ISSN 2181-1296

ILMIY AXBOROTNOMA

НАУЧНЫЙ ВЕСТНИК **SCIENTIFIC JOURNAL**

2022-yil, 3-son (133) **ANIQ FANLAR SERIYASI**

Matematika, Mexanika, Informatika, Fizika

Samarqand viloyat matbuot boshqarmasida ro‘yxatdan o‘tish tartibi 09-25.
Jurnal 1999-yildan chop qilina boshlagan va OAK ro‘yxatiga kiritilgan.

Obuna indeksi – yakka tartbidagi obunachilar uchun **- 5583,**
tashkilot, korxonalar uchun **- 5584**

MUNDARIJA / СОДЕРЖАНИЕ / CONTENTS

MATEMATIKA / МАТЕМАТИКА / MATHEMATICS

Absalamov A.T., Ziyadinov B.A.
On one of the family of evolution operators.. 4-9
Хасанов Т.Г.
Задача Коши для Уравнения кортевега-де фриза с нагруженными членами и источником в классе быстроубывающих функций.. 10-19
Akramova D.I., Soleev A.
Newton polyhedra in estimates for the Fourier transform of characteristic functions and convolution operators.. 20-32
Alladustova I.U.
The number of eigenvalues of the discrete schrödinger operator associated to a system of two bosons on a lattice 33-38
Almuratov F.M.
Asymptotics for eigenvalues of schrödinger operator associated to one-particle systems on one dimensional lattice.. 39-46
Akhmadova M.O.
On the existence of roots of some matrix-type functions, estimates for the roots and their number 47-50
Akhmedov A.B., Kuldibaeva L.A., Kholmanov N.Y.
Modified non-classical theory of plate bending.. 51-57
Ибрагимов А.М., Неъматов А.Б.
Спектральные свойства оператора дирака с периодическим потенциалом на всей прямой 58-66
Ochilov Z.Kh.
Integral geometry problems with perturbation in the family of parabolas with a weight function of a special form.. 67-73
Икромов И.А., Сафаров А.Р., Вахобова Г.Б.
Об оценке тригонометрических интегралов с неполной квадратичной фазой.................... 74-81
Собиров Ш.К.
Интегрирование нагруженного модифицированного уравнения кортевега-де фриза с самосогласованным источником интегрального типа в случае простых собственных значений.. 82-89

MEXANIKA / МЕХАНИКА / MECHANICS

Хужаёров Б.Х., Сайдуллаев У.Ж., Шадманов И.Э.
Численное решение осесимметричной задачи фильтрации суспензий с образованием упруго - пластического кейк-слоя.. 90-95

INFORMATIKA / ИНФОРМАТИКА / INFORMATICS

Nazarov F.M., Yarmatov Sh.Sh.
Artificial intellectual mechanisms in control systems and their application............................... 96-105

FIZIKA / ФИЗИКА / PHYSICS

Jumabayev A., Xushvaqtov H., Absanov A., Sharifov G‘., Ernazarov Z., Odilova S.
Meta-ksilol va uning eritmalarida molekulalararo ta’sirlarni kombinatsion sochilish spektrlarida namoyon bo‘lishi.. 106-110
Кувандиков О.К., Ражабов Р.М., Шодиев З.М., Амонов Б.У., Сулайманов О.А., Ахтамов Ж.Ш.
Магнитные свойства редкоземельных хромоборатов $RCr_3(BO_3)_4$ (R=Dy, Ho) при высоких температурах.. 111-114
Ахмеджанов Ф.Р., Тугалов Ф.Б.
Характеристики акустических волн в кристаллах германата-лития……………………….. 115-118

Vapayev M.E., Sobirov B.R., Kamalov Sh.R., Davletov I.Y., Boltaev G.S.
Lazer nurlanishi ta'sirida mis sirtida hosil qilingan plazmasining optik xususiyati……………. 119-124

Шаронов И.А., Алибеков А.С., Ахмедов Я.А., Салимов М.И., Сандибоев А.Н., Холматов А.
Нейтронно-активационный и гамма-спектроскопический анализ при исследовании некоторых объектов археометаллургии.. 125-138

Mamatqulov O.B., To'xtayev U.U., Tog'ayev B.S., Umirzaqov E.A., Amonova R.O'., Temirov Sh.S., Sayfiyev Sh.Sh., Qilichev A.Q.
Lalmi ekinlarining ayrim donli mahsulotlarida tabiiy ^{40}k radionuklidining to'planishini o'rganish………………………………………………………………………………………. 139-144

Mustafayev T.Sh., Axmedjanov F.R., Eshniyozov O.I.
Kvars kristallarida akustik to'lqinlarning so'nish anizotropiyasi……………………………… 145-151

Mamatqulov O.B., Safarov A.N., To'xtayev U.U., Tog'ayev B.S., Umirzaqov E.A., Nurmurodov L.T., Amonova R.O'.,Yusupov M.N., Haydarov U.E., Sayfiyev Sh.Sh., Temirov Sh.S.
Samarqand viloyati Nurobod tumani ayrim hududlaridagi ichimlik suvlarining radioaktivligi 152-156

Рахматуллаев И.А., Чернега Н.В., Давронов М.Х., Хайдаров Х.С.
Одно- и двухфотонно-возбуждаемая люминесценция микропорошков диоксида титана при лазерном возбуждении…………………………………………………………………. 157-163

Ergashev K.E, Tojiyev R.I., Jurayev B.Sh., Shaymanov A.M., Boyqobilov T.I.
(16447) Vauban asteroidining optik kuzatuvlari va faza funksiyasi tahlili……………………… 164-170

Payzullayev A.N., Gafurova M.V., Allayev B.A., Tellyayev S.K., Mirzayev S.Z.
Nanosuyuqlikning qovushqoqligi va barqarorligining kremniy dioksidi nanozarrachalari chiziqli o'lchamiga bog'liqligi.. 171-175

Mualliflarga

УДК: 517.518.5

NEWTON POLYHEDRA IN ESTIMATES FOR THE FOURIER TRANSFORM OF CHARACTERISTIC FUNCTIONS AND CONVOLUTION OPERATORS

D.I.Akramova, A.Soleev

Самаркандский государственный университет

Annotation. In this paper, we prove analog of Randol Theorem, on estimates for the Fourier transform of indicator functions of convex domains with analytic boundaries, for some classes of non-convex domains. Also, we obtain estimates for convolution operators related to solutions to the strictly hyperbolic equations.

Keywords: Newton polyhedra, oscillatory integrals, height of function, convolution operator, boundedness.

Xarakteristik funksiyalar Fur'e almashtirishlarini va o'rama operatorlarini baholashda Nyuton ko'pyoqliklari

Annotatsiya. Ushbu maqolada, analitik chegarali kompakt sohalar indikator funksiyasi Fur'e almashtirishining baholari haqidagi Rendol teoremasining analogi ba'zi noqavariq sohalar singi uchun isbotlangan. Bundan tashqari, biz qat'iy giperbolik tenglamalar yechimlari bilan bog'langan o'rama tipidagi operatorlar uchun ham baholar olamiz.

Kalit so'zlar: Nyuton ko'pyoqligi, tebranuvchan integral, funksiyaning balandligi, o'rama operatori, chegaralanganlik.

Многогранники Ньютона в оценках преобразования Фурье характеристических функций и сверточных операторов

Аннотация. В этой работе, мы докажем аналог теоремы Рендела об оценках преобразования Фурье индикатора выпуклых компактных областей с аналитической границей, для некоторых классов невыпуклых областей. А также мы получим оценки для сверточных операторов, связанных с решениями строго гиперболических уравнений.

Ключевая слова: Многогранник Ньютона, осцилляторный интеграл, высота функции, оператор свертки, ограниченность.

Introduction. Let $D \subset \mathbb{R}^{n+1}$ be a compact domain with C^∞ boundary and $u \in C^\infty(\mathbb{R}^{n+1})$ be a smooth function. We consider the integral

$$\hat{u}_D(\xi) = \int_D u(x) e^{i(x,\xi)} dx,$$

where (x,ξ) is the inner product of the vectors x and ξ, e.g. $(x,\xi) := x_1\xi_1 + x_2\xi_2 + ... + x_{n+1}\xi_{n+1}$.

If $u(x) = 1$ for any $x \in D$ then $\hat{u}_D$ coincides with the Fourier transform of the indicator function of the set D.

In this paper, we will assume that ∂D is a smooth compact and connected hypersurface. Note that the case when ∂D is a finite union of such hypersurfaces can be treated by similar arguments.

If D is a strictly convex (e.g. convex and the Gaussian curvature of the hypersurface does not vanish) then due to the classical Hlawka [3] result the following

$$\hat{u}_D(\xi) = O(|\xi|^{-\frac{n+2}{2}}) \quad (as \quad |\xi| \to \infty)$$

asymptotic relation holds true. But, for the general domains by using integration by parts arguments one can get only the relation

$$\hat{u}_D(\xi) = O(|\xi|^{-1}) \quad (as \quad |\xi| \to \infty).$$

By B. Randol had been introduced [6] the following maximal function:

$$M(\omega) := \sup_{r>0} r^{\frac{n+2}{2}} |\hat{u}_D(r\omega)|, \tag{1}$$

in order to investigate behavior of the function $\hat{u}_D(\xi)$, when $|\xi|$ gets large, where S^n is the unit sphere in $\mathbb{R}^{n+1}$ centered at the origin and $\omega \in S^n$, also we write ξ in polar coordinates system e.g $\xi = r\omega$.

The function defined by the relation (1) is called to be a Randol maximal function.

It is easy to see that M is a Borel's measurable function (see section 2).

In this paper, we consider the problem: Find LUP (least upper bound) of the set:

$$\{p \in [0,\infty) : M \in L^p(S^n)\}.$$

Also we apply the estimates for the Fourier transform of measures to a boundedness problem for convolution operators.

B. Randol [6] proved that if D is convex compact domain with analytic boundary then there exists an $\varepsilon > 0$ such that $M \in L^{2+\varepsilon}(S^n)$. Further, I. Svensson [11] obtain analogical result for convex domain with C^∞ boundary of which has finite line type. It means that any tangent hyperplane to ∂D does not contain any straight line having contact of infinite order with ∂D. Note that surface in $\mathbb{R}^3$ may be convex but, not finite line type. For example, the usual cylinder with a circle base is convex but, not finite line type. Surely, one can construct compact convex domain with C^∞ boundary and the boundary has no finite line type. Surely, if ∂D is convex and analytic then it has finite line type.

The paper organized as follows: In the next section 2 we reduce some facts about Randol maximal functions. The section 3 contains results on L^p-estimates for the Randol's maximal functions. Then we consider estimates for the convolution operators in the last section of the paper.

2. Preliminaries

In this section we will give necessary definitions and notions. These definitions will be used through the paper. More general notion of Newton polyhedra and normal cones are given in the book [2] (also see [10] for applications of Newton polyhedrons to problems related to behavior of solutions to ODE).

Let ϕ be a smooth function. Consider the associated Taylor series

$$\phi(x) : \sum_{|\alpha|=0}^{\infty} c_\alpha x^\alpha \tag{2}$$

of ϕ centered at the origin. Where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \setminus \{0\}, \mathbb{Z}_0 := \{0\} \cup \mathbb{N}, |\alpha| := \alpha_1 + \ldots + \alpha_n, x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The set

$$T(\phi) := \{\alpha \in \mathbb{Z}_+^n \setminus \{0\} : c_\alpha = \frac{1}{\alpha_1! \ldots \alpha_n!} \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} \phi(0) \neq 0\}$$

will be called the Taylor support of ϕ at the origin. If the function ϕ is analytic at the origin then the last series converge, provided $|x|$ is sufficiently small. Otherwise, the series (2) will be considered as a formal power series.

Similarly, the Taylor support of a function ϕ at the point x^0 denoting by $T_{x^0}(\phi)$, which is defined on a neighborhood of the point x^0.

The Newton polyhedron $\mathcal{N}(\phi)$ of ϕ at the origin is defined to be the convex hull of the union of all the octants $\alpha+\mathbb{R}^n_+$ in $\mathbb{R}^n$, with $\alpha \in T(\phi)$. The associated Newton diagram $\mathcal{N}_d(\phi) \subset \mathcal{N}(\phi)$ in the sense of Varchenko [13] is the union of all compact faces of the Newton polyhedron; here, by a face, we shall mean a face of dimension bigger than one or a vertex.

We shall use coordinates t for points in the space $\mathbb{R}^n_t$ containing the Newton polyhedron, in order to distinguish this space from the $\mathbb{R}^n_x$ - space.

The Newton distance in the sense of Varchenko, or shorter distance, $d = d(\phi)$ between the Newton polyhedron and the origin is given by the coordinate d of the point $(d,\dots,d)$ at which the bi-sectrix $t_1 = t_2 = \dots = t_n$ intersects the boundary of the Newton polyhedron.

The principal face $\pi(\phi)$ of the Newton polyhedron of ϕ is the face of minimal dimension containing the point $(d,\dots,d)$. Deviating from the notation in [13], we shall call the series

$$\phi_{pr}(x) := \sum_{\alpha\in\pi(\phi)} c_\alpha x^\alpha$$

the principal part of ϕ. In case that $\pi(\phi)$ is compact, ϕ_{pr} is a mixed homogeneous polynomial; otherwise, we shall consider ϕ_{pr} as a formal power series.

Note that the distance between the Newton polyhedron and the origin depends on the chosen local coordinate system in which ϕ is expressed. By a local coordinate system (at the origin) we shall mean a smooth coordinate system defined near the origin which preserves 0. The height of the smooth function ϕ is defined by

$$h(\phi) := sup\{d_y\},$$

where the supremum is taken over all local coordinate systems y at the origin, and where d_y is the distance between the Newton polyhedron and the origin in the coordinates y.

A given coordinate system x is said to be adapted to ϕ if $h(\phi) = d_x$. If restrict ourselves with linear coordinates system then we came linear adapted coordinates system and a linear height $h_{lin}(\phi)$. By definition we have $d(\phi) \le h_{lin}(\phi) \le h(\phi)$. If ϕ is a smooth convex function then there exists an adapted coordinates system, moreover $h_{lin}(\phi) = h(\phi)$ (see [8] and also see [5] for more general case of convex functions).

In [4] it is proved that one can always find an adapted local coordinate system in two dimensions, thus generalizing the fundamental work by Varchenko [13] who worked in the setting of real-analytic functions ϕ without multiple components.

Notice that if the principal face of the Newton polyhedron $\mathcal{N}(\phi)$ is a compact face, then it lies on a unique principal hyperplane

$$L := \{t \in \mathbb{R}^n : (\kappa, t) = 1\},$$

with $\kappa_j > 0, j = 1,\dots,n$. The weight $\kappa = (\kappa_1,\dots,\kappa_n)$ will be called the principal weight associated to ϕ. It induces dilations $\delta_r(x) := (r^{\kappa_1}x_1,\dots,r^{\kappa_n}x_n)$, $r > 0$, on $\mathbb{R}^n$, so that the principal part ϕ_{pr} of ϕ is κ-homogeneous of degree one with respect to these dilations, i.e., $\phi_{pr}(\delta_r(x)) = r\phi_{pr}(x)$ for every $r > 0$, and we find that

$$d = \frac{1}{\kappa_1+\dots+\kappa_n} = \frac{1}{|\kappa|},$$

where we used notation: $|\kappa| := \kappa_1+\dots+\kappa_n$.

More generally, we assume that $\kappa = (\kappa_1, \ldots, \kappa_n)$ is any weight with $0 < \kappa_j, j = 1, \ldots, n$ such that the hyperplane $L_\kappa := \{t \in \mathbb{R}^n : (\kappa, t) = 1\}$ is a supporting hyperplane to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ (recall that a supporting hyperplane to a convex set K in the Euclidean space is a hyperplane such that K is contained in one of the two closed half-spaces into which the hyperplane divides the space and such that this hyperplane intersects the boundary of K). Then $L_\kappa \bigcap \mathcal{N}(\phi)$ is a face of $\mathcal{N}(\phi)$, i.e., either a compact face or a vertex, and the κ-principal part of ϕ defined by

$$\phi_\kappa(x) := \sum_{\alpha \in L_\kappa} c_\alpha x^\alpha$$

is a non-trivial polynomial which is κ-homogeneous of degree 1 with respect to the dilations associated to this weight as before. By definition, we then have $\phi(x) = \phi_\kappa(x) +$ terms of higher κ-degree. The same notions can be defined for arbitrary dimension n. However, the adapted coordinates system exists only for the case when $n = 2$. It is worth to note that, any local coordinates system is adapted in the case $n = 1$. But, for the case $n \geq 3$ as had been shown by A.N. Varchenko adapted coordinates system does not exist. However, if ϕ is a convex smooth function having finite line type then linearly adapted coordinates system exists and moreover $h_{lin}(\phi) = h(\phi)$. The result belongs to H. Schultz. More precisely, by H. Schultz [8] it is proved existence of adapted coordinates system for smooth convex functions having finite line type. More general smooth convex functions had been considered in the paper [5].

Let ϕ be a smooth function defined in a neighborhood of the origin and $\phi(0) = 0$ and $\nabla\phi(0) = 0$.

Definition. *A rank of the critical point $x = 0$ is called to be the rank of the hessian matrix at the origin, e.g. rank of the matrix $\{\partial_j \partial_k \phi(0)\}_{j,k=1}^n$, in particular, if $rank\{\partial_j \partial_k \phi(0)\}_{j,k=1}^n = n$ then $x = 0$ is called to be a non-degenerate critical point.*

The classical Morse Lemma yields:

Lemma 2.1 *If a smooth function ϕ has critical point at the origin with rank k then there exists a diffeomorphic map $\varphi(y)$ defined in a neighborhood of the origin such that $\varphi(0) = 0$ and the following relation holds true:*

$$\phi(\varphi(y)) = (By', y') + \phi_1(y''), \tag{3}$$

where $y' = (y_1, \ldots, y_k), y'' = (y_{k+1}, \ldots, y_n)$, and B is a $k \times k$ non-degenerate symmetric matrix and also ϕ_1 is a smooth function of y'' having order 3 at the origin, i.e. all derivatives up to order 3 at the origin vanish.

The following result holds:

Proposition 2.2 *Let ϕ be a smooth function defined in a neighborhood of the origin of $\mathbb{R}^n$, satisfying the conditions $\phi(0) = 0$ and $\nabla\phi(0) = 0$. If rank of the critical point at the origin is $n-2$ then there exists a smooth adapted coordinates system.*

Indeed, by the Lemma 2.1 we first, reduce the function to the form (3) and then applying result of the paper [4] we came to a proof of the Proposition 2.2.

Remark. Note that in the general case $n \geq 3$ adapted coordinates system does not exist. So, in the statement 2.2 the "rank" condition is essential.

Theorem 2.3 *If D is a compact domain with smooth boundary then M is Borel's measurable function and finite for a.e. $\omega \in S^n$.*

Indeed, one can define the following sequence of functions:

$$M_k(\omega) = \max_{0 \leq r \leq k} r^{\frac{n+2}{2}} |\hat{u}_D(r\omega)| \quad k = 1, 2, \ldots,.$$

Since $r^{\frac{n+2}{2}} \,|\, \hat{u}_D(r\omega) \,|$ is a continuous function on $S^n \times \mathbb{R}_+$, so is M_k on S^n. Consequently M_k is a Borel's measurable function. On the other hand M_k is a monotone increasing sequence. Hence, there exists the limit

$$\lim_{k\to\infty} M_k(\omega) = M(\omega).$$

Now, we show that the limit is finite for a.e. $\omega \in S^n$.

Let $N : \partial D \to S^n$ be the Gaussian normal map. Then $x \in \partial D$ is the critical point of the map N if and only if $K(x) = 0$ (where $K(x)$ is the Gaussian curvature of the hypersurface at the point $x \in \partial D$).

Since by the classical A. Sard's theorem [7] the set of critical values has zero Lebesque measure. Consequently, the preimage $N^{-1}(\omega)$ does not contain any critical point for a.e. $\omega \in S^n$. By the inverse mapping Theorem $N^{-1}(\omega)$ is a discrete set (any point of the set is isolated) for any regular value of $\omega \in S^n$ and since ∂D is a compact set, hence the cardinality of the set $N^{-1}(\omega)$ is finite.

Let $\omega \in S^n$ be a regular value of the normal map. Thus, for any $x^0 \in N^{-1}(\omega)$ we have $K(x^0) \neq 0$. It means that the phase function $(x,\omega)\,|_{\partial D}$ has non-degenerate critical point at $x = x^0$. By stationary phase method if $x^0 \in \partial D$ is a non-degenerate critical point and φ is a smooth function concentrated in a sufficiently small neighborhood of the point $(x^0;\omega) \in \partial D \times S^n$ then the following asymptotic relation

$$\int_{\partial D} \varphi(x,\omega) e^{ir(x,\omega)} dS(x) = O(r^{-\frac{n}{2}}), \quad as \quad r \to \infty$$

holds true. Moreover, the last asymptotic relation is locally uniform with respect to $\omega \in S^n$. More precisely, if $\omega_0 \in S^n$ is not any critical value of the normal map N then there exists a neighborhood $V(\omega_0)$ of ω_0 and a constant $c > 0$ such that for any $\omega \in V(\omega_0)$ the inequality

$$|\, O(r^{-\frac{n}{2}}) \,| \le \frac{C}{r^{n/2}}$$

holds true.

Moreover, if $\omega_0 \in S^n$ is not critical value and $supp\tilde{\varphi}(\cdot,\omega)$ contains all critical points and $\tilde{\varphi}(x,\omega) \equiv 1$ for $\omega \in V(\omega_0)$ in a neighborhood of the critical set $N^{-1}(\omega_0)$.

Then

$$\int_{\partial D} a(x,\omega)(1-\tilde{\varphi}(x,\omega)) e^{ir(x,\omega)} dS = O(r^{-N}) \quad (as \quad |\, r \,| \to \infty),$$

where $a \in C^\infty(\partial D \times S^n)$ and N is a natural number. The number N can be chosen as large as we wish, so, we may assume $N \ge \frac{n}{2}$.

Thus, we came to the following conclusion: If $\omega_0 \in S^n$ is not a critical value of the normal map then there exists a neighborhood $V(\omega_0)$ of the point ω_0 such that for any $a \in C^\infty(\partial D \times S^n)$ and for any $\omega \in V(\omega_0)$ the asymptotic relation

$$\int_{\partial D} a(x,\omega) e^{ir(x,\omega)} dS(x) = O(r^{-\frac{n}{2}}) \quad (as \quad r \to \infty)$$

holds true.

Proposition 2.4 *If* ∂D *is a bounded closed smooth hypersurface then for a.e.* $\omega \in S^n$ $M(\omega)$ *is finite and*

$$M(\omega) = \lim_{k\to\infty} M_k(\omega).$$

The last limit exist for a.e. $\omega \in S^n$. Since M_k is a continuous function then M is a Borel's measurable function.

Proof. We use the following divergence Theorem: If $F : \overline{D} \to \mathbb{R}^{n+1}$ is a smooth vector function (vector fields) then

$$\int_D div(F)dx = \int_{\partial D}(F,\nu)dS,$$

where ν is the unit outer normal to the hypersurface ∂D and $div(F) := (\nabla, F)$. Let's define the vector function $F(x,\xi) := e^{i(\xi,x)}u(x)\frac{\xi}{i\,|\,\xi\,|^2}$ assuming $\xi \neq 0$. Then, we have

$$div(F) = \sum_{j=1}^{n+1}\frac{\partial F_j}{\partial x_j} = \frac{1}{i\,|\,\xi\,|^2}\sum_{j=1}^{n+1}\xi_j\frac{\partial}{\partial x_j}\left(e^{i(\xi,x)}u(x)\right) = e^{i(\xi,x)}u(x) + \frac{i(\xi,x)}{i\,|\,\xi\,|}(\omega,\nabla u),$$

where $\omega := \frac{\xi}{|\,\xi\,|}$.

Now, we define the new function $u_1(x,\omega) := (\omega,\nabla u)$ and have

$$divF = e^{i(\xi,x)}u(x) + \frac{e^{i(\xi,x)}}{i\,|\,\xi\,|}u_1(x,\omega).$$

Therefore

$$\int_D e^{i(\xi,x)}u(x)dx = \frac{1}{i\,|\,\xi\,|}\int_{\partial D}e^{i(\xi,x)}u(x)(\omega,\nu)dS - \frac{1}{i\,|\,\xi\,|}\int_D e^{i(\xi,x)}u_1(x,\omega)dx.$$

By a recursion formula we define

$$F_1(x,\omega) = e^{i(\xi,x)}u_1(x,\omega)\frac{\xi}{i\,|\,\xi\,|^2}.$$

Again by using divergence Theorem we get

$$\int_D e^{i(\xi,x)}u_1(x,\omega)dx = \frac{1}{i\,|\,\xi\,|}\int_{\partial D}e^{i(\xi,x)}u_1(x,\omega)(\omega,\nu)dS - \frac{1}{i\,|\,\xi\,|}\int_D e^{i(\xi,x)}u_2(x,\omega)dx,$$

where $u_2(x,\omega) := (\omega,\nabla u_1(x,\omega))$. Continuing the procedure we obtain:

$$\int_D e^{i(\xi,x)}u(x)dx = \frac{1}{i\,|\,\xi\,|}\int_{\partial D}e^{i(\xi,x)}u(x)(\omega,\nu)dS + \sum_{j=1}^{N}\frac{(-1)^j}{(i\,|\,\xi\,|)^{j+1}}\int_{\partial D}e^{i(\xi,x)}u_j(x,\omega)(\omega,\nu)dS +$$

$$+\frac{(-1)^{N+1}}{(i\,|\,\xi\,|)^{N+1}}\int_D e^{i(\xi,x)}u_N(x,\omega)dx = \frac{1}{i\,|\,\xi\,|}\int_{\partial D}e^{i(\xi,x)}a\left(x,\frac{1}{|\,\xi\,|},\omega\right)dS +$$

$$+\frac{(-1)^{N+1}}{(i\,|\,\xi\,|)^{N+1}}\int_D e^{i(\xi,x)}u_N(x,\omega)dx,$$

where

$$a\left(x,\frac{1}{|\,\xi\,|},\omega\right) = u(x)(\omega,\nu) + (\omega,\nu)\sum_{j=1}^{N}\frac{(-1)^j}{(i\,|\,\xi\,|)^j}u_j(x,\omega).$$

Thus, since $\hat{u}_D(\xi)$ is an analytic function, then taking $N+1 \geq \frac{n+2}{2}$ we have

$$M(\omega) \leq \sup_{r\geq 1} r^{\frac{n+2}{2}}\,|\,\hat{u}_D(r\omega)\,| + \sup_{0\leq r\leq 1}|\,\hat{u}_D(r\omega)\,|$$

and

$$M(\omega) \leq \sup_{r\geq 1} r^{\frac{n}{2}}\left|\int_{\partial D}e^{ir(\omega,x)}a(x,\frac{1}{r},\omega)dS(x)\right| + C(\omega), \tag{4}$$

where $C(\omega) := \sup_{0\le r\le 1} |\hat{u}_D(r\omega)|$ is a bounded function.

In particular, if $\omega_0 \in S^n$ is not a critical value of the normal map then by stationary phase method we have

$$\int_{\partial D} e^{ir(\omega_0, x)} a(x, \frac{1}{r}, \omega_0) dS = O(r^{-\frac{n}{2}}), \quad (as \quad r \to \infty)$$

and hence by (4) $M(\omega_0)$ is a finite number. Proposition 2.4 is proved.

For the convenience of readers we reduce simple fact from the classical differential geometry.

Proposition 2.5 *Let ∂D be an analytic compact hypersurface, then $K(x) \not\equiv 0$.*

Proof. Since ∂D is a compact set then there exists a ball $B(R)$ of radius R such that $\partial D \subset B(R)$. Also, we can choose $B(R)$ the ball with minimal radius. Then there exists a point $x^0 \in \partial D \cap B(R)$ and $\partial D \subset B(R)$. Thus by the definition of the Gaussial curvature $|K(x^0)| \ge \frac{1}{R^n} > 0$, where $K(x_0)$ is the Gaussian curvature of the hypersurface ∂D at the point $x_0 \in \partial D$. Since $K(x)$ is an analytic function and ∂D is a connected set then by uniqueness Theorem for real analytic functions we have $K(x) \not\equiv 0$. Because K is a real analytic function defined on ∂D for fixed first and second fundamental forms. Proposition 2.5 is proved.

3. Integrability of the Randol maximal functions

In this section we prove results on the Randol maximal functions. Through the section we use notation of the paper [1] (also see [9] for related results on Randol maximal functions).

The following result holds.

Theorem 3.1 *Let $D \subset \mathbb{R}^{n+1}$ be a compact domain with C^∞ boundary. If ∂D is a smooth hypersurface of type I then for any $u \in C^\infty(\mathbb{R}^{n+1})$ there exists an $\varepsilon > 0$ such that the associated maximal operator M belongs to the space $L^{2+\varepsilon}(S^n)$, e.g. $M \in L^{2+\varepsilon}(S^n)$.*

Proof. Since $u \in C^\infty(\mathbb{R}^{n+1})$ then by (4) we have

$$M(\omega) \le \sup_{r\ge 1} r^{\frac{n}{2}} \left| \int_{\partial D} e^{ir(\omega,x)} a(x, \frac{1}{r}, \omega) dS(x) \right| + C(\omega),$$

where C is a bounded function. By the Theorem 1 of the paper [1] there exists an $\varepsilon > 0$ such that

$$\sup_{r\ge 1} r^{\frac{n}{2}} \left| \int_{\partial D} e^{ir(\omega,x)} a(x, \frac{1}{r}, \omega) dS(x) \right| \in L^{2+\varepsilon}(S^n).$$

Indeed, note that order k is an upper semi-continuous function defined on the hypersurface ∂D. Since ∂D is a compact set then there exists K such that for any $x_0 \in \partial D$ order of the hypersurface ∂D is less or equal to K. Then by the Theorem 1 of the paper [1] we have $M \in L^{2+\varepsilon}(S^n)$ for any $\varepsilon < \frac{2}{K-1}$. Theorem 3.1 is proved.

Corollary 3.2 *If D is a compact domain with C^∞ boundary and ∂D is a smooth hypersurface of type I and also order of any points of the boundary is bounded by k then for any $u \in C^\infty(\mathbb{R}^{n+1})$ the relation $M \in L^{\frac{2k}{k-1}-0}(S^n)$ holds true.*

A proof of the Corollary 3.2 follows from the Theorem 3.1.

Theorem 3.3 *Let D be a compact domain and ∂D is an analytic hypersurface and also $\partial D \in E$ then for any smooth function $u \in C(\mathbb{R}^{n+1})$ the associated Randol maximal function M belongs to $L^{4-0}(S^n)$.*

Proof. Again we use the relation (4). Since C is a bounded function. Then the Theorem 2 of the paper [1] yields the relation $M \in L^{4-0}(S^n)$.

Theorem 3.4 *Let D be a compact domain with analytic boundary. If $\partial D \subset \Lambda_{n-1}$ then there exists an $\varepsilon > 0$ such that for any $u \in C^{\infty}(\mathbb{R}^{n+1})$ the inclusion: $M \in L^{2+\varepsilon}(S^n)$ holds true.*

Proof of Theorem 3.4. We remark that if D is a convex domain then the statement of the Theorem 3.4 follows from more general Randol Theorem [6]. Note that the Randol Theorem holds true for any convex compact domain with analytic boundary. So, may by $\partial D \not\subset \Lambda_{n-1}$. But, on the other hand from the condition $\partial D \subset \Lambda_{n-1}$ does not follow convexity of the set D. For example, any compact domain with analytic boundary satisfies the conditions of Theorem 3.4. For example, suppose D is a compact domain defined by $D := \{x_1^4 + x_1^2 x_2^2 + x_2^4 \le 1\}$, having analytic boundary $\partial D = \{(x_1, x_2) : x_1^4 + x_1^2 x_2^2 + x_2^4 = 1\}$. But, D is not a convex domain. Indeed in a neighborhood of $(0,1)$ boundary has the form

$$x_2 = 1 - \frac{1}{4} x_1^2 + O(x_1^4).$$

Hence D is not a convex domain.

First, due to the Proposition 2.5 we see that $K(x) \not\equiv 0$. Then Theorem 3 of the paper [1] yields the following: there exists an $\varepsilon > 0$ such that the inclusion $M \in L^{2+\varepsilon}(S^n)$ holds true. Theorem 3.4 is proved.

Note that for the C^{∞} boundary such kind of result does not hold. More precisely, there exists a compact plane domain with C^{∞} boundary such that $M \notin L^2(S^1)$.

4. Boundedness problem for convolution operators

In this section we consider $L^p \to L^{p'}$ boundedness problem for the convolution operators, related to strictly hyperbolic operators. The convolution kernel of the operator is given by the following relation

$$M_k = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F, \tag{5}$$

where F is the Fourier transform $\varphi \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$ is a smooth homogeneous function of order 1 $a_k \in C^{\infty}(\mathbb{R}^n)$ is homogeneous of order $-k$ for large $|\xi|$. It means that for $r > 0$ and $|\xi| > 1$ and $r|\xi| > 1$ the relation $a_k(r\xi) = r^{-1} a_k(\xi)$ holds true.

The relation (5) can be written as

$$(M_k f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} e^{i(\xi,x)} e^{i\varphi(\xi)} a_k(\xi) \int_{\mathbb{R}^n_y} e^{-i(\xi,y)} f(y) dy,$$

where f is a smooth fast decay function with all derivatives which belongs to the Schwartz class of functions. So, the kernel function denoting by $K_k(x)$ is a distribution defined by

$$K_k = F^{-1}(e^{i\varphi(\cdot)} a_k(\cdot)). \tag{6}$$

Such kind of convolution operator arises as a solution (up to a smooth function) to the following Cauchy problem:

$$P(D_t, D_x)u = 0, \quad here \quad D_t := \frac{\partial}{i\partial t}, \quad D_x := \left(\frac{\partial}{i\partial x_1}, \dots, \frac{\partial}{i\partial x_n} \right) \tag{7}$$

$$D_t^k u|_{t=0} = g_k, \quad D_t^l u|_{t=0} = 0, \quad (l = 0,1,\dots m-1, l \ne k), \tag{8}$$

where $P(D_t, D_x)$ is a homogeneous constant coefficient partial differential operator of degree m in the time t and space $x \in \mathbb{R}^n$. Moreover, we will assume that $P(D_t, D_x)$ is strictly hyperbolic. It means that for any $\xi \in \mathbb{R}^n \setminus \{0\}$ the associated polynomial $P(\tau, \xi)$ has m different real zeros, so we have the following factorization

$$P(\tau, \xi) = (\tau - \varphi_1(\xi)) \dots (\tau - \varphi_n(\xi)),$$

where $\varphi_1, \dots, \varphi_m$ are real-valued real analytic functions on $\mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $\varphi_k(\xi) \ne \varphi_l(\xi)$, where $k \ne l$.

Let's look at simple example:

$$P(D_t, D_x) := \frac{\partial^2}{\partial t^2} - \Delta,$$

where Δ is the usual Laplas operator. Then we have $P(\tau,\xi) = -\tau^2 + |\xi|^2$. Hence $m = 2$ and $\varphi_1(\xi) = |\xi| = \sqrt{\xi_1^2 + \ldots + \xi_n^2}$ and $\varphi_2(\xi) = -|\xi|$.

Analogically, we can consider the pseudo-differential operator $P(D_t, D_x)$. Moreover, one can consider hyperbolic system of equations.

The solution to the Cauchy Problem (7) is a linear combination of operators of Fourier multiplier of the type (5) and the phase function φ is one of $\{t\varphi_l(\xi)\}_{l=1}^m$. Further, for the sake of being definite we will assume that φ is smooth (real analytic) function definite on $\mathbb{R}^n \setminus \{0\}$ and $\varphi(\xi) > 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Then $\Sigma := \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\}$ is a smooth hypersurface due to the following Euler's homogeneity relation

$$\sum_{j=1}^n \xi_j \frac{\partial \varphi(\xi)}{\partial \xi_j} = \varphi(\xi) = 1 \quad on \quad \Sigma$$

and by the classical implicit function Theorem. Indeed, at any point ξ of Σ we have $\nabla\varphi(\xi) \neq 0$.

First, we use A.N.Varchenko [14] result on average decay of the Fourier transform and get estimate for the convolution operator in the case of arbitrary smooth hyper-surface Σ (compare with Theorem 4.2) .

Theorem 4.1 *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth compact closed hypersurface. If $3/4 \leq p \leq 2$ and $k_p > 2(n+1)(\frac{1}{p} - \frac{1}{2})$ then the convolution operator M_k is $L^p(\mathbb{R}^n) \mapsto L^{p'}(\mathbb{R}^n)$ bounded, where p, p' are conjugate exponents, e.g. $1/p + 1/p' = 1$.*

Proof. A proof of the Theorem 4.1 is based on result on average decay Fourier transform of surface-carried measure.

For completeness, let us formulate a well-known statement on the estimate of the averaging of an oscillatory integral (for more information, see [14]). Consider the oscillatory integral

$$J(\lambda, s, a) := \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,s,a)} \varphi(x,s,a) dx. \tag{9}$$

Denote by $\Sigma_\Phi(a)$ the critical set of the phase:

$$\Sigma_\Phi(a) := \{(x,s) \in \mathbb{R}^n \times \mathbb{R}^m : \partial\Phi(x,s,a)/\partial x_j = 0, j = 1,\ldots,n\}.$$

Theorem 4.2 *Suppose that $(0,0) \in \Sigma_\Phi(0)$, the differentials*

$$d_{(x,s)}(\partial\Phi(x,s,a)/\partial x_j), j = 1,\ldots,n$$

are linearly independent at the point $(0,0,0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$ and the support φ lies in a sufficiently small neighborhood $U \times V \times W$ of the origin. Then the following uniform with respect to $a \in W$ asymptotic relation is valid:

$$\int_V |J(\lambda, s, a)|^2 \, ds = O(|\lambda|^{-\frac{n}{2}}), \quad as \quad |\lambda| \mapsto +\infty. \tag{10}$$

Actually, the following Proposition hold:

Proposition 4.3 *Let Σ be a smooth surface defined by $\Sigma := \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\}$, where φ is homogeneous of order one a smooth function defined on $\mathbb{R}^{n+1} \setminus \{0\}$ then the following asymptotic relation*

$$\int_{S^n} \left| \int_\Sigma e^{i\lambda(\xi,\omega)} \psi(\xi,\omega) d\sigma(\xi) \right|^2 d\sigma(\omega) = O(\lambda^{-n}) \quad as \quad \lambda \to +\infty \tag{11}$$

holds true, where $d\sigma(\xi)$ is the induced surface-carried measure on Σ and $d\sigma(\omega)$ is the induced surface-carried measure on S^n (here S^n is the unit sphere centered at the origin).

Proof of Proposition 4.3. In order to prove the Proposition 4.3 we fix $\omega=\omega_0\in S^n$. For the sake of being defined we will assume that $\omega_0=(0,\ldots,0,1)$. Since $\xi\in\Sigma$ is a critical point for the function (ξ,ω_0) if and only if the normal $n_\xi=\nabla\varphi(\xi)/|\nabla\varphi(\xi)|=\pm\omega_0$. Thus, we get the critical set

$$C_\Sigma:=\left\{\xi\in\Sigma:\frac{\partial\varphi(\xi)}{\partial\xi_j}=0,\quad for\quad j=1,\ldots,n\right\}. \tag{12}$$

The C_Σ is a closed set and hence it is a compact set. Let χ be a smooth cut-off such that $\chi(\xi)=1$ in a neighborhood of the set C_Σ defined by (12). Then there exists a positive number $\delta>0$ such that we have the following localization relation

$$\int_\Sigma e^{i\lambda(\xi,\omega)}\psi(\xi,\omega)d\sigma(\xi)=\int_\Sigma e^{i\lambda(\xi,\omega)}\psi(\xi,\omega)\chi(\xi)d\sigma(\xi)+O(\lambda^{-N})\quad as\quad \lambda\to+\infty, \tag{13}$$

where N is a large number, we can chose N as large as we wish for example we can choose $N\geq n/2$.

Further, we can write the principal part of the integral (e.g. integral over the support of χ) as a finite sum of oscillatory integral with small support.

Later, possible after rotation in the space $\mathbb{R}^{n+1}_\xi$ WLOG we may assume that $\xi^0:=(0,\ldots,0,1)\in\Sigma$ is the critical point and then $\partial_{n+1}\varphi(\xi^0)=1\neq0$. By implicit function Theorem there exists a neighborhood $V(\xi^0)$ of the point ξ^0 such that in the neighborhood $V(\xi^0)$ the hypersurface Σ can be written as the graph of a smooth function ϕ:

$$V(\xi^0)\cap\Sigma=\{\xi\in\mathbb{R}^{n+1},\quad \xi_{n+1}=\phi(\xi_1,\ldots,\xi_n)\},$$

where ϕ is a smooth function satisfying the condition $\phi(0)=1$ and $\nabla\phi(0)=0$.

Then we have the phase function

$$\Phi(\xi,\omega):=\frac{\omega_1}{\omega_{n+1}}\xi_1+\ldots+\frac{\omega_n}{\omega_{n+1}}\xi_n+\phi(\xi_1,\ldots,\xi_n). \tag{14}$$

We use local coordinates $s_j:=\omega_j/\omega_{n+1}\, j=1,\ldots,n$ and a little abuse of notation we write $\Phi(\xi,s)$ instead $\Phi(\xi,\omega(s))$. Thus, we have

$$\Sigma_\Phi:=\{(\xi,s)\in\mathbb{R}^n\times\mathbb{R}^n: s=-\nabla\phi(\xi_1,\ldots,\xi_n)\}.$$

It is obvious that

$$d_{(\xi,s)}(\partial\Phi(\xi,s)/\partial\xi_j), j=1,\ldots,n$$

are linearly independent, more precisely $\det\{\partial_{\xi_j}\partial_{s_k}\Phi(0,0)\}_{j,k=1}^n\neq0$. Consequently, the phase function (14) satisfies the condition of the Theorem 4.2.

Due to the Theorem 4.2 we obtain a proof of the Proposition 4.3.

Let's use notation

$$I(\lambda,\omega):=\int_\Sigma e^{i\lambda(\xi,\omega)}\psi(\xi,\omega)d\sigma(\xi).$$

Now, for the convenience of the readers we reformulate the proposition proved in the paper [12] (compare with Proposition 1, page no. 385).

Proposition 4.4 *Let* $q\geq2$ *and* $\alpha\geq0$. *Suppose, for all* $\psi\in C^\infty(\Sigma\times S^n)$ *and* $\lambda>0$,

$$\|I(\lambda;\cdot)\|_{L^q(S^n)}\leq C_\psi\lambda^{-\alpha} \tag{15}$$

where C_ψ is independent of λ. Then $K_k\in L^q(\mathbb{R}^{n+1})$; hence M_k is $L^p(\mathbb{R}^n)\mapsto L^{p'}(\mathbb{R}^n)$ bounded for $p=2q/(2q-1)$, if $k>n+1-\alpha-1/q$.

We first, consider the function defined by (6). We write the kernel function as a sum of two functions:

$$K_k = F^{-1}(e^{i\varphi(\cdot)}\chi_0 a_k) + F^{-1}(e^{i\varphi(\cdot)}(1-\chi_0)a_k) =: K_k^0 + K_k^1, \tag{16}$$

where χ is a cut-off function such that $\chi_0(x) = 1$ in a neighborhood of the origin. Corresponding convolution operators are denoted by M_k^0 and M_k^1 respectively.

Lemma 4.5 *The following inclusion: $K_k^0 \in L^p(\mathbb{R}^{n+1})$ holds true for any $p > 1$. Hence for any $p > 1$ the convolution operator M_k^0 is bounded as an operator $L^p(\mathbb{R}^{n+1}) \mapsto L^{p'}(\mathbb{R}^{n+1})$.*

Proof of Lemma 4.5. Since $\chi_0 a_k$ is a smooth function with compact support then K_k^0 is a smooth function. Therefore it is enough to show that $K_k^0 \in L^p(\mathbb{R}^{n+1} \setminus B_N(0))$, where $B_N(0)$ is the ball of radius N centered at the origin. The number N will be chosen later.

Let χ_1 be a smooth function supported in the annulus $D := \{\xi \in \mathbb{R}^{n+1} : 1/2 \le |\xi| \le 2|\}$ satisfying the condition

$$\sum_{l=-L}^{\infty} \chi_1(2^l \xi) = 1, \quad for \quad \xi \in B_R(0) \setminus \{0\},$$

where L is a number depending on R. Hence we have

$$K_k^0(x) = \sum_{l=-L}^{\infty} F^{-1}(e^{i\varphi(\cdot)}\chi_1(2^l \cdot)\chi_0 a_k) =: K_{k,l}^0(x).$$

So, we have

$$K_{k,l}^0(x) = \int_{\mathbb{R}^{n+1}} e^{i(\varphi(\xi)-(\xi,x))}\chi_1(2^l\xi)\chi_0(\xi)a_k(\xi)d\xi.$$

We use change of variables given by the shift $2^l\xi \to \xi$ and obtain:

$$K_{k,l}^0(x) = 2^{-l(n+1)}\int_{\mathbb{R}^{n+1}} e^{i2^l(\varphi(\xi)-(\xi,x))}\chi_1(\xi)\chi_0(2^{-l}\xi)a_k(2^{-l}\xi)d\xi.$$

Note that $\varphi \in C^\infty(D)$ and since D is a compact set we may assume that $|\nabla\varphi(\xi)| \le N/2$. Therefore the phase function $\varphi(\xi) - (x,\xi)$ has no critical point on D, whenever $x \in \mathbb{R}^{n+1} \setminus B_N(0)$ and moreover $|\nabla\varphi(\xi) - x| \ge |x|/2$. Hence by using integration by parts argument we have

$$|K_{k,l}^0(x)| \le C_1 2^{-l(n+1)}(1 + 2^{-l}|x|)^{-(n+2)},$$

where C is a constant depending on n. Consequently, we get

$$\left(\int_{\mathbb{R}^{n+1}\setminus B_N(0)} |K_{k,l}^0(x)|^p \, dx\right)^{\frac{1}{p}} \le C_2 2^{(n+1)l(1-\frac{1}{p})},$$

where C_2 is a constant depending on n, p. Therefore, the series

$$\sum_{l=-L}^{\infty} K_{k,l}^0(\cdot),$$

converge in the space $L^p(\mathbb{R}^{n+1} \setminus B_N(0))$, whenever $p > 1$, because the space is complete. Then we get $K_k^0 \in L^p(\mathbb{R}^{n+1})$. Hence the due to Young's inequality the convolution operator M_k^0 is bounded as operator $L^p(\mathbb{R}^{n+1}) \mapsto L^{p'}(\mathbb{R}^{n+1})$ for any $p > 1$. Lemma 4.5 is proved.

Now, a proof of Proposition 4.4 follows from the arguments proved in the Proposition 1 (see page no. 385) of the paper by M. Sugumoto [12].

Proof of the Theorem 4.1. From the Proposition 4.4 it follows that for $p = 4/3$, if $k > n + 1 - \frac{n}{2} - 1/2 = (n+1)/2$ the operator M_k is bounded. We define the following family of convolution operator $M_k(z)$ defined by:

$$a_k(\xi, z) := (1 - \chi(|\xi|))\varphi(\xi)^{zk_0} a_{k_0}(\xi), \quad where \quad k_0 = \frac{n+1}{2}, \tag{17}$$

with a smooth cut-off function χ satisfying:

$$\chi(x)=\begin{cases}1, & |x|\le 1\\ 0 & |x|\ge 2.\end{cases}$$

The operator $M_k(z)$ is bounded $L^2(\mathbb{R}^{n+1})\mapsto L^2(\mathbb{R}^{n+1})$ for $\Re(z)=1$ by Plansharel identity and due to the Proposition 4.4 $M_k(z)$ is bounded $L^{4/3}(\mathbb{R}^{n+1})\mapsto L^4(\mathbb{R}^{n+1})$ for $\Re(z)=0$. Hence by Stein analytic interpolation Theorem M_k is bounded $L^p(\mathbb{R}^{n+1})\mapsto L^{p'}(\mathbb{R}^{n+1})$ for $k_p>2(n+1)(\frac{1}{p}-\frac{1}{2})$, where $4/3\le p\le 2$. The Theorem 4.1 is proved.

Now, we will assume that Σ is an analytic hypersurface.

Theorem 4.6 *Let $\Sigma\subset\mathbb{R}^{n+1}$ be an analytic compact hypersurface. If $\Sigma\in\Lambda_{n-2}$ and*

$$h_\Sigma:=\max_{\xi\in\Sigma}h_\Sigma(\xi).$$

Then the convolution operator M_k is bounded $L^p(\mathbb{R}^{n+1})\mapsto L^{p'}(\mathbb{R}^{n+1})$ for $1\le p\le 2$ provided that $k_p>(n+4-\frac{2}{h})(\frac{1}{p}-\frac{1}{2})$.

Remark. Note that if $n\ge 3$ and $\Sigma\in\Lambda_{n-2}$ then index $\gamma_0(\Sigma)$ introduced by M. Sugumoto (see page no. 383) is equal to 2, e.g. $\gamma_0(\Sigma)=2$. Thus, from the results by M. Sugumoto we obtain $k>(2n+2-1)(1/p-1/2)=(2n+1)(1/p-1/2)$ (see Theorem B, page no. 383). Thus, we get better lower bound. Moreover, if $n=2$ then we the inequality $\gamma_0(\Sigma)\ge h_\Sigma$ and the relation $\gamma_0(\Sigma)=h_\Sigma$ holds true for for $n=1$. Hence for the case $n=1$ the result of the Theorem 4.6 coincides with result by Sugumoto. Consequently, we obtain more sharp result for general $n\ge 2$.

Proof of the Theorem 4.6. A proof of theorem 4.6 is based on the following Proposition 2 (see Sugumoto page no. 386).

Proposition 4.7 *Let $\alpha\ge 0$. Suppose, for all $\psi\in C^\infty(\Sigma\times S^n)$ and $\lambda>0$,*

$$\|I(\lambda;\cdot)\|_{L^\infty(S^n)}\le C_\psi\lambda^{-\alpha} \tag{18}$$

where C_ψ is independent of λ. Then M_k is $L^p(\mathbb{R}^n)\mapsto L^{p'}(\mathbb{R}^n)$ bounded for $1\le p\le 2$, provided $k>(2n+2-2\alpha)(\frac{1}{p}-\frac{1}{2})$.

Remark. Note that the notion of height is defined for any hypersurface $\Sigma\in\mathbb{R}^{n+1}$. If $\Sigma\in\Lambda_{n-2}$ then any critical point of the associated phase function $(X,\omega)|_\Sigma$ has singularity of rank at least $n-2$. Hence, by the classical Varchenko result there exists adapted coordinates system.

The following statement follows from more general Karpushkin's Theorem.

Lemma 4.8 *Let $\Sigma\subset\mathbb{R}^{n+1}$ be an analytic compact hypersurface. If $\Sigma\in\Lambda_{n-2}$ and*

$$h_\Sigma:=\max_{\xi\in\Sigma}h_\Sigma(\xi).$$

Then the following estimate

$$\left|\int_\Sigma e^{i(X,\xi)}a(X)dS(x)\right|\le C_a(1+|\xi|)^{-\frac{n-2}{2}-\frac{1}{h_\Sigma}+\varepsilon}$$

holds true, where $a\in C^\infty(\Sigma)$ and C_a a constant depending on C^n norm of the function a where $\varepsilon>0$ is a small number. Moreover, we can choose the $\varepsilon>0$ as small as we wish.

Finally, a proof of Theorem 4.6 follows from the Lemma (4.8) combined with Proposition 4.7. Surely, in the case when $\Sigma \in \Lambda_{n-2}$ it is possible $h = \infty$.

Theorem 4.9 *Let* $\Sigma \subset \mathbb{R}^{n+1}$ *be a smooth compact hypersurface. If* $\Sigma \in \Lambda_k$ *(where* $k \ge n$*) then the convolution operator* M_k *is bounded* $L^p(\mathbb{R}^{n+1}) \mapsto L^{p'}(\mathbb{R}^{n+1})$ *for* $1 \le p \le 2$ *provided that*

$k_p > (2n-k)(\frac{1}{p} - \frac{1}{2})$.

Remark. If $k = 0$ then the result of the Theorem 4.9 is worst then the Theorem B of the paper by Sugumoto. However, if $k \ge 1$ then we have $\gamma_0(\Sigma) = 2$ and hence we get better result whenever $k \ge 1$.

From the classical Walter Theorem it follows the following estimate:

$$\left| \int_\Sigma e^{i(X,\xi)} a(X) dS(x) \right| \le C_a (1 + |\xi|)^{-\frac{k}{2}},$$

where $a \in C^\infty(\Sigma)$ and C_a a constant depending on C^k norm of the function a

Then again by using the Proposition 4.7 we obtain a proof of Theorem 4.9.

References

1. Akramova D. I. and Ikromov I. A., Randol Maximal Functions and the Integrability of the Fourier Transform of Measures, Mathematical Notes, 2021, Vol. 109, No. 5, pp. 3–20.
2. Бруно А.Д. Степенная геометрия в алгебраических и дифференциальных уравнениях. М.:Наука. Физматлит. 1998.-288 с.
3. Hlawka E. Über Integrale auf convexen Körpern. I. Monatsh. Math. 54, (1950), 1-36.
4. Ikromov I. A., Müller D. On adapted coordinate systems., Trans. Amer. Math. Soc., 363, (2011), no. 6, 28212848.
5. Ikromov I.A., Soleev A. On construction of the Shultz base, Uzbek Math. Journal, No.4 (2008), p. 75-89.
6. Randol B. "On the asymptotic behavior of the Fourier transform of the indicator function of a convex set", Trans. AMS, 139 (1970), 278–285.
7. Sard A. The measure of the critical values of differentiable maps, Bull. Amer. Math. Soc., 48 (1942), 883-890.
8. Schulz, H., Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms. Indiana Univ. Math. J., 40 (1991), 1267–1275. [35] Sogge, C. D., Fourier Integrals in Classical Analysis. Cambridg.
9. Soleeva N.A., L^p-bound for the Fourier Transform of Surface-Carried Measures Supported on Hypersurfaces with D_∞ Type Singularities. // Jour. of 118 Siberian Federal Un, Math. and Physics. Volume 13, Issue 3, 2020. P. 350-359.
10. Soleev A., Soleeva N.A. Power geometry for finding periodic solutions in one system of ODE. // Malaysian journal of mathematical sciences, Volume 8, Issue 2. 2014. P. 255-264.
11. Svensson I. "Estimates for the Fourier transform of the characteristic function of a convex set". Ark. Mat., 9:1 (1970), 11-22.
12. Sugimoto M. "Estimates fo Hyperbolic Equations of Space Dimension 3". J. Funct. Anal. 160 (No. 2), 382-407 (1998).
13. Varchenko, A. N. Newton polyhedra and estimates of oscillating integrals. *Funkcional. Anal. i Priložen*, 10 (1976), 13–38 (Russian); English translation in *Funktional Anal. Appl.*, 18 (1976), 175–196.
14. Varchenko A.N. On numbers of integer points in homothetic domains in $\mathbb{R}^n$. Functional anal. and appl. 1983, T.17, no. 2, p. 1-6.