Gamma(1+\alpha)},\frac{d_7}{\Gamma(1+\alpha)}T^{\alpha}E_{\alpha}(q_0T^{\alpha}),\bar{q}_0\right\}$. Applying the successive approximation method to inequality (11) with the help of the scheme

$$|\tilde{v}(x,t)|_0\le d_9\left[|\tilde{u}_1|^{\gamma+2}+|\tilde{u}_2|^{\gamma_1+2}+\|\tilde{f}\|^{\gamma_2+2}+\|\tilde{q}\|^{\alpha}\right],$$

$$|\tilde{v}(x,t)|_n\le\bar{q}_0\int_0^t d\tau\int_{\mathbb{R}^2} Y(x-\xi,t-\tau)|\tilde{v}(\xi,\tau)|_{n-1}d\xi,\quad n=1,2,...,$$

we arrive at the estimate

$$|\tilde{v}(x,t)|\le d_8\left[|\tilde{u}_1|^{\gamma+2}+|\tilde{u}_2|^{\gamma_1+2}+\|\tilde{f}\|^{\gamma_2+2}+\|\tilde{q}\|^{\alpha}\right], \tag{50}$$

where $d_8 := d_9 d_{00}$, which will be used to solve the inverse problem. Indeed the expression (50) is the stability estimate for the solution to the Cauchy problem (43) and (44). The uniqueness for this solution follows from (50).

Now, we investigate the inverse problem (43)-(45). Setting in (46) $x = 0$ and using additional condition (45), after simple converting, we get the following integral equation for determining $q(x_1,t)$:

$$q(x_1,t) = q_0(x_1,t) - \frac{1}{g(x_1,t)} \int_0^t d\tau \int_{\mathbb{R}^2} Y(x_1-\xi_1, \xi_2, t-\tau) q(\xi_1,\tau) v(\xi,\tau) d\xi, \tag{51}$$

$$q_0(x_1,t) = \frac{1}{g(x_1,t)} \left[v_0(x_1,0,t) - \left({}^C\mathscr{D}_t^\alpha g\right)(x_1,t) + \partial_{x_1}^2 g_1(x_1,t) \right],$$

where $v_0(x_1,0,t)$ is defined by (47) when $x_2 = 0$.

The solution of integral equation (46) depends on q, i.e. $v = v(x,t;q)$.

We introduce an operator $\mathscr{A}$ defining it by the right hand side of (51).

$$\mathscr{A}[q](x_1,t) = q_0(x_1,t) - \frac{1}{g(x_1,t)} \int_0^t d\tau \int_{\mathbb{R}^2} Y(x_1-\xi_1, \xi_2, t-\tau) q(\xi_1,\tau) v(\xi,\tau;q) d\xi.$$

Then the equation (51) is written in a more convenient form as

$$q(x_1,t) = \mathscr{A}[q](x_1,t). \tag{52}$$

Let $d_{10} := \|q_0\|^\alpha$. Fix a number $\rho > 0$ and consider the ball

$$B_T[q_0,r] := \{q \in C(H^\alpha(\mathbb{R}),[0,T]) : \|q-q_0\|^\alpha \le r\}, \alpha \in (1,2).$$

Theorem 1. *Let $f(x,t) \in C(H^{\gamma_2+2}(\mathbb{R}^2);[0,T])$, $u_1(x) \in H^{\gamma+2}(\mathbb{R}^2)$, $u_2(x) \in H^{\gamma_1+2}(\mathbb{R}^2)$, $g(x_1,t) \in C^2(H^\alpha(\mathbb{R});[0,T])$, $u_i(x_1,0) = g(x_1,0)$, $i = 1,2$ and the condition*

$$\|g(x_1,t)\|^\alpha \ge g_0 > 0, \quad x_1 \in \mathbb{R}, \quad t \in [0,T]$$

is fulfilled, then at sufficiently small $T_0 \in (0,T)$ the solution to the inverse problem (1)-(4) on the intercept $\mathbb{R} \times [0,T]$ exists, unique and belongs to the class $C(H^\alpha(\mathbb{R});[0,T_0])$.

Proof. Let us first prove that the operator $\mathscr{A}$, defined by (52), is a contraction on the Banach space $B_T[q_0,r]$ into itself, if the final time $T > 0$ is small enough. Indeed, for any continuous function $q(x_1,t)$, the function $\mathscr{A}[q](x_1,t)$ calculated using formula (52) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\|A[q]-q_0\|^\alpha \le \frac{g_0^{-1}T^\alpha}{\Gamma(1+\alpha)} d_{00} \|q\|^\alpha,$$

here we have used the estimate (48). Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $q(x_1,t)$ belongs to the ball $B_T[q_0,r]$ implies the inequality

$$\|q\|^\alpha \le r + d_{10}. \tag{53}$$

Therefore, we only strengthen the inequality if we replace $\|q\|^\alpha$ in this inequality with the expression $r + d_{10}$. Performing these replacements, we obtain the estimate

$$\|A[q]-q_0\|^\alpha \le \frac{g_0^{-1}d_7}{\Gamma(1+\alpha)} T^\alpha (r+d_{10}) E_\alpha((r+d_{10})T^\alpha).$$

Let T_1 be a positive root of the equation

$$r_1(T) = \frac{g_0^{-1}d_7}{\Gamma(1+\alpha)} T^\alpha (r+d_{10}) E_\alpha((r+d_{10})T^\alpha) = r.$$

Then for $T \in [0,T_1]$ we have $A[q](x_1,t) \in B_T[q_0,r]$.

Now consider two functions $q(x_1,t)$ and $\bar{q}(x_1,t)$ belonging to the ball $B_T[q_0,r]$ and estimate the distance between their images $A[q](x_1,t)$ and $A[\bar{q}](x_1,t)$ in the space $C(H^\alpha(\mathbb{R}),[0,T])$. The function $\bar{v}(x,t)$ corresponding to $\bar{q}(x_1,t)$

satisfies the integral equation (46) with the functions $(u_1)_{x_2x_2} = (\bar{u}_1)_{x_2x_2}$, $(u_2)_{x_2x_2} = (\bar{u}_2)_{x_2x_2}$, $f_{x_2x_2} = \bar{f}_{x_2x_2}$. Composing the difference $A[q](x_1,t) - A[\bar{q}](x_1,t)$ with the help of equation (46) and then estimating its norm, we obtain

$$\|A[q](x_1,t) - A[\bar{q}](x_1,t)\|^{\alpha} \leq \frac{g_0^{-1}}{\Gamma(1+\alpha)} T^{\alpha} \Big[\|v\| \|q-\bar{q}\|^{\alpha} + \|\bar{q}\|^{\alpha} \|v-\bar{v}\| \Big]. \tag{54}$$

Using inequality (48) and the estimate (49) with $(u_1)_{x_2x_2} = (\bar{u}_1)_{x_2x_2}$, $(u_2)_{x_2x_2} = (\bar{u}_2)_{x_2x_2}$, $f_{x_2x_2} = \bar{f}_{x_2x_2}$, we continue the previous inequality in the following form:

$$\|A[q](x_1,t) - A[\bar{q}](x_1,t)\|^{\alpha} \leq \frac{g_0^{-1} d_7}{\Gamma(1+\alpha)} T^{\alpha} \Big[E_{\alpha}(q_0 T^{\alpha}) + d_9 \bar{q}_0 \Big] \|q-\bar{q}\|^{\alpha}.$$

The functions $q(x_1,t)$ and $\bar{q}(x_1,t)$ belong to the ball $B_T[q_0,r]$, and hence for each of these functions one has inequality (53). Note that the function on the right-hand side in inequality (54) at the factor $\|q-\bar{q}\|^{\alpha}$ is monotone increasing with $\|q\|^{\alpha}$, $\|\bar{q}\|$ and T. Consequently, replacing and in inequality (54)(including in d_9) with $r+d_{10}$ will only strengthen the inequality. Thus, we have

$$\|A[q](x_1,t) - A[\bar{q}](x_1,t)\|^{\alpha} \leq \frac{g_0^{-1} d_7}{\Gamma(1+\alpha)} T^{\alpha} \Big[E_{\alpha}((r+d_{10}) T^{\alpha}) + d_9 (r+d_{10}) \Big] \|q-\bar{q}\|^{\alpha}.$$

Let T_2 be a positive root of the equation

$$r_2(T) = \frac{g_0^{-1} d_7}{\Gamma(1+\alpha)} T^{\alpha} \Big[E_{\alpha}((r+d_{10}) T^{\alpha}) + d_9 (r+d_{10}) \Big] = 1.$$

Then for $T \in [0,T_2]$ we have that the distance between the functions $A[q](x_1,t)$ and $A[\bar{q}](x_1,t)$ in the function space $C(H^{\alpha}(\mathbb{R}),[0,T])$ is not greater than the distance between the functions $q(x_1,t)$ and $\bar{q}(x_1,t)$ multiplied by $r_2(T) < 1$. Consequently, if we choose $T_0 = \min\{T_1,T_2\}$, then the operator $\mathscr{A}$ is a contraction in the ball $B_T[r,q_0]$. However, in accordance with the Banach theorem, the operator A has a unique fixed point in the ball $B_T[r,q_0]$; i.e., there exists a unique solution of the equation (52). Theorem 1 is proven.

Let T be a positive fixed number. Consider the set $\Omega_1(\mu_1)$ (μ_1 is some positive fixed number) of the given functions (f,u_1,u_2,g) for which all conditions from Theorem 1 are fulfilled and so that $\max\{\|f\|^{l+2}, |u_1|^{\gamma+2}, |u_2|^{\gamma+2}, \|g\|^{\alpha}\}$. By $\Omega_2(\mu_2)$ we denote the class of functions $q(x_1,t) \in C(H^{\alpha}(\mathbb{R}),[0,T])$, satisfying the inequality $\|q\|^{\alpha} \leq \mu_2$ with some fixed positive number μ_2.

Theorem 2. *Suppose (f,u_1,u_2,g) and $(\bar{f},\bar{u}_1,\bar{u}_2,\bar{g})$ are in $\Omega_1(\mu_1)$. Then for the solution of the inverse problem (43)-(45) the following stability estimate is valid:*

$$\|q-\bar{q}\|^{\alpha} \leq d_{11} \Big(|u_1-\bar{u}_1|^{\gamma+2} + |u_2-\bar{u}_2|^{\gamma_1+2} + \|f-\bar{f}\|^{\gamma_2+2} + \|g-\bar{g}\|^{\alpha} \Big),$$

where the constant d_{11} depends only on T,α,μ_1,μ_2.

The proof of Theorem 2 is identical to the proof of Theorem 2 in [28]. So we don't repeat it and the following uniqueness theorem for any $T > 0$ follows from Theorem 2:

Theorem 3. *Let the functions q,f,u_1,u_2,g and $\bar{q},\bar{f},\bar{u}_1,\bar{u}_2,\bar{g}$ have the same meaning as in Theorem 2. Besides, if $f=\bar{f}$, $u_1=\bar{u}_1$, $u_2=\bar{u}_2$, $g=\bar{g}$ for D_T, then $q(x_1,t) \equiv q(x_1,t)$, $(x_1,t) \in \mathbb{R}\times[0,T]$.*

CONCLUSION

In this paper, we investigated an inverse space and time-dependent source problem of a two dimensional time-fractional wave equation with additional condition. The existence and uniqueness of a solution for the direct problem are obtained by the integral equation theory. Then, the well-posedness for the inverse problem are obtained by the Contraction Theorem.

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