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Optimal Quadrature Formulas for Computing of Fourier Integrals in $W_2^{(m,m-1)}$ Space

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Abstract. The present paper is devoted to construction of an optimal quadrature formulas for approximation of Fourier integrals in the Hilbert space $W_2^{(m,m-1)}(0,1)$ of non-periodic, complex valued functions. Here the quadrature sum consists of linear combination of the given function values on the uniform grid. The difference between integral and quadrature sum is estimated by the norm of the error functional. The optimal quadrature formula is obtained by minimizing the norm of the error functional with respect to coefficients. In addition, analytic formulas for optimal coefficients are obtained using the discrete analogue of the differential operator $d^m/dx^m - d^{m-1}/dx^{m-1}$. Moreover, the order of convergence of the optimal quadrature formula is studied. The constructed quadrature formulas are applied for reconstruction of CT images.

INTRODUCTION AND THE STATEMENT OF THE PROBLEM

There are many results about optimal quadrature formulas, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] for a recent articles. This work also contains important results for the approximation of Fourier coefficients of non periodic functions, mainly for equidistant nodes. As a general survey for the computation of oscillatory integrals we recommend PhD theses Andreas Asheim and Olver [12, 13].

In [14] and [15], based on Sobolev method, the problem of construction of optimal quadrature formulas for numerical calculation of Fourier integrals

$$I(\varphi) = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx, \tag{1}$$

with $\omega \in \mathbb{Z}$ was studied in Hilbert spaces $L_2^{(m)}(0,1)$ and $W_2^{(m,m-1)}(0,1)$, respectively. In these works explicit formulas of optimal coefficients were obtained for $m \geq 1$. In particular, for the case $m = 1$ the order of convergence of optimal quadrature formulas was studied.

In [16], authors studied approximate computation of univariate Fourier coefficients for the standard Sobolev spaces H^s of periodic and non-periodic functions with an arbitrary integer $s \geq 1$. They found matching lower and upper bounds on the minimal worst case error of algorithms that use function or derivative values in n points. They also found sharp bounds on the information complexity which is the minimal n for which the absolute or normalized error is at most ε.

It should be noted that in practice due to the fact that we have discrete values of an integrand function of the Fourier transforms are reduced to approximation of integrals of type (1) with $\omega \in \mathbb{R}$. For example, the problem of X-ray Computed Tomography (CT) is to reconstruct the function from its Radon transform. One of the widely used analytic methods of CT reconstruction is the filtered back-projection method in which the Fourier transforms are used (see [17]).

Authors in [18] was studied the problem of construction of optimal quadrature formulas in the sense of Sard for approximate calculation of Fourier integrals of the form (1) with $\omega \in \mathbb{R}$ in the space $L_2^{(1)}$.

It should be noted for numerical evaluation of integrals (1) with real ω a quadrature formula with explicit coefficients is required. Therefore, in the present work we study the problem of construction of optimal quadrature formulas in the sense of Sard for approximate calculation of Fourier integrals of the form (1) with $\omega \in \mathbb{R}$ in the space $W_2^{(m,m-1)}$.

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We obtain explicit formulas for optimal coefficients and calculate the norm of the error functional of the obtained optimal quadrature formula. We note that the obtained optimal quadrature formula can be used for approximation of Fourier integrals and reconstruction of a function from its discrete Radon transform.

We consider the following quadrature formula

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \cdot \varphi(x_\beta), \tag{2}$$

accompanied by the error

$$(\ell,\varphi) = \int_0^1 e^{2\pi\omega ix}\varphi(x)\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta\varphi(x_\beta), \tag{3}$$

where

$$(\ell,\varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)\,\mathrm{d}x,$$

and respective error functional

$$\ell(x) = e^{2\pi i\omega x}\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta\delta(x-x_\beta). \tag{4}$$

Here, C_β are coefficients and x_β are nodes of quadrature formula (2), $i^2=-1$, $\omega\in\mathbb{R}$ with $\omega\neq 0$, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval [0,1], and $\delta(x)$ is Dirac's delta-function.

Function φ belongs to the linear space $W_2^{(m,m-1)}[0,1]$ which is defined as

$$W_2^{(m,m-1)}[0,1] = \{\varphi:[0,1]\to\mathbb{C}\,|\varphi^{(m-1)} \text{ is abs.cont, and } \varphi^{(m)}\in L_2[0,1]\}.$$

The linear space $W_2^{(m,m-1)}[0,1]$ under the pseudo-inner product

$$\langle\varphi,\psi\rangle = \int_0^1 (\varphi^{(m)}(x)+\varphi^{(m-1)}(x))(\bar{\psi}^{(m)}(x)+\bar{\psi}^{(m-1)}(x))dx \tag{5}$$

is the Hilbert space of complex valued functions. Here $\bar{\psi}$ is the complex conjugate function to the function ψ and the semi-norm of the function φ is defined by the formula

$$\|\varphi\|_{W_2^{(m,m-1)}[0,1]} = \langle\varphi,\varphi\rangle^{1/2}$$

provided that

$$\int_0^1 (\varphi^{(m)}(x)+\varphi^{(m-1)}(x))(\bar{\varphi}^{(m)}(x)+\bar{\varphi}^{(m-1)}(x))dx < \infty.$$

We note that the coefficients C_β of the formula (2) depends on ω, h and m, i.e., $C_\beta = C_\beta(\omega,h,m)$.

The error (3) of the quadrature formula (2) is a linear functional in $W_2^{(m,m-1)*}[0,1]$, where $W_2^{(m,m-1)*}[0,1]$ is the conjugate space to the space $W_2^{(m,m-1)}[0,1]$.

The norm of the error functional (4) defined by

$$\|\ell\|_{W_2^{(m,m-1)*}[0,1]} = \sup_{\|\varphi\|_{W_2^{(m,m-1)}[0,1]}\neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|}. \tag{6}$$

From (6) for the absolute value of the error (3) we get the following inequality

$$|(\ell,\varphi)| \le \|\varphi\|_{W_2^{(m,m-1)}[0,1]}\cdot\|\ell\|_{W_2^{(m,m-1)*}[0,1]}. \tag{7}$$

Hence we conclude that the absolute value of the error (3) is estimated by the norm of the error functional. It is natural that in order to get more accurate formula we should minimise (by coefficients or by nodes) the right-hand side of the inequality (7). The minimum of the norm of the error functional ℓ by coefficients C_β with fixed nodes gives us *the optimal quadrature formula* of the form (2) in the sense of Sard [19] . We note that here distances between neighbor nodes of the formula (2) are equal, that is $x_\beta = h\beta, \quad h = 1/N, \ N \in \mathbb{N}$.

Therefore, for constructing optimal quadrature formulas of the form (2) in the sense of Sard in the space $W_2^{(m,m-1)}[0,1]$ we need to solve the following problems.

Problem 1. Find the norm of the error functional (3) of quadrature formula (2) in the space $W_2^{(m,m-1)}[0,1]$.

Problem 2. Find the coefficients C_β that give infimum value to $\|\ell\|_{W_2^{(m,m-1)*}[0,1]}$, and calculate

$$\left\|\overset{\circ}{\ell}\right\|_{W_2^{(m,m-1)*}[0,1]} = \inf_{C_\beta} \|\ell\|_{W_2^{(m,m-1)*}[0,1]}.$$

This problem, for the quadrature formulas of the form (2) with $\omega = 0$, was first studied in the space $W_2^{(m,m-1)}$ for any natural number m in [20].

These type problems for optimal interpolation were investigated in the works [21, 22, 23, 24, 25].

THE NORM OF THE ERROR FUNCTIONAL (4)

In order to solve Problem 1 first we calculate the norm (6) the error functional ℓ. For this we use *the extremal function*

$$\psi_\ell(x) = (-1)^m \bar{\ell}(x) * G_m(x) + P_{m-2}(x) + de^{-x}, \tag{8}$$

which was found in (see, [20]) and we get

$$\|\ell\|^2_{W_2^{(m,m-1)*}} = (\ell, \psi_\ell) = (-1)^m \left[\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_\beta \overline{C}_\gamma G_m(h\beta - h\gamma) - \sum_{\beta=0}^{N} \int_0^1 \left(\overline{C}_\beta e^{2\pi i \omega x} + C_\beta e^{-2\pi i \omega x} \right) G_m(x - h\beta) dx \right.$$
$$\left. + \int_0^1 \int_0^1 e^{2\pi i \omega x} \cdot e^{-2\pi i \omega y} G_m(x-y) dx dy \right]. \tag{9}$$

Furthermore, for the error functional (4) to be defined on the space $W_2^{(m,m-1)}[0,1]$ it should be imposed the following conditions:

$$\begin{aligned} &(\ell, x^\alpha) = 0, \ \ \alpha = 0,1,2,\ldots,m-2, \\ &(\ell, e^{-x}) = 0. \end{aligned} \tag{10}$$

From this, it is clear that for the existence of quadrature formulas of the form (2), the condition $N+1 \geq m$. Equalities (10) mean that our quadrature formula is exact for e^{-x} and for any polynomial of degree $m-2$.

Thus, Problem 1 is solved.

Further, in the next sections, we deal with the solution of Problem 2.

The square of the norm (9) of the error functional (4) is a multivariable quadratic function of the coefficients C_β. To find the minimum of the norm (9) with respect to the coefficients C_β under the conditions (10), we use the Lagrange method for finding the conditional extremum. Then for the coefficients of optimal quadrature formulas of the form (2), we get the following system of $N+m+1$ linear equations

$$\sum_{\gamma=0}^{N} C_\gamma \cdot G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha (h\beta)^\alpha + de^{-h\beta} = f_m(h\beta), \quad \beta = 0,1,\ldots,N, \tag{11}$$

$$\sum_{\gamma=0}^{N} C_\gamma (h\gamma)^\alpha = g_\alpha, \quad \alpha = 0,1,\ldots,m-2, \tag{12}$$

$$\sum_{\gamma=0}^{N} C_\gamma \cdot e^{-h\gamma} = \int_0^1 e^{2\pi i \omega x} e^{-x} dx, \tag{13}$$

where $f_m(h\beta)=\int_0^1 e^{2\pi i\omega x}G_m(x-h\beta)dx$, $G_m(x)=\frac{\mathrm{sgn}x}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{k=1}^{m-1}\frac{x^{2k-1}}{(2k-1)!}\right)$, $g_\alpha=\int_0^1 e^{2\pi i\omega x}x^\alpha dx$. Here C_β ($\beta=0,1,\ldots,N$), a_α ($\alpha=\overline{0,m-2}$) and d are $N+m+1$ unknowns.

It should be noted that system (11) - (13) has a unique solution for $N+1\geq m$ and this solution gives the minimum to the square of the norm (9) of the error functional (4) under the conditions (12) and (13). The uniqueness of the solution to this system is obtained from Theorems 3.1 and 3.2 in [20]. As mentioned above the quadrature formula with coefficients $\overset{\circ}{C}_\beta$ ($\beta=\overline{0,N}$), corresponding to this minimum is called *the optimal quadrature formula in the sense of Sard* and $\overset{\circ}{C}_\beta$ ($\beta=\overline{0,N}$) are called *the optimal coefficients*. For convenience, the optimal coefficients $\overset{\circ}{C}_\beta$ will be denoted as C_β.

In such a setting, it is necessary to express the coefficients C_β, we need an operator $D_m(h\beta)$ that satisfies the equality

$$D_m(h\beta)*G_m(h\beta)=\delta_d(h\beta), \tag{14}$$

here $\delta_d(h\beta)$ is the discrete delta-function.

In [26, 27] the operator $D_m(h\beta)$ which satisfies equation (14) is constructed and its some properties are studied.

The following theorems are proved in works [26, 27].

Theorem 1. *The discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}}-\frac{d^{2m-2}}{dx^{2m-2}}$ satisfying the equation (14) has the form*

$$D_m(h\beta)=\frac{1}{p_{2m-2}}\begin{cases}\sum\limits_{k=1}^{m-1}A_k\lambda_k^{|\beta|-1}, & |\beta|\geq 2,\\ -2e^h+\sum\limits_{k=1}^{m-1}A_k, & |\beta|=1,\\ 2C+\sum\limits_{k=1}^{m-1}\frac{A_k}{\lambda_k}, & \beta=0,\end{cases} \tag{15}$$

where

$$C=1+(2m-2)e^h+e^{2h}+\frac{e^h\cdot p_{2m-3}}{p_{2m-2}},$$

$$A_k=\frac{2(1-\lambda_k)^{2m-2}[\lambda_k(e^{2h}+1)-e^h(\lambda_k^2+1)]p_{2m-2}}{\lambda_k\,\mathscr{P}'_{2m-2}(\lambda_k)},$$

$$\mathscr{P}_{2m-2}(\lambda)=\sum_{s=0}^{2m-2}p_s\lambda^s=(1-e^{2h})(1-\lambda)^{2m-2}-2(\lambda(e^{2h}+1)-e^h(\lambda^2+1))$$
$$\times\left[h(1-\lambda)^{2m-4}+\frac{h^3(1-\lambda)^{2m-6}}{3!}E_2(\lambda)+\ldots+\frac{h^{2m-3}E_{2m-4}(\lambda)}{(2m-3)!}\right], \tag{16}$$

p_{2m-2}, p_{2m-3} are the coefficients of the polynomial $\mathscr{P}_{2m-2}(\lambda)$ defined by equality (16), λ_k are the roots of the polynomial $\mathscr{P}_{2m-2}(\lambda)$ which absolute values less than 1, $E_k(\lambda)$ is the Euler-Frobenius polynomial of degree k (the definition of the Euler-Frobenius polynomial is given, for example, in [28]).

Theorem 2. *The discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}-\frac{d^{2m-2}}{dx^{2m-2}}$ satisfies the following equalities*

1) $D_m(h\beta)*e^{h\beta}=0$,
2) $D_m(h\beta)*e^{-h\beta}=0$,
3) $D_m(h\beta)*(h\beta)^n=0$, for $n=0,1,...,2m-3$,
4) $D_m(h\beta)*G_m(h\beta)=\delta_d(h\beta)$,

here $G_m(h\beta)$ is the function of discrete argument corresponding to the function $G_m(x)$ and $\delta_d(h\beta)$ is the discrete delta-function.

Further, we find analytical solution of system (11) - (13). For this the next theorem is very important.

Theorem 3. *The coefficients of optimal quadrature formulas* (2) *in the sense of Sard in the space $W_2^{(m,m-1)}(0,1)$ have the following form*

$$C_\beta=D_m(h\beta)*f_m(h\beta)+\sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta+b_k\lambda_k^{N-\beta}\right),\quad \beta=1,2,...,N-1, \tag{17}$$

where a_k and b_k are unknowns, λ_k are roots of the polynomial $\mathscr{P}_{2m-2}(\lambda)$ which is defined by equality (16) *and* $|\lambda_k| < 1$.

This theorem is obtained from Theorem 3 of the work [29].

Now we need the value of convolution $D_m(h\beta) * f_m(h\beta)$ from the formula (17).

$$D_m(h\beta) * f_m(h\beta) = D_m(h\beta) * \left(\frac{e^{-h\beta}}{4} \cdot \frac{e^{2\pi i\omega+1} - 2e^{(2\pi i\omega+1)h\beta} + 1}{2\pi i\omega + 1} - \frac{e^{h\beta}}{4} \cdot \frac{e^{2\pi i\omega-1} - 2e^{(2\pi i\omega-1)h\beta} + 1}{2\pi i\omega - 1} \right.$$
$$\left. + \sum_{k=1}^{m-1} \left[-\frac{e^{2\pi i\omega h\beta}}{(2\pi i\omega)^{2k}} + \sum_{\alpha=0}^{2k-1} \left\{ \frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!} \cdot \left(\frac{(-1)^\alpha g_\alpha}{2\alpha!} + \frac{1}{(2\pi i\omega)^{\alpha+1}} \right) \right\} \right] \right).$$

Hence using properties of discreet analogue $D_m(h\beta)$, and after some simplification, we have

$$D_m(h\beta) * f_m(h\beta) = e^{2\pi i\omega h\beta} \left(-\frac{1}{2(2\pi i\omega+1)} + \frac{1}{2(2\pi i\omega-1)} - \sum_{k=1}^{m-1} \frac{1}{(2\pi i\omega)^{2k}} \right) \sum_{\gamma=-\infty}^{\infty} D_m(h\gamma) e^{-2\pi i\omega h\gamma}$$
$$= e^{2\pi i\omega h\beta} \cdot K_{\omega,m},$$

where

$$K_{\omega,m} = \left(-\frac{1}{2(2\pi i\omega+1)} + \frac{1}{2(2\pi i\omega-1)} - \sum_{k=1}^{m-1} \frac{1}{(2\pi i\omega)^{2k}} \right) \sum_{\gamma=-\infty}^{\infty} D_m(h\gamma) e^{-2\pi i\omega h\gamma}.$$

So, we rewrite the formula (17)

$$C_\beta = e^{2\pi i\omega h\beta} \cdot K_{\omega,m} + \sum_{k=1}^{m-1} \left(a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \quad \beta = 1,2,...,N-1. \tag{18}$$

It is clear from Theorem 3 that in order to obtain explicit expressions for the optimal coefficients C_β in $W_2^{(m,m-1)}(0,1)$ space, it suffices to find the unknowns a_k and b_k ($k = \overline{1,m-1}$). Substituting equality (18) into (11), we obtain an identity with respect to $(h\beta)$. Whence, equating the corresponding coefficients of the left and right sides of equation (11), we find unknowns a_k, b_k and an explicit form of $K_{\omega,m}$ from (18).

Now we consider equality (11)

$$\sum_{\gamma=0}^{N} C_\gamma \cdot G_m(h\beta - h\gamma) + P_{m-2}(h\beta) + de^{-h\beta} = f_m(h\beta), \quad \beta = 0,1,\ldots,N\ , \tag{19}$$

where

$$G_m(h\beta - h\gamma) = \frac{\mathrm{sgn(h\beta - h\gamma)}}{2} \left(\frac{e^{h\beta-h\gamma} - e^{h\gamma-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!} \right),$$

$$f_m(h\beta) = \frac{e^{-h\beta}}{4} \cdot \frac{e^{2\pi i\omega+1} - 2e^{(2\pi i\omega+1)h\beta} + 1}{2\pi i\omega + 1} - \frac{e^{h\beta}}{4} \cdot \frac{e^{2\pi i\omega-1} - 2e^{(2\pi i\omega-1)h\beta} + 1}{2\pi i\omega - 1} +$$
$$+ \sum_{k=1}^{m-1} \left[-\frac{e^{2\pi i\omega h\beta}}{(2\pi i\omega)^{2k}} + \sum_{\alpha=0}^{2k-1} \left\{ \frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!} \cdot \left(\frac{(-1)^\alpha g_\alpha}{2\alpha!} + \frac{1}{(2\pi i\omega)^{\alpha+1}} \right) \right\} \right], \tag{20}$$

$$g_\alpha = \int_0^1 e^{2\pi i\omega x} x^\alpha dx = e^{2\pi i\omega} \cdot \left(\sum_{k=1}^{\alpha-1} (-1)^k \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{(2\pi i\omega)^{k+1}} + \frac{1}{2\pi i\omega} \right) +$$
$$+(-1)^\alpha \frac{\alpha!}{(2\pi i\omega)^{\alpha+1}} \cdot \left(e^{2\pi i\omega} - 1 \right), \quad \alpha = 0,1,\ldots. \tag{21}$$

Introduce the following denotation

$$g(h\beta)=\sum_{\gamma=0}^{N}C_\gamma\frac{\mathrm{sgn}(\mathrm{h}\beta-\mathrm{h}\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-\sum_{k=1}^{m-1}\frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!}\right).$$

Then for $g(h\beta)$ we get the following

$$g(h\beta)=C_0\left(\frac{e^{h\beta}-e^{-h\beta}}{2}-\sum_{k=1}^{m-1}\frac{(h\beta)^{2k-1}}{(2k-1)!}\right)+g_1(h\beta)+g_2(h\beta), \tag{22}$$

where

$$g_1(h\beta)=\sum_{\gamma=1}^{\beta-1}C_\gamma\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-\sum_{k=1}^{m-1}\frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!}\right), \tag{23}$$

$$g_2(h\beta)=-\frac{1}{2}\sum_{\gamma=0}^{N}C_\gamma\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-\sum_{k=1}^{m-1}\frac{(h\beta-h\gamma)^{2k}}{(2k-1)!}\right). \tag{24}$$

First we consider $g_1(h\beta)$. Changing variable $\beta-\gamma$ to γ in (23), we have

$$g_1(h\beta)=\sum_{\gamma=1}^{\beta-1}C_{\beta-\gamma}\left(\frac{e^{h\gamma}-e^{-h\gamma}}{2}-\sum_{k=1}^{m-1}\frac{(h\gamma)^{2k-1}}{(2k-1)!}\right).$$

Hence, using (18), we obtain

$$\begin{aligned}g_1(h\beta)=&\sum_{\gamma=1}^{\beta-1}\left(K_{\omega,m}e^{2\pi i\omega(\beta-\gamma)}+\sum_{k=1}^{m-1}(a_k\lambda_k^{\beta-\gamma}+b_k\lambda_k^{N-\beta+\gamma})\right)\cdot\frac{e^{h\gamma}}{2}\\&-\sum_{\gamma=1}^{\beta-1}\left(K_{\omega,m}e^{2\pi i\omega(\beta-\gamma)}+\sum_{k=1}^{m-1}(a_k\lambda_k^{\beta-\gamma}+b_k\lambda_k^{N-\beta+\gamma})\right)\cdot\frac{e^{-h\gamma}}{2}\\&-\sum_{\ell=1}^{m-1}\sum_{\gamma=1}^{\beta-1}\left(K_{\omega,m}e^{2\pi i\omega(\beta-\gamma)}+\sum_{k=1}^{m-1}(a_k\lambda_k^{\beta-\gamma}+b_k\lambda_k^{N-\beta+\gamma})\right)\cdot\frac{(h\gamma)^{2\ell-1}}{(2\ell-1)!}.\end{aligned}$$

Using

$$\sum_{\gamma=0}^{n-1}q^\gamma\gamma^k=\frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i\Delta^i0^k-\frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i\Delta^i\gamma^k|_{\gamma=n}, \tag{25}$$

where $\Delta^i\gamma^k$ is finite order difference, γ^k, q are denominator of geometric progression, and $\Delta^i0^k=\sum_{l=1}^{i}(-1)^{i-1}C_i^l l^k$ and after some simplifications, we come

$$\begin{aligned}g_1(h\beta)=&\frac{K_{\omega,m}e^{2\pi i\omega\beta}}{2}\left[\frac{e^{(h-2\pi i\omega h)\beta}-e^{h-2\pi i\omega h}}{e^{h-2\pi i\omega h}-1}-\frac{e^{(-h-2\pi i\omega h)\beta}-e^{-h-2\pi i\omega h}}{e^{-h-2\pi i\omega h}-1}\right.\\&\left.-\sum_{l=1}^{m-1}\frac{2h^{2l-1}}{(2l-1)!}\left(\frac{e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\cdot\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha0^{2l-1}}{(e^{2\pi i\omega h}-1)^\alpha}-\frac{e^{-2\pi i\omega h\beta}\cdot e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\cdot\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha\beta^{2l-1}}{(e^{2\pi i\omega h}-1)^\alpha}\right)\right]\\&+\sum_{k=1}^{m-1}\left(a_k\cdot\frac{\lambda_k^\beta e^h-\lambda_k e^{h\beta}}{2(\lambda_k-e^h)}+b_k\cdot\frac{\lambda_k^{N-\beta+1}e^h-\lambda_k^N e^{h\beta}}{2(1-\lambda_k e^h)}\right)\\&-\sum_{k=1}^{m-1}\left(a_k\cdot\frac{\lambda_k^\beta-\lambda_k e^{h-h\beta}}{2(\lambda_k e^h-1)}+b_k\cdot\frac{\lambda_k^{N-\beta+1}-\lambda_k^N e^{h-h\beta}}{2(e^h-\lambda_k)}\right)\\&-\sum_{l=1}^{m-1}\frac{h^{2l-1}}{(2l-1)!}\sum_{k=1}^{m-1}\left[a_k\left(\frac{\lambda_k^{\beta+1}}{\lambda_k-1}\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha0^{2l-1}}{(\lambda_k-1)^\alpha}-\frac{\lambda_k}{\lambda_k-1}\cdot\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha\beta^{2l-1}}{(\lambda_k-1)^\alpha}\right)\right.\\&\left.+b_k\left(\frac{\lambda_k^{N-\beta}}{1-\lambda_k}\sum_{\alpha=0}^{2l-1}\left(\frac{\lambda_k}{1-\lambda_k}\right)^\alpha\Delta^\alpha0^{2l-1}-\frac{\lambda_k^N}{1-\lambda_k}\cdot\sum_{\alpha=0}^{2l-1}\left(\frac{\lambda_k}{1-\lambda_k}\right)^\alpha\Delta^\alpha\beta^{2l-1}\right)\right].\end{aligned}$$

In the last expression of $g_1(h\beta)$ the coefficients of terms λ_k^β and $\lambda_k^{N-\beta}$ are expressed by the values of the polynomial $\mathscr{P}_{2m-2}(\lambda)$, which is defined by equality (16), at λ_k. Since λ_k are roots of the polynomial $\mathscr{P}_{2m-2}(\lambda)$, then the coefficients of in front of λ_k^β and $\lambda_k^{N-\beta}$ are equal to zero. There from last equation for $g_1(h\beta)$, we get

$$\begin{aligned}
g_1(h\beta) &= \frac{e^{h\beta}}{2}\left[\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^h-e^{2\pi i\omega h}}+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{e^h-\lambda_k}+b_k\frac{\lambda_k^N}{\lambda_k e^h-1}\right)\right]\\
&-\frac{e^{-h\beta}}{2}\left[\frac{K_{\omega,m}e^{2\pi i\omega h+h}}{1-e^{2\pi i\omega h+h}}+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k e^h}{1-e^h\lambda_k}+b_k\frac{\lambda_k^N e^h}{\lambda_k-e^h}\right)\right]\\
&-\frac{e^{2\pi i\omega h\beta}}{2}\left[\frac{K_{\omega,m}e^h}{e^h-e^{2\pi i\omega h}}-\frac{K_{\omega,m}}{1-e^{2\pi i\omega h+h}}+\frac{2K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\ell l=1}^{m-1}\frac{h^{2l-1}}{(2l-1)!}\cdot\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha 0^{2l-1}}{(e^{2\pi i\omega h}-1)^\alpha}\right]\\
&+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{l=1}^{m-1}\sum_{j=1}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\cdot\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{(e^{2\pi i\omega h}-1)^\alpha}\\
&+\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\sum_{k=1}^{m-1}a_k\frac{\lambda_k}{\lambda_k-1}\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{(\lambda_k-1)^\alpha}\\
&+\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\sum_{k=1}^{m-1}b_k\frac{\lambda_k^N}{1-\lambda_k}\sum_{\alpha=0}^{2l-1-j}\left(\frac{\lambda_k}{1-\lambda_k}\right)^\alpha\Delta^\alpha 0^{2l-1-j}. \qquad (26)
\end{aligned}$$

Now, we consider the function $g_2(h\beta)$. Using the binomial formula and equalities (12), (13), from (24) we obtain

$$\begin{aligned}
g_2(h\beta) &= -\frac{1}{2}\left[\frac{e^{h\beta}}{2}\frac{e^{2\pi i\omega-1}-1}{2\pi i\omega-1}-\frac{e^{-h\beta}}{2}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}\right]+\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}\\
&+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!2\alpha!}\sum_{\gamma=0}^{N}C_\gamma(h\gamma)^\alpha. \qquad (27)
\end{aligned}$$

From equality (20) for $f_m(h\beta)$ applying the binomial formula, we have

$$\begin{aligned}
f_m(h\beta) &= \frac{e^{-h\beta}}{4}\cdot\frac{e^{2\pi i\omega+1}-2e^{(2\pi i\omega+1)h\beta}+1}{2\pi i\omega+1}-\frac{e^{h\beta}}{4}\cdot\frac{e^{2\pi i\omega-1}-2e^{(2\pi i\omega-1)h\beta}+1}{2\pi i\omega-1}-\sum_{k=1}^{m-1}\frac{e^{2\pi i\omega h\beta}}{(2\pi i\omega)^{2k}}\\
&+\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\left(\frac{(-1)^\alpha g_\alpha}{2\alpha!}+\frac{1}{(2\pi i\omega)^{\alpha+1}}\right)\\
&+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\left(\frac{(-1)^\alpha g_\alpha}{2\alpha!}+\frac{1}{(2\pi i\omega)^{\alpha+1}}\right)\\
&+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\left(\frac{(-1)^\alpha g_\alpha}{2\alpha!}+\frac{1}{(2\pi i\omega)^{\alpha+1}}\right). \qquad (28)
\end{aligned}$$

Taking into account (26) and (27), substituting (22) and (28) into (19), we obtain

$$
\begin{aligned}
&C_0\left[\frac{e^{h\beta}-e^{-h\beta}}{2}-\sum_{k=1}^{m-1}\frac{(h\beta)^{2k-1}}{(2k-1)!}\right]+\frac{e^{h\beta}}{2}\left[\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^h-e^{2\pi i\omega h}}+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{e^h-\lambda_k}+b_k\frac{\lambda_k^N}{\lambda_k e^h-1}\right)\right]\\
&-\frac{e^{-h\beta}}{2}\left[\frac{K_{\omega,m}e^{2\pi i\omega h+h}}{1-e^{2\pi i\omega h+h}}+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k e^h}{1-e^h\lambda_k}+b_k\frac{\lambda_k^N e^h}{\lambda_k-e^h}\right)\right]\\
&-\frac{e^{2\pi i\omega h\beta}}{2}\left[\frac{K_{\omega,m}e^h}{e^h-e^{2\pi i\omega h}}-\frac{K_{\omega,m}}{1-e^{2\pi i\omega h+h}}+\frac{2K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\ell l=1}^{m-1}\frac{h^{2l-1}}{(2l-1)!}\cdot\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha 0^{2l-1}}{\left(e^{2\pi i\omega h}-1\right)^\alpha}\right]\\
&+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\cdot\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{\left(e^{2\pi i\omega h}-1\right)^\alpha}\\
&+\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\sum_{k=1}^{m-1}a_k\frac{\lambda_k}{\lambda_k-1}\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{(\lambda_k-1)^\alpha}\\
&+\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\sum_{k=1}^{m-1}b_k\frac{\lambda_k^N}{1-\lambda_k}\sum_{\alpha=0}^{2l-1-j}\left(\frac{\lambda_k}{1-\lambda_k}\right)^\alpha\Delta^\alpha 0^{2l-1-j}+P_{m-2}(h\beta)+2d\frac{e^{-h\beta}}{2}\\
&-\frac{1}{2}\left[\frac{e^{h\beta}}{2}\frac{e^{2\pi i\omega-1}-1}{2\pi i\omega-1}-\frac{e^{-h\beta}}{2}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}\right]+\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}\\
&+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!2\alpha!}\sum_{\gamma=0}^{N}C_\gamma(h\gamma)^\alpha\\
&\quad=\frac{e^{-h\beta}}{2}\cdot\frac{e^{2\pi i\omega+1}+1}{2(2\pi i\omega+1)}-\frac{e^{h\beta}}{2}\cdot\frac{e^{2\pi i\omega-1}+1}{2(2\pi i\omega-1)}-\frac{e^{2\pi i\omega h\beta}}{2}\cdot\left(\frac{1}{2\pi i\omega+1}-\frac{1}{2\pi i\omega-1}+\sum_{k=1}^{m-1}\frac{2}{(2\pi i\omega)^{2k}}\right)\\
&\quad+\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}+\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\frac{1}{(2\pi i\omega)^{\alpha+1}}\\
&\quad+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\frac{1}{(2\pi i\omega)^{\alpha+1}}\\
&\quad+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\left(\frac{(-1)^\alpha g_\alpha}{2\alpha!}+\frac{1}{(2\pi i\omega)^{\alpha+1}}\right).
\end{aligned}
\tag{29}
$$

Since the last equation is an identity with respect to $(h\beta)$ then from (29) equating the coefficients of $e^{2\pi i\omega h\beta}$ we get that

$$
K_{\omega,m}=\frac{\frac{1}{2\pi i\omega+1}-\frac{1}{2\pi i\omega-1}+\sum_{k=1}^{m-1}\frac{2}{(2\pi i\omega)^{2k}}}{\frac{e^h}{e^h-e^{2\pi i\omega h}}-\frac{1}{1-e^{2\pi i\omega h+h}}+\frac{2e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{l=1}^{m-1}\frac{h^{2l-1}}{(2l-1)!}\cdot\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha 0^{2l-1}}{\left(e^{2\pi i\omega h}-1\right)^\alpha}}.
\tag{30}
$$

Keeping in the mind (30), after some simplifications from (29), we obtain

$$\frac{e^{h\beta}}{2}\left[C_0+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^h-e^{2\pi i\omega h}}+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{e^h-\lambda_k}+b_k\frac{\lambda_k^N}{\lambda_k e^h-1}\right)-\frac{e^{2\pi i\omega-1}-1}{2(2\pi i\omega-1)}\right]$$
$$+\frac{e^{-h\beta}}{2}\left[-C_0-\frac{K_{\omega,m}e^{2\pi i\omega h+h}}{1-e^{2\pi i\omega h+h}}-\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k e^h}{1-e^h\lambda_k}+b_k\frac{\lambda_k^N e^h}{\lambda_k-e^h}\right)+\frac{1}{2}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}+2d\right]$$
$$-C_0\sum_{k=1}^{m-1}\frac{(h\beta)^{2k-1}}{(2k-1)!}+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\cdot\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{(e^{2\pi i\omega h}-1)^\alpha}$$
$$+\sum_{l=1}^{m-1}\sum_{j=0}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{\lambda_k-1}\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{(\lambda_k-1)^\alpha}+b_k\frac{\lambda_k^N}{1-\lambda_k}\sum_{\alpha=0}^{2l-1-j}\left(\frac{\lambda_k}{1-\lambda_k}\right)^\alpha\Delta^\alpha 0^{2l-1-j}\right)$$
$$+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!2\alpha!}\sum_{\gamma=0}^{N}C_\gamma(h\gamma)^\alpha+P_{m-2}(h\beta)$$
$$=\frac{e^{-h\beta}}{2}\cdot\frac{e^{2\pi i\omega+1}+1}{2(2\pi i\omega+1)}+\frac{e^{h\beta}}{2}\cdot\frac{e^{2\pi i\omega-1}+1}{2(1-2\pi i\omega)}+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha g_\alpha}{(2k-1-\alpha)!2\alpha!}$$
$$+\sum_{k=1}^{m-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}}{(2k-1-\alpha)!}\frac{1}{(2\pi i\omega)^{\alpha+1}}. \quad (31)$$

As mentioned above, equality (31) is an identity of $(h\beta)$. From (31), equating the corresponding coefficients at $e^{h\beta}$, $e^{-h\beta}$, $(h\beta)^\alpha$, $\alpha=\overline{0,2m-3}$ we obtain a system of linear equations in unknowns a_k, b_k $(k=\overline{1,m-1})$, $P_{m-2}(h\beta)$ and d.

Equating the coefficients at $e^{h\beta}$ in both sides of (31), we obtain the following equation for the coefficient C_0

$$C_0=\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-e^h}+\frac{1}{1-2\pi i\omega}+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{\lambda_k-e^h}+b_k\frac{\lambda_k^N}{1-\lambda_k e^h}\right). \quad (32)$$

Now, from equations (12) with $\alpha=0$ using identity (25) taking into account (18) after some simplifications, for the coefficient C_N we obtain the following expression

$$C_N=\frac{e^{2\pi i\omega}-1}{2\pi i\omega}+C_0-\sum_{\gamma=1}^{N-1}C_\gamma=\frac{e^{2\pi i\omega}-1}{2\pi i\omega}-\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-e^h}-\frac{1}{1-2\pi i\omega}-\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{\lambda_k-e^h}+b_k\frac{\lambda_k^N}{1-\lambda_k e^h}\right)$$
$$-K_{\omega,m}\frac{e^{2\pi i\omega}-e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}-\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k^N-\lambda_k}{\lambda_k-1}+b_k\frac{\lambda_k^N-\lambda_k}{\lambda_k-1}\right).$$

Hence

$$C_N=K_{\omega,m}\cdot\frac{e^h\left(e^{2\pi i\omega}-e^{2\pi i\omega h}\right)+e^{2\pi i\omega h}\left(1-e^{2\pi i\omega}\right)}{\left(e^{2\pi i\omega h}-1\right)\left(e^{2\pi i\omega h}-e^h\right)}+\frac{(2\pi i\omega-1)e^{2\pi i\omega}+1}{2\pi i\omega(2\pi i\omega-1)}$$
$$+\sum_{k=1}^{m-1}(a_k\frac{\lambda_k^N(e^h-\lambda_k)+\lambda_k(1-e^h)}{(1-\lambda_k)(e^h-\lambda_k)}+b_k\frac{\lambda_k^{N+1}(e^h-1)+\lambda_k(1-\lambda_k e^h)}{(1-\lambda_k)(\lambda_k e^h-1)}). \quad (33)$$

From equation (13), using expressions (18), (32) and (33) of coefficients, we get the following linear equation for the unknowns a_k and b_k.

$$\sum_{k=1}^{m-1}(a_k\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)(e^h-\lambda_k)}+b_k\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)(\lambda_k e^h-1)})=\frac{e^{2\pi i\omega}-1}{2\pi i\omega(2\pi i\omega-1)(1-e^h)}$$
$$+K_{\omega,m}\left[\frac{e^{2\pi i\omega}-e^{2\pi i\omega h}}{(e^{2\pi i\omega h}-1)(1-e^h)}+\frac{e^{2\pi i\omega h}-e^{2\pi i\omega+h}}{(e^{2\pi i\omega h}-e^h)(1-e^h)}\right].(34)$$

Further, in (31), we take the terms that contain $(h\beta)^\alpha$, $\alpha=\overline{m-1,2m-3}$ and we obtain the equation

$$\sum_{l=\left[\frac{m+1}{2}\right]}^{m-1}\left[-C_0\frac{(h\beta)^{2l-1}}{(2l-1)!}+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{j=m-1}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\cdot\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{\left(e^{2\pi i\omega h}-1\right)^\alpha}+\sum_{j=m-1}^{2l-1}\frac{(h\beta)^j h^{2l-1-j}}{j!(2l-1-j)!}\right.$$
$$\left.\times\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{\lambda_k-1}\sum_{\alpha=0}^{2l-1-j}\frac{\Delta^\alpha 0^{2l-1-j}}{(\lambda_k-1)^\alpha}+b_k\frac{\lambda_k^N}{1-\lambda_k}\sum_{\alpha=0}^{2l-1-j}\left(\frac{\lambda_k}{1-\lambda_k}\right)^\alpha\Delta^\alpha 0^{2l-1-j}\right)\right]=\sum_{l=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{j=m-1}^{2l-1}\frac{(h\beta)^j}{j!(2\pi i\omega)^{2l-j}} \quad (35)$$

From here we collect similar terms with respect to degrees of $(h\beta)$. For this we consider two cases, when m is even and m is odd, separately. In the case when m is even, i.e., $m=2p$ for $p=1,2,\ldots$, from (35), we have

$$\sum_{k=1}^{2p-1}a_k\left[\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\sum_{\alpha=0}^{2l-2}\frac{\lambda_k\Delta^\alpha 0^{2l-2}}{(\lambda_k-1)^{\alpha+1}}-\frac{\lambda_k}{\lambda_k-e^h}\right]$$
$$+\sum_{k=1}^{2p-1}b_k\left[\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\sum_{\alpha=0}^{2l-2}\frac{\lambda_k^{N+\alpha}\Delta^\alpha 0^{2l-2}}{(1-\lambda_k)^{\alpha+1}}-\frac{\lambda_k^N}{1-\lambda_k e^h}\right]=\frac{1}{1-2\pi i\omega}+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-e^h}$$
$$-\sum_{l=1}^{j}\left(\frac{h^{2l-2}}{(2l-2)!}\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\alpha=0}^{2l-2}\left(\frac{1}{e^{2\pi i\omega h}-1}\right)^\alpha\Delta^\alpha 0^{2l-2}-\frac{1}{(2\pi i\omega)^{2l-1}}\right),\quad j=\overline{1,p}, \quad (36)$$

$$\sum_{k=1}^{2p-1}a_k\left[\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\sum_{\alpha=0}^{2l-1}\frac{\lambda_k\Delta^\alpha 0^{2l-1}}{(\lambda_k-1)^{\alpha+1}}\right]+\sum_{k=1}^{2p-1}b_k\left[\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\sum_{\alpha=0}^{2l-1}\frac{\lambda_k^{N+\alpha}\Delta^\alpha 0^{2l-1}}{(1-\lambda_k)^{\alpha+1}}\right]$$
$$=-\sum_{l=1}^{j}\left[\frac{h^{2l-1}}{(2l-1)!}\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\alpha=0}^{2l-1}\left(\frac{1}{e^{2\pi i\omega h}-1}\right)^\alpha\Delta^\alpha 0^{2l-1}-\frac{1}{(2\pi i\omega)^{2l}}\right],\quad j=\overline{1,p-1}, \quad (37)$$

Now we consider the case when m is odd, i.e., $m=2p+1$, $p=1,2,\ldots$. Then from (35), we get

$$\sum_{k=1}^{2p}a_k\left[\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\sum_{\alpha=0}^{2l-2}\frac{\lambda_k\Delta^\alpha 0^{2l-2}}{(\lambda_k-1)^{\alpha+1}}-\frac{\lambda_k}{\lambda_k-e^h}\right]$$
$$+\sum_{k=1}^{2p}b_k\left[\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\sum_{\alpha=0}^{2l-2}\frac{\lambda_k^{N+\alpha}\Delta^\alpha 0^{2l-2}}{(1-\lambda_k)^{\alpha+1}}-\frac{\lambda_k^N}{1-\lambda_k e^h}\right]=\frac{1}{1-2\pi i\omega}+\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-e^h}$$
$$-\sum_{l=1}^{j}\left(\frac{h^{2l-2}}{(2l-2)!}\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\alpha=0}^{2l-2}\left(\frac{1}{e^{2\pi i\omega h}-1}\right)^\alpha\Delta^\alpha 0^{2l-2}-\frac{1}{(2\pi i\omega)^{2l-1}}\right),\quad j=\overline{1,p}, \quad (38)$$

$$\sum_{k=1}^{2p}a_k\left[\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\sum_{\alpha=0}^{2l-1}\frac{\lambda_k\Delta^\alpha 0^{2l-1}}{(\lambda_k-1)^{\alpha+1}}\right]+\sum_{k=1}^{2p}b_k\left[\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\sum_{\alpha=0}^{2l-1}\frac{\lambda_k^{N+\alpha}\Delta^\alpha 0^{2l-1}}{(1-\lambda_k)^{\alpha+1}}\right]$$
$$=-\sum_{l=1}^{j}\left[\frac{h^{2l-1}}{(2l-1)!}\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\alpha=0}^{2l-1}\left(\frac{1}{e^{2\pi i\omega h}-1}\right)^\alpha\Delta^\alpha 0^{2l-1}-\frac{1}{(2\pi i\omega)^{2l}}\right],\quad j=\overline{1,p-1}. \quad (39)$$

Lemma 1. *The conditions* (12) *is equivalent to the system of equations*

$$\sum_{k=1}^{m-1}a_k\left[h^\alpha\sum_{j=0}^{\alpha}\frac{\lambda_k^i-\lambda_k^{N+i}}{(1-\lambda_k)^{j+1}}\Delta^j 0^\alpha-\sum_{j=0}^{\alpha-1}h^j\binom{\alpha}{j}\sum_{i=0}^{j}\frac{\lambda_k^{N+i}}{(1-\lambda_k)^{j+1}}\Delta^i 0^j+\frac{\lambda_k^N(e^h-\lambda_k)+\lambda_k(1-e^h)}{(1-\lambda_k)(e^h-\lambda_k)}\right]$$
$$+\sum_{k=1}^{m-1}b_k\left[h^\alpha\sum_{j=0}^{\alpha}\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)^{j+1}}\Delta^j 0^\alpha-\sum_{j=0}^{\alpha-1}h^j\binom{\alpha}{j}\sum_{i=0}^{j}\frac{\lambda_k}{(\lambda_k-1)^{j+1}}\Delta^i 0^j+\frac{\lambda_k^{N+1}(e^h-1)+\lambda_k(1-\lambda_k e^h)}{(1-\lambda_k)(\lambda_k e^h-1)}\right]$$
$$=g_\alpha-K_{\omega,m}\cdot\frac{e^h\left(e^{2\pi i\omega}-e^{2\pi i\omega h}\right)+e^{2\pi i\omega h}\left(1-e^{2\pi i\omega}\right)}{\left(e^{2\pi i\omega h}-1\right)\left(e^{2\pi i\omega h}-e^h\right)}-\frac{(2\pi i\omega-1)e^{2\pi i\omega}+1}{2\pi i\omega(2\pi i\omega-1)}$$
$$-h^\alpha K_{\omega,m}\left[\sum_{j=0}^{\alpha}\frac{e^{2\pi i\omega hj}\Delta^j 0^\alpha-e^{2\pi i\omega h(j+N)}\Delta^j 0^\alpha}{\left(1-e^{2\pi i\omega h}\right)^{j+1}}-\sum_{l=1}^{\alpha-1}h^{l-\alpha}\binom{\alpha}{l}\sum_{j=0}^{l}\frac{e^{2\pi i\omega h(N+j)}\Delta^j 0^l}{\left(1-e^{2\pi i\omega h}\right)^{j+1}}\right],\quad \alpha=\overline{1,m-2}.$$

where g_α is defined by (21) and $K_{\omega,m}$ is quoted in (30).

Proof. Now from (12) for the values $\alpha = 1,2,\ldots,m-2$. Using the expression (18) of coefficients C_β, $\beta = 1,2,\ldots,N-1$, we have

$$\sum_{\gamma=0}^{N} C_\gamma (h\gamma)^\alpha = \int_0^1 e^{2\pi i\omega x} x^\alpha dx = g_\alpha,$$

$$h^\alpha \sum_{\gamma=1}^{N-1}\left(K_{\omega,m} e^{2\pi i\omega h\gamma}\gamma^\alpha + \sum_{k=1}^{m-1}\left(a_k\lambda_k^\gamma \gamma^\alpha + b_k \lambda_k^{N-\gamma}\gamma^\alpha\right)\right) + C_N = g_\alpha,\ \ \alpha = 1,2,\ldots,m-2.$$

Hence, using the formula (25), we have

$$h^\alpha K_{\omega,m}\left[\frac{1}{1-e^{2\pi i\omega h}}\sum_{j=0}^{\alpha}\left(\frac{e^{2\pi i\omega h}}{1-e^{2\pi i\omega h}}\right)^j \Delta^j 0^\alpha - \frac{e^{2\pi i\omega}}{1-e^{2\pi i\omega h}}\sum_{j=0}^{\alpha}\left(\frac{e^{2\pi i\omega h}}{1-e^{2\pi i\omega h}}\right)^j \Delta^j N^\alpha\right]$$
$$+h^\alpha\sum_{k=1}^{m-1} a_k\left[\frac{1}{1-\lambda_k}\sum_{j=0}^{\alpha}\left(\frac{\lambda_k}{1-\lambda_k}\right)^j\Delta^j 0^\alpha - \frac{\lambda_k^N}{1-\lambda_k}\sum_{j=0}^{\alpha}\left(\frac{\lambda_k}{1-\lambda_k}\right)^j \Delta^j N^\alpha\right]$$
$$+h^\alpha\sum_{k=1}^{m-1} b_k\lambda_k^N\left[\frac{\lambda_k}{\lambda_k-1}\sum_{j=0}^{\alpha}\left(\frac{1}{\lambda_k-1}\right)^j\Delta^j 0^\alpha - \frac{\lambda_k^{1-N}}{\lambda_k-1}\sum_{j=0}^{\alpha}\left(\frac{1}{\lambda_k-1}\right)^j\Delta^j N^\alpha\right] + C_N = g_\alpha.$$

Using the following formula

$$\Delta^j N^\alpha = \sum_{i=0}^{\alpha}\binom{\alpha}{i}\Delta^j 0^i N^{\alpha-i} = \sum_{i=0}^{\alpha}\binom{\alpha}{i}\Delta^j 0^i h^{i-\alpha},$$

we get

$$h^\alpha K_{\omega,m}\left[\sum_{j=0}^{\alpha}\frac{e^{2\pi i\omega hj}\Delta^j 0^\alpha - e^{2\pi i\omega h(j+N)}\Delta^j 0^\alpha}{(1-e^{2\pi i\omega h})^{j+1}} - \sum_{l=1}^{\alpha-1} h^{l-\alpha}\binom{\alpha}{l}\sum_{j=0}^{l}\frac{e^{2\pi i\omega h(N+j)}\Delta^j 0^l}{(1-e^{2\pi i\omega h})^{j+1}}\right]$$
$$+\sum_{k=1}^{m-1} a_k\left[h^\alpha\sum_{j=0}^{\alpha}\frac{\lambda_k^i - \lambda_k^{N+i}}{(1-\lambda_k)^{j+1}}\Delta^j 0^\alpha - \sum_{j=0}^{\alpha-1} h^j\binom{\alpha}{j}\sum_{i=0}^{j}\frac{\lambda_k^{N+i}}{(1-\lambda_k)^{j+1}}\Delta^i 0^j\right]$$
$$+\sum_{k=1}^{m-1} b_k\left[h^\alpha\sum_{j=0}^{\alpha}\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)^{j+1}}\Delta^j 0^\alpha - \sum_{j=0}^{\alpha-1} h^j\binom{\alpha}{j}\sum_{i=0}^{j}\frac{\lambda_k}{(\lambda_k-1)^{j+1}}\Delta^i 0^j\right]$$
$$+K_{\omega,m}\cdot\frac{e^h\left(e^{2\pi i\omega}-e^{2\pi i\omega h}\right)+e^{2\pi i\omega h}\left(1-e^{2\pi i\omega}\right)}{\left(e^{2\pi i\omega h}-1\right)\left(e^{2\pi i\omega h}-e^h\right)} + \frac{(2\pi i\omega-1)e^{2\pi i\omega}+1}{2\pi i\omega(2\pi i\omega-1)}$$
$$+\sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k^N(e^h-\lambda_k)+\lambda_k(1-e^h)}{(1-\lambda_k)(e^h-\lambda_k)} + b_k\frac{\lambda_k^{N+1}(e^h-1)+\lambda_k(1-\lambda_k e^h)}{(1-\lambda_k)(\lambda_k e^h-1)}\right) = g_\alpha.$$

From here we get the system which is given in the statement of the Lemma. Lemma 1 is proved.

In this section we give the solution of Problem 2 for any $m\in\mathbb{N}$ and $N\geq m-1$. We consider the cases $m=1$ and $m\geq 2$ separately.

From (18), (32), (33), (34), (36), (37), (38) and (39) formulas and from result of Lemma 1 have proved the following theorem.

Main results of this work.

Theorem 4. *Coefficients of optimal quadrature formulas of the form* (2) *with the error functional* (4) *for equally spaced nodes in the space* $W_2^{(m,m-1)}(0,1)$ *when* $m\geq 2$, $N+1\geq m$ *and* $\omega\in\mathbb{R}$ *with* $\omega\neq 0$ *expressed by formulas*

$$C_0 = \frac{K_{\omega,m}\cdot e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1} + \frac{1}{1-2\pi i\omega} + \sum_{k=1}^{m-1}\left(a_k\frac{\lambda_k}{\lambda_k-e^h} + b_k\frac{\lambda_k^N}{1-e^h\lambda_k}\right),$$

$$C_\beta = e^{2\pi i\omega h\beta}K_{\omega,m} + \sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \qquad \beta = 1,2,...,N-1,$$

$$C_N = K_{\omega,m}\cdot\frac{e^h\left(e^{2\pi i\omega}-e^{2\pi i\omega h}\right)+e^{2\pi i\omega h}\left(1-e^{2\pi i\omega}\right)}{\left(e^{2\pi i\omega h}-1\right)\left(e^{2\pi i\omega h}-e^h\right)}+\frac{(2\pi i\omega-1)e^{2\pi i\omega}+1}{2\pi i\omega(2\pi i\omega-1)}$$
$$+\sum_{k=1}^{m-1}(a_k\frac{\lambda_k^N(e^h-\lambda_k)+\lambda_k(1-e^h)}{(1-\lambda_k)(e^h-\lambda_k)}+b_k\frac{\lambda_k^{N+1}(e^h-1)+\lambda_k(1-\lambda_k e^h)}{(1-\lambda_k)(\lambda_k e^h-1)}),$$

here a_k and b_k ($k=\overline{1,m-1}$) satisfy the following system $2m-2$ linear equations

$$\sum_{k=1}^{m-1}a_k\left[\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\sum_{i=0}^{2l-2}\frac{\lambda_k\Delta^i 0^{2l-2}}{(\lambda_k-1)^{i+1}}-\frac{\lambda_k}{\lambda_k-e^h}\right]+\sum_{k=1}^{m-1}b_k\left[\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\sum_{i=0}^{2l-2}\frac{\lambda_k^{N+i}\Delta^i 0^{2l-2}}{(1-\lambda_k)^{i+1}}-\frac{\lambda_k^N}{1-\lambda_k e^h}\right]$$
$$=\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-e^h}+\frac{1}{1-2\pi i\omega}-\sum_{l=1}^{j}\frac{h^{2l-2}}{(2l-2)!}\left[\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\alpha=0}^{2l-2}\frac{\Delta^\alpha 0^{2l-2}}{(e^{2\pi i\omega h}-1)^\alpha}-\frac{1}{(2\pi i\omega)^{2l-1}}\right],\quad j=\overline{1,[\frac{m}{2}]},$$

$$\sum_{k=1}^{m-1}a_k\left[\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\sum_{i=0}^{2l-1}\frac{\lambda_k\Delta^i 0^{2l-1}}{(\lambda_k-1)^{i+1}}\right]+\sum_{k=1}^{m-1}b_k\left[\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\sum_{i=0}^{2l-1}\frac{\lambda_k^{N+i}\Delta^i 0^{2l-1}}{(1-\lambda_k)^{i+1}}\right]$$
$$=-\sum_{l=1}^{j}\frac{h^{2l-1}}{(2l-1)!}\left[\frac{K_{\omega,m}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}\sum_{\alpha=0}^{2l-1}\frac{\Delta^\alpha 0^{2l-1}}{(e^{2\pi i\omega h}-1)^\alpha}-\frac{1}{(2\pi i\omega)^{2l}}\right],\qquad j=\overline{1,[\frac{m-1}{2}]},$$

$$\sum_{k=1}^{m-1}a_k\left[h^j\sum_{i=0}^{j}\frac{\lambda_k^i-\lambda_k^{N+i}}{(1-\lambda_k)^{i+1}}\Delta^i 0^j-\sum_{l=0}^{j-1}h^l C_j^l\sum_{i=0}^{l}\frac{\lambda_k^{N+i}\Delta^i 0^l}{(1-\lambda_k)^{i+1}}+\frac{\lambda_k^N(e^h-\lambda_k)+\lambda_k(1-e^h)}{(1-\lambda_k)(e^h-\lambda_k)}\right]$$
$$+\sum_{k=1}^{m-1}b_k\left[h^j\sum_{i=0}^{j}\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)^{i+1}}\Delta^i 0^j-\sum_{l=0}^{j-1}h^l C_j^l\sum_{i=0}^{l}\frac{\lambda_k\Delta^i 0^l}{(\lambda_k-1)^{i+1}}+\frac{\lambda_k^{N+1}(e^h-1)+\lambda_k(1-\lambda_k e^h)}{(1-\lambda_k)(\lambda_k e^h-1)}\right]$$
$$=g_j-\frac{e^{2\pi i\omega}}{2\pi i\omega}-\frac{1}{2\pi i\omega(2\pi i\omega-1)}-K_{\omega,m}h^j\sum_{\alpha=0}^{j}\frac{\left(e^{2\pi i\omega h}\right)^\alpha\left(1-e^{2\pi i\omega}\right)}{\left(1-e^{2\pi i\omega h}\right)^{\alpha+1}}\Delta^\alpha 0^j$$
$$+K_{\omega,m}\sum_{l=1}^{j-1}h^l C_j^l\sum_{\alpha=0}^{l}\frac{\left(e^{2\pi i\omega h}\right)^{N+\alpha}\Delta^\alpha 0^l}{(\lambda_k-1)^{\alpha+1}}-K_{\omega,m}\frac{e^h\left(e^{2\pi i\omega}-e^{2\pi i\omega h}\right)+e^{2\pi i\omega h}\left(1-e^{2\pi i\omega}\right)}{\left(e^{2\pi i\omega h}-1\right)\left(e^{2\pi i\omega h}-e^h\right)},\quad j=\overline{1,m-2},$$

$$\sum_{k=1}^{m-1}\left[a_k\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)(e^h-\lambda_k)}+b_k\frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)(\lambda_k e^h-1)}\right]$$
$$=K_{\omega,m}\left[\frac{e^{2\pi i\omega}-e^{2\pi i\omega h}}{\left(e^{2\pi i\omega h}-1\right)\left(1-e^h\right)}+\frac{e^{2\pi i\omega h}-e^{2\pi i\omega+h}}{\left(e^{2\pi i\omega h}-e^h\right)\left(1-e^h\right)}\right]+\frac{e^{2\pi i\omega}-1}{2\pi i\omega(2\pi i\omega-1)(1-e^h)}.$$

Here λ_k are roots of polynomial (16) *and* $|\lambda_k|<1$.

It should be noted that in case $m=1$ does not arise from the general case. Therefore, we reflect on this case separately.

We consider construction of optimal quadrature formula for the case $m=1$.

Therefore, for constructing optimal quadrature formulas of the form (3) in the sense of Sard in the space $W_2^{(1,0)}[0,1]$ we need to solve the following problem.

Problem 3. Find the coefficients $\mathring{C}_\beta$ that give inf value to $\|\ell\|_{W_2^{(1,0)*}[0,1]}$, and calculate

$$\left\|\mathring{\ell}\right\|_{W_2^{(1,0)*}[0,1]}=\inf_{C_\beta}\|\ell\|_{W_2^{(1,0)*}[0,1]}. \tag{40}$$

For the solution of Problem 3 we get the following system of $N+2$ linear equations

$$\sum_{\gamma=0}^{N}C_\gamma G_1(h\beta-h\gamma)+de^{-h\beta}=\int_0^1 e^{2\pi i\omega x}G_1(x-h\beta)dx,\qquad \beta=0,...,N, \tag{41}$$

$$\sum_{\gamma=0}^{N}C_\gamma\cdot e^{-h\gamma}=\frac{e^{2\pi i\omega-1}-1}{2\pi i\omega-1}, \tag{42}$$

where $G_1(x) = \frac{sgn(x)}{2} \cdot \frac{e^x - e^{-x}}{2}$.

We consider the coefficients C_β as a discrete argument function and assume $C_\beta = 0$ for $\beta < 0$ and $\beta > N$. Then the system (41) and (42) can be rewriting as follows:

$$C_\beta * G_1(h\beta) + de^{-h\beta} = f(h\beta), \;\; \beta = 0, 1, ..., N, \tag{43}$$

$$\sum_{\beta=0}^{N} C_\beta e^{-h\beta} = g_0, \tag{44}$$

where

$$f(h\beta) = \frac{e^{-h\beta}}{4} \cdot \frac{e^{2\pi i\omega+1}+1}{2\pi i\omega+1} - \frac{e^{h\beta}}{4} \cdot \frac{e^{2\pi i\omega-1}+1}{2\pi i\omega-1} + \frac{e^{2\pi i\omega h\beta}}{(2\pi i\omega+1)(2\pi i\omega-1)}, \tag{45}$$

$$g_0 = \frac{e^{2\pi i\omega-1}-1}{2\pi i\omega-1}. \tag{46}$$

Now we have the following problem.

Problem 4. Find C_β $(\beta = 0, 1, .., N)$ and d which satisfy the system (43)-(44) for given $f(h\beta)$ and g_0.

Note that Problem 4 is equivalent to Problem 3. For this case we have the following results.

Theorem 5. (Theorem 2.3 in [30]) *Coefficients of the optimal quadrature formulas of the form (2) in the sense of Sard for $\omega \in \mathbb{R}$ with $\omega \neq 0$ in the space $W_2^{(1,0)}[0,1]$ have the form*

$$C_0 = \frac{1+e^{2h}+2\pi i\omega(e^{2h}-1)-2e^{(2\pi i\omega+1)h}}{(e^{2h}-1)((2\pi\omega)^2+1)},$$

$$C_\beta = \frac{2(1+e^{2h}-2e^h\cos(2\pi\omega h))}{(e^{2h}-1)((2\pi\omega)^2+1)} \cdot e^{2\pi i\omega h\beta}, \; \beta = 1, 2, ..., N-1,$$

$$C_N = \frac{e^{2\pi i\omega}(1+e^{2h}-2\pi i\omega(e^{2h}-1)-2e^{(1-2\pi i\omega)h})}{(e^{2h}-1)((2\pi\omega)^2+1)}.$$

Furthermore, for the square of the norm of the error functional (4) of the optimal quadrature formula (2) on the space $W_2^{(1,0)*}[0,1]$ the following holds

$$\left\|\mathring{\ell}\right\|^2_{W_2^{(1,0)*}} = \frac{1}{(4\pi^2\omega^2+1)^2}\left(4\pi^2\omega^2+1-\frac{2(1+e^{2h}-2e^h\cos(2\pi\omega h))}{h(e^{2h}-1)}\right). \tag{47}$$

Proof. We consider the following discrete argument function

$$u(h\beta) = C_\beta * G_1(h\beta) + de^{-h\beta}.$$

Then, taking Theorem 1 and Theorem 2 into account, we have

$$C_\beta = D(h\beta) * u(h\beta). \tag{48}$$

For calculating the convolution (48) we need the representation of the function $u(h\beta)$ for all integer values of β. From (43) we have

$$u(h\beta) = f(h\beta) \text{ for } \beta = 0, 1, ..., N. \tag{49}$$

Now we should find the representation of $u(h\beta)$ for $\beta < 0$ and $\beta > N$.

For $\beta \leq 0$ and $\beta \geq N$, using $G_1(x) = \frac{sgn(x)}{2} \cdot \frac{e^x - e^{-x}}{2}$ and (44), respectively, we get the following

$$u(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}\, g_0 + \left(\frac{1}{4}\sum\limits_{\gamma=0}^{N} C_\gamma e^{h\gamma} + d\right) e^{-h\beta}, & \beta \leq 0, \\ \frac{e^{h\beta}}{4}\, g_0 + \left(-\frac{1}{4}\sum\limits_{\gamma=0}^{N} C_\gamma e^{h\gamma} + d\right) e^{-h\beta}, & \beta \geq N, \end{cases} \tag{50}$$

where g_0 is defined by (46), $\sum_{\gamma=0}^{N} C_\gamma e^{h\gamma}$ and d are unknowns. We denote

$$a^- = \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma e^{h\gamma} + d,$$
$$a^+ = -\frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma e^{h\gamma} + d.$$

Then from (50) when $\beta = 0$ and $\beta = N$ for these unknowns we obtain the system of two linear equations

$$a^- - \frac{1}{4}g_0 = f(0),$$
$$a^+ e^{-1} + \frac{e}{4}g_0 = f(1).$$

Hence, solving this system, using (45) and (46), we get

$$a^- = \frac{e^{2\pi i\omega+1}-1}{4(2\pi i\omega+1)},\ \ a^+ = -\frac{e^{2\pi i\omega+1}-1}{4(2\pi i\omega+1)}.$$

Hence

$$d = 0, \tag{51}$$

$$\frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma e^{h\gamma} = \frac{e^{2\pi i\omega+1}-1}{4(2\pi i\omega+1)}. \tag{52}$$

Keeping (51) and (52) in mind, and combining (49) and (50) we get

$$u(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}\cdot\frac{e^{2\pi i\omega-1}-1}{2\pi i\omega-1} + \frac{e^{-h\beta}}{4}\cdot\frac{e^{2\pi i\omega+1}-1}{2\pi i\omega+1}, & \beta \le 0,\\ f(h\beta), & 0 \le \beta \le N,\\ \frac{e^{h\beta}}{4}\cdot\frac{e^{2\pi i\omega-1}-1}{2\pi i\omega-1} - \frac{e^{-h\beta}}{4}\cdot\frac{e^{2\pi i\omega+1}-1}{2\pi i\omega+1}, & \beta \ge N. \end{cases}$$

Now, taking Theorem 1 and Theorem 2 into account, using the last representation of $u(h\beta)$, and from (48) by direct calculation for the optimal coefficients C_β, $\beta = 0,1,...,N$, we obtain analytic formulas Theorem 5.

Now we go to get (47). We have the following

$$\begin{aligned}\|\mathring{\ell}\|^2 = -\Bigg[&\sum_{\beta=0}^{N} C_\beta^R \left(\sum_{\gamma=0}^{N} C_\gamma^R G(h\beta-h\gamma) - \int_0^1 \cos(2\pi\omega x)\ G(x-h\beta)dx\right)\\ &+\sum_{\beta=0}^{N} C_\beta^I \left(\sum_{\gamma=0}^{N} C_\gamma^I G(h\beta-h\gamma) - \int_0^1 \sin(2\pi\omega x)\ G(x-h\beta)dx\right)\\ &-\sum_{\beta=0}^{N} C_\beta^R \int_0^1 \cos(2\pi\omega x)\ G(x-h\beta)dx - \sum_{\beta=0}^{N} C_\beta^I \int_0^1 \sin(2\pi\omega x)\ G(x-h\beta)dx\\ &+\int_0^1\int_0^1 \cos[2\pi\omega(x-y)]G(x-y)dxdy\Bigg].\end{aligned} \tag{53}$$

Since $d = d^R + id^I$, taking (51) into account, we have

$$d^R = 0 \text{ and } d^I = 0.$$

Therefore, using these last two equalities, from (41) and (42) we get the following equalities

$$\sum_{\gamma=0}^{N} C_\gamma^R G(h\beta - h\gamma) - \int_0^1 \cos(2\pi\omega x)\, G(x-h\beta)dx = 0,\ \beta = 0,...,N,$$

and

$$\sum_{\gamma=0}^{N} C_\gamma^I G(h\beta - h\gamma) - \int_0^1 \sin(2\pi\omega x)\, G(x-h\beta)dx = 0,\ \beta = 0,...,N.$$

Then the expression (53) for $\|\mathring{\ell}\|^2$ takes the form

$$\|\mathring{\ell}\|^2 = \sum_{\beta=0}^{N} C_\beta^R \int_0^1 \cos(2\pi\omega x)\, G(x-h\beta)dx + \sum_{\beta=0}^{N} C_\beta^I \int_0^1 \sin(2\pi\omega x)\, G(x-h\beta)dx$$
$$- \int_0^1 \int_0^1 \cos[2\pi\omega(x-y)]G(x-y)dxdy.$$

Hence calculating the definite integrals, keeping $C_\beta = C_\beta^R + iC_\beta^I$ in mind and using equations of Theorem 5, after some simplifications we get (47). Theorem 5 is proved. □

We note that in Theorem 8 the formulas for the optimal coefficients C_β are decomposed into two parts – real and imaginary parts. Therefore from the formulas of Theorem 5 we get the following results.

Corollary 1. *Coefficients for the optimal quadrature formula of the form*

$$\int_0^1 \cos(2\pi\omega x)\cdot\varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta^R \varphi(h\beta)$$

in the sense of Sard in $W_2^{(1,0)}[0,1]$ *for* $\omega \in \mathbb{R}$ *with* $\omega \neq 0$ *have the form*

$$C_0^R = \frac{1+e^{2h}-2e^h\cos(2\pi\omega h)}{(e^{2h}-1)((2\pi\omega)^2+1)},$$
$$C_\beta^R = \frac{2(1+e^{2h}-2e^h\cos(2\pi\omega h))}{(e^{2h}-1)((2\pi\omega)^2+1)}\cdot\cos(2\pi\omega h\beta),\ \beta = 1,2,...,N-1,$$
$$C_N^R = \frac{(1+e^{2h})\cos(2\pi\omega) - 2e^h\cos(2\pi\omega(1+h)) + 2\pi\omega\sin(2\pi\omega)(e^{2h}-1)}{(e^{2h}-1)((2\pi\omega)^2+1)}.$$

Corollary 2. *Coefficients for the optimal quadrature formula of the form*

$$\int_0^1 \sin(2\pi\omega x)\cdot\varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta^I \varphi(h\beta)$$

in the sense of Sard in $W_2^{(1,0)}[0,1]$ *for* $\omega \in \mathbb{R}$ *with* $\omega \neq 0$ *have the form*

$$C_0^I = \frac{2\pi\omega(e^{2h}-1)-2e^h\sin(2\pi\omega h)}{(e^{2h}-1)((2\pi\omega)^2+1)},$$
$$C_\beta^I = \frac{2(1+e^{2h}-2e^h\cos(2\pi\omega h))}{(e^{2h}-1)((2\pi\omega)^2+1)}\cdot\sin(2\pi\omega h\beta),\ \beta = 1,2,...,N-1,$$
$$C_N^I = \frac{(1+e^{2h})\sin(2\pi\omega) - 2e^h\sin(2\pi\omega(1+h)) - 2\pi\omega\cos(2\pi\omega)(e^{2h}-1)}{(e^{2h}-1)((2\pi\omega)^2+1)}.$$

It is easy to see that for $\omega \to 0$ from Theorem 5 we get the optimal of the trapezoidal quadrature formula in $W_2^{(1,0)}[0,1]$ [20].

Corollary 3. *Coefficients of the optimal quadrature formula of the form*

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta) \tag{54}$$

in the space $W_2^{(1,0)}[0,1]$ *have the form*

$$\begin{aligned} C_0 &= \frac{e^h-1}{e^h+1}, \\ C_\beta &= \frac{2(e^h-1)}{e^h+1},\ \beta = 1,2,...,N-1, \\ C_N &= \frac{e^h-1}{e^h+1} \end{aligned}$$

and for the square of the norm of the error functional of the optimal quadrature formula (54) on the space $W_2^{(1,0)*}[0,1]$ *the following holds*

$$\left\|\mathring{\ell}\right\|^2_{W_2^{(1,0)*}} = 1 - \frac{2(e^h-1)}{e^h+1}.$$

Remark 1. It should be noted that for fixed ω from (47) we get

$$\|\mathring{\ell}\|^2 = \frac{1}{12}h^2 - \frac{4\pi^2\omega^2+3}{360}h^4 + O(h^6),$$

i.e., the order of convergence of the optimal quadrature formula of the form (2) is $O(h)$.

Remark 2. In particular, from Theorem 5 in the case $\omega \in \mathbb{Z}$ with $\omega \neq 0$, we get the results of [15].

Remark 3. The equality (52) means that the optimal quadrature formula of the form (2) with coefficients in Theorem 5 is exact to $\varphi(x) = e^x$, because

$$\int_0^1 e^{2\pi i\omega x} e^x\, dx = \frac{e^{2\pi i\omega+1}-1}{2\pi i\omega+1}.$$

The equalities (52) and (44) provide exactness of our optimal quadrature formula to e^x and e^{-x}, respectively.

Optimal quadrature formulas for the interval [a,b]

Here from the results of the previous section by a linear transform we get optimal quadrature formulas for the interval $[a,b]$.

We consider construction of optimal quadrature formula of the form

$$\int_a^b e^{2\pi i\omega x}\varphi(x)\, dx \cong \sum_{\beta=0}^{N} C_{\beta,\omega}[a,b]\varphi(x_\beta) \tag{55}$$

in the Hilbert space $W_2^{(1,0)}[a,b]$. Here $C_{\beta,\omega}[a,b]$ are coefficients and $x_\beta = h\beta + a$ ($\in [a,b]$) are nodes of the formula (2), $\omega \in \mathbb{R}$, $i^2 = -1$, $h = \frac{b-a}{N}$, $N \in \mathbb{N}$.

Now by linear transformation $x = (b-a)y + a$, where $0 \le y \le 1$, we get

$$\int_a^b e^{2\pi i\omega x}\varphi(x)\, dx = (b-a)e^{2\pi i\omega a}\int_0^1 e^{2\pi i\omega(b-a)y}\varphi((b-a)y+a)dy.$$

Finally, applying Theorem 8 and Corollary 3 to the integral on the right-hand side of the last equality, we get the following result.

For $m = 2$ from Theorem 5 we get the following result which is Theorem 3 of [31].

Corollary 4. (Theorem 3 of [31]). *For real ω with $\omega h \notin \mathbb{Z}$, the coefficients of optimal quadrature formulas of the form* (2) *in the space $W_2^{(2,1)}(0,1)$ are expressed by formulas*

$$
\begin{aligned}
C_0 &= \frac{K_{\omega,2}\, e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1} - \frac{1}{2\pi i\omega} + a_1\frac{\lambda_1}{\lambda_1 - 1} + b_1\frac{\lambda_1^N}{1-\lambda_1},\\
C_\beta &= e^{2\pi i\omega h\beta} K_{\omega,2} + a_1\lambda_1^{\beta} + b_1\lambda_1^{N-\beta}, \qquad \beta = 1,2,\dots,N-1,\\
C_N &= K_{\omega,2}\left(\frac{e^{2\pi i\omega}e^h}{e^h - e^{2\pi i\omega h}} + \frac{e^{2\pi i\omega h+1}(1-e^h)}{(e^{2\pi i\omega h}-1)(e^h - e^{2\pi i\omega h})}\right) + \frac{e^{2\pi i\omega}}{2\pi i\omega - 1} + a_1\frac{\lambda_1^N e^h}{e^h - \lambda_1} + b_1\frac{\lambda_1 e^h}{\lambda_1 e^h - 1},
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{(\lambda_1 - 1)(\lambda_1 - e^h)}{\lambda_1(1-\lambda_1^{2N})}\left(\frac{1-\lambda_1^N e^{2\pi i\omega}}{2\pi i\omega(2\pi i\omega - 1)(e^h - 1)} + \frac{K_{\omega,2}\lambda_1^N(e - e^{2\pi i\omega})e^{2\pi i\omega h}}{(e^{2\pi i\omega h}-1)(e^h - e^{2\pi i\omega h})}\right),\\
b_1 &= \frac{(\lambda_1 - 1)(1-\lambda_1 e^h)}{\lambda_1(1-\lambda_1^{2N})}\left(\frac{\lambda_1^N - e^{2\pi i\omega}}{2\pi i\omega(2\pi i\omega - 1)(e^h - 1)} + \frac{K_{\omega,2}(e - e^{2\pi i\omega})e^{2\pi i\omega h}}{(e^{2\pi i\omega h}-1)(e^h - e^{2\pi i\omega h})}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
K_{\omega,2} &= \frac{(e^h - e^{2\pi i\omega h})(1-e^{2\pi i\omega h+h})(1-e^{2\pi i\omega h})^2}{2\pi^2\omega^2(4\pi^2\omega^2+1)e^{2\pi i\omega h}\left((1-e^{2h})(1-e^{2\pi i\omega h})^2 + 2h(e^h - e^{2\pi i\omega h})(1-e^{2\pi i\omega h+h})\right)},\\
\lambda_1 &= \frac{h(e^{2h}+1) - e^{2h} + 1 - (e^h - 1)\sqrt{h^2(e^h+1)^2 + 2h(1-e^h)}}{1-e^{2h}+2he^h}, \quad |\lambda_1| < 1,
\end{aligned}
$$

$h = 1/N$, N is a natural number.

NUMERICAL RESULTS

Application of optimal quadrature formulas for CT image reconstruction

Here we give numerical results of CT image reconstruction using the optimal quadrature formula (2) in the space $W_2^{(2,1)}$.

For the numerical experiment, we use three simulated phantoms (see Figure 1): a checkerboard (CB), the Shepp-Logan (SL) and concentric circles (CC). They are of size 128×128 and the sinograms are generated using half rotation sampling with sampling angle 1°.

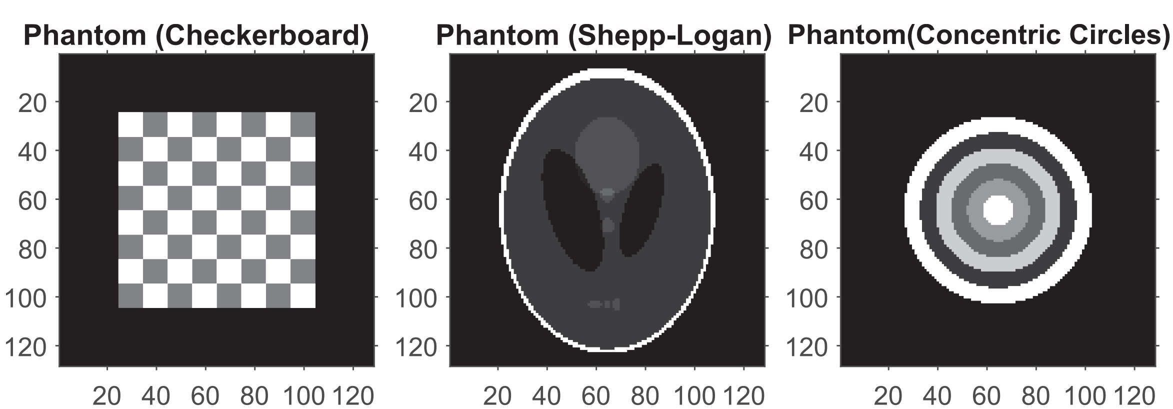


Figure 1. Original phantom images of size 128×128: checkerboard (first), the Shepp-Logan (second) and concentric circles (third).

Algorithm 1 is the pseudo code of the algorithm for CT image reconstruction using optimal quadrature formulas. For *Step 2* and *Step 3*, the optimal quadrature formulas are used for approximating Fourier integrals.

Algorithm 1. Reconstruction algorithm with optimal quadrature formula

1. A given projection data (sinogram):

$$P(t,\theta) = \int_{\ell_{t,\theta}} f(x,y)ds, \quad 0 \leq \theta \leq \pi, \quad a \leq t \leq b.$$

2. The approximation of the Fourier transform $S(\omega,\theta)$ of $P(t,\theta)$ using optimal quadrature formula:

$$S(\omega,\theta) \cong S(\omega,\theta_k) = \sum_{m=0}^{M} \overset{\circ}{C}_{m,-\omega} P(t_m,\theta_k), \quad \omega \in \mathrm{R}.$$

3. The approximation of the inverse Fourier transform $Q(t,\theta)$ of $S(\omega,\theta)\,|\omega|$ using optimal quadrature formula:

$$Q(t,\theta) \cong Q(t,\theta_k) = \sum_{n=0}^{N} \overset{\circ}{C}_{n,t} S(\omega_n,\theta_k)\,|\omega_n|\,.$$

4. Back projection to reconstruct the CT image:

$$f(x,y) = \int_0^{\pi} Q(t,\theta)d\theta \cong \frac{\pi}{K}\sum_{k=0}^{K-1} Q(t,\theta_k).$$

Quantitative image analysis for the CT image reconstruction using the second order optimal quadrature formula in Corollaries 4 has been done. All metrics obtained by using optimal quadrature formula of the second order compared with metrics of iradon. A built-in function of MATLAB 2019a, *iradon*, is used for the comparison of CT image reconstruction, and it is well known that *iradon* uses fft and $ifft$ for Fourier integrals.

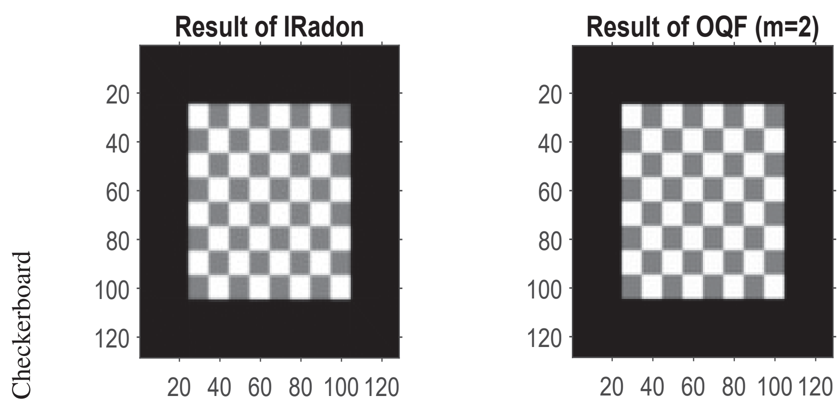
For the image quality analysis, we compare maximum error ($E_{\max}$), mean squared error (MSE), and the peak signal-to-noise ratio ($PSNR$):

$$E_{\max}(I) = \max_{i,j} \left| I(i,j) - I_{ref}(i,j) \right|,$$

$$MSE(I) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \left| I(i,j) - I_{ref}(i,j) \right|^2,$$

$$PSNR(I) = 10\log_{10}\left(\frac{I_{\max}^2}{MSE(I)}\right),$$

where $I_{\max}$ is the maximum pixel value of the image I. Images of original simulated phantoms are denoted by I_{ref} . Figures 2 and 3 show the results of CT image reconstruction using optimal quadrature formulas of the second order and profiles of the results, respectively.



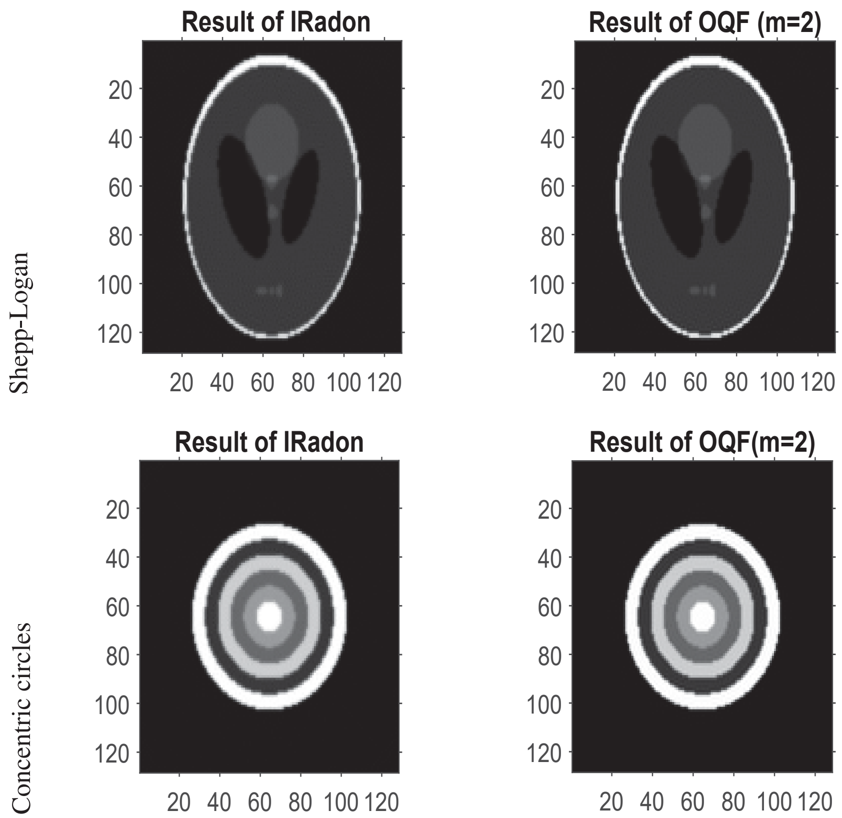
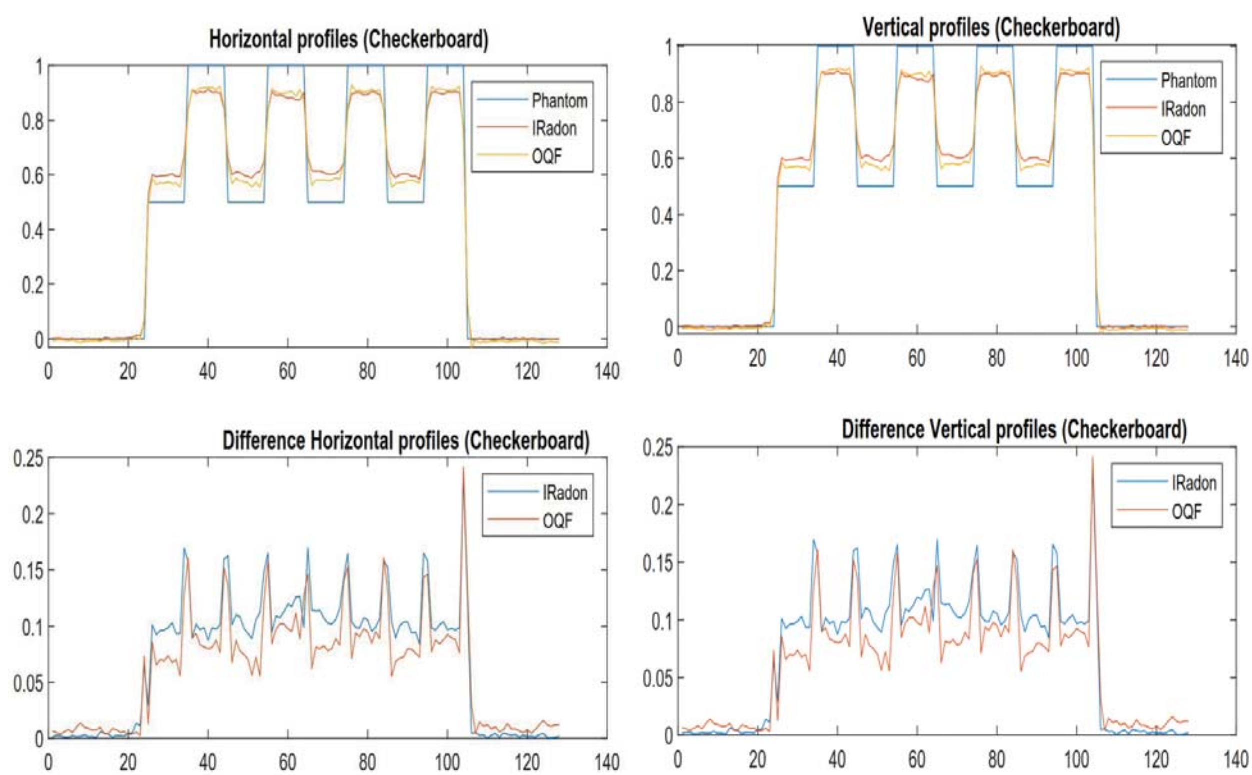


Figure 2. Reconstruction results using the MATLAB command *iradon* (the first column), using the second order optimal quadrature formula (the second column).



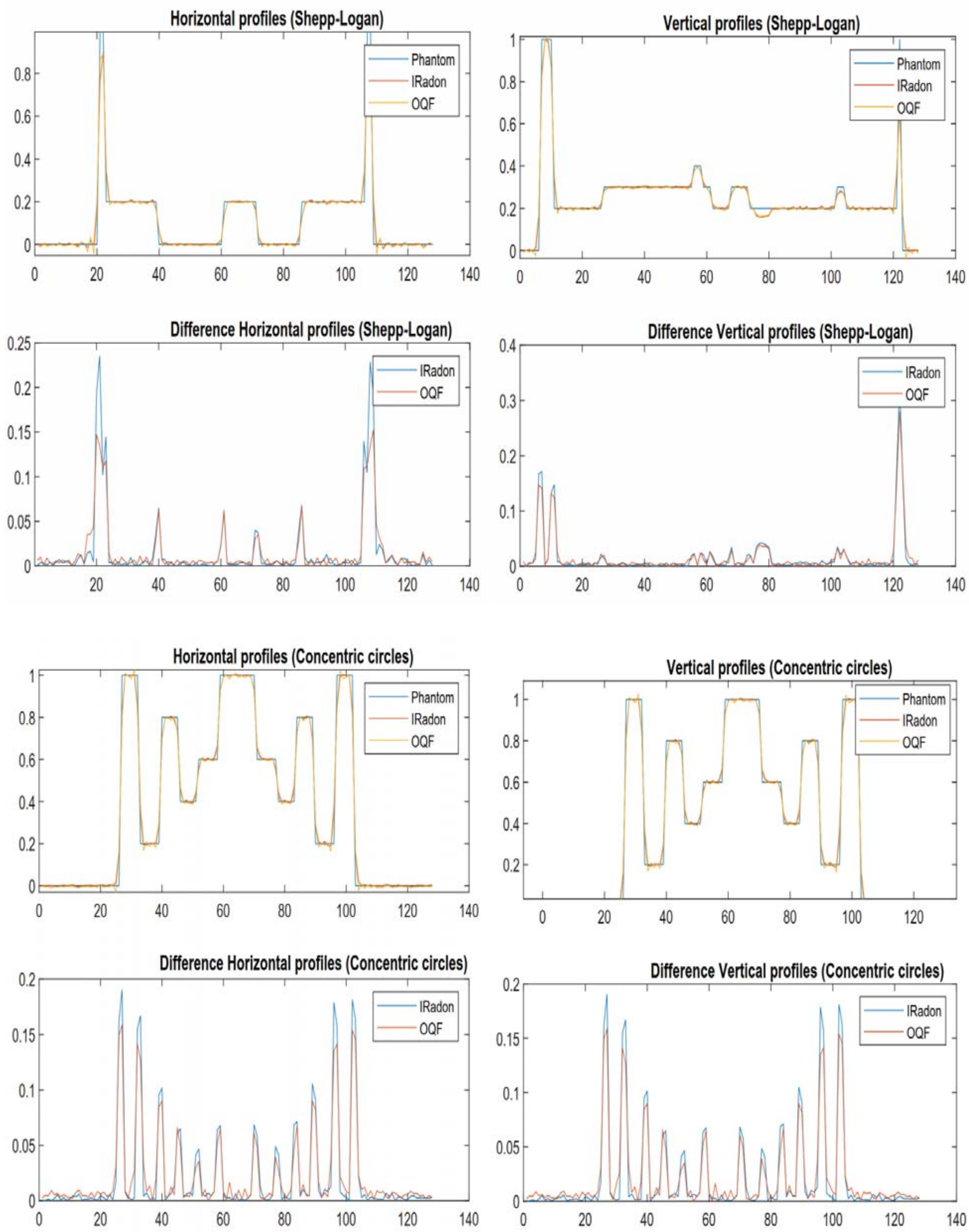


Figure 3. Profiles of the reconstructed images of a checkerboard (top), Shepp-Logan (center) and concentric circles(bottom): horizontal (left) and vertical (right).

As shown in Figure 2, the optimal quadrature formulas produce slightly clearer reconstruction results than iradon while having the same structure as the results of *iradon*. As shown in Figure 3, for the case of the Shepp-Logan, the profile lines of *iradon* and optimal quadrature formula almost coincide, however, for the checkerboard case, the second order optimal quadrature formula gives almost the same profiles as *iradon*.

Table I shows $E_{\max}$, MSE, and $PSNR$ for the reconstruction results using the second order optimal quadrature formulas. Even though no image processing technique has been applied to our image reconstruction algorithm, the second order optimal quadrature formula produce more improved quality of image quantitatively than *iradon* in MATLAB.

	A checkerboard		Shepp-Logan		Concentric circles	
	MATLAB *iradon*	proposed $m=2$	MATLAB *iradon*	proposed $m=2$	MATLAB *iradon*	proposed $m=2$
$E_{\max}$	**0.3058**	**0.2964**	**0.3601**	**0.3358**	**0.3653**	**0.3249**
MSE	**0.0022**	**0.0017**	**0.0036**	**0.0028**	**0.0029**	**0.0024**
PSNR	**26.6692**	**27.6974**	**24.4305**	**25.5878**	**25.3881**	**26.1924**

TABLE I. $E_{\max}$, MSE, and $PSNR$ for the reconstruction results using iradon and the second order optimal quadrature formulas.

Remark 4. It should be noted that since the coefficients of the optimal quadrature formulas of the form (2) are continuous functions of the parameter ω, it is not needed any interpolation in Algorithm 1.

CONCLUSION

Here for approximation of Fourier integrals in the space $W_2^{(m,m-1)}[0,1]$ the optimal quadrature formulas in the sense of Sard are constructed. By linear transformation the results are extended to the case of arbitrary interval $[a,b]$. That is, for approximation of Fourier integrals in the space $W_2^{(1,0)}[a,b]$ the optimal quadrature formula is obtained. The obtained optimal quadrature formula is applied to approximate reconstruction of Computed Tomography images from projections.

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