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Содержание

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OPTIMAL QUADRATURE FORMULA FOR THE APPROXIMATION OF THE RIGHT RIEMANN-LIOUVILLE INTEGRAL*

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In the present article, the problem of construction of the optimal quadrature formula in the sense of Sard is discussed for numerical integration of the right Riemann-Liouville integral in the Hilbert space of real-valued functions. Initially, the norm of the error functional is found using the extremal function of the error functional of the quadrature formula. Since the error functional is defined on the Hilbert space, the quadrature formula that we are constructing is exact for zeros of this space, that is, we have the conditions that the influence of the error functional on these functions is equal to zero. Then, the Lagrange function is constructed to find the conditional extremum of the error functional. Thereby, a system of linear equations is obtained for the coefficients of the optimal quadrature formula. The existence and uniqueness of the solution of the obtained system are studied.

Keywords: optimal quadrature formula, the extremal function, the error functional, optimal coefficient, Lagrange function.

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1 Introduction and statement of the problem

It is known that calculus means integration and differentiation. Fractional calculus, as its name suggests, refers to fractional integration and fractional differentiation. Fractional integration often means Riemann–Liouville integral. But for fractional differentiation, there are several kinds of fractional derivatives. In the following definition is introduced [6, 8, 9].

Definition 1.(Definition 1 in [3]) The left fractional integral (or the left Riemann–Liouville integral) and right fractional integral (or the right Riemann–Liouville integral) with order $\alpha > 0$ of the given function $\varphi(t)$, $t \in (0, 1)$ are defined as

$$D_{0,t}^{-\alpha}\varphi(t) = {}_{RL}D_{0,t}^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\varphi(x)dx, \tag{1}$$

and

$$D_{t,1}^{-\alpha}\varphi(t) = {}_{RL}D_{t,1}^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_t^1 (t-x)^{\alpha-1}\varphi(x)dx, \tag{2}$$

respectively, where $\Gamma(\cdot)$ is Eulier's gamma function.

The study of fractional integrals (1) and (2) is a two hundred year old subject that is part of a branch of mathematical analysis called Fractional Calculus [6, 7, 9]. Recently, due to its many applications in science and engineering, there has been an increase of interest in the study of fractional calculus. Fractional integrals appear naturally in many different contexts, e.g., when dealing with fractional variational problems or fractional optimal control. As is frequently

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observed, solving such equations analytically can be a difficult task, even impossible in some cases. One way to overcome the problem consists of applying numerical methods, e.g., using Riemann sums to approximate the fractional operators.

In this work we construct a quadrature formula for approximation of the right Riemann–Liouville integral (2). We consider a quadrature formula of the following form

$$\int_t^1 \frac{\varphi(x)}{(x-t)^{1-\alpha}}dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta), \tag{3}$$

where, $0<\alpha<1$, $h=\frac{1-t}{N}$, $t\leqslant 1$, $x_\beta = h\beta + t$ and function φ belongs to the linear space $W_2^{(m,m-1)}[t,1]$ which is defined as (see [1, 2])

$$W_2^{(m,m-1)}[t,1] = \{\varphi : [t,1]\to\mathbb{R} | \varphi^{(m-1)} \text{ is abs. cont, and } \varphi^{(m)} \in L_2[t,1]\}.$$

The following difference is called the error of the quadrature formula (3)

$$(\ell,\varphi) = \int_t^1 \frac{\varphi(x)}{(x-t)^{1-\alpha}}dx - \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx.$$

The error functional for quadrature formula (3) has the following form

$$\ell(x) = \frac{\varepsilon_{[t,1]}(x)}{(x-t)^{1-\alpha}} - \sum_{\beta=0}^{N} C_\beta \delta(x-x_\beta).$$

Here, C_β are coefficients of quadrature formula (3), $\varepsilon_{[t,1]}(x)$ is the characteristic function of the interval $[t,1]$, and $\delta(x)$ is Dirac's delta-function.

The inner product of two functions $\varphi(x)$ and $\psi(x)$ in the space $W_2^{(m,m-1)}[t,1]$ is defined as

$$\langle\varphi,\psi\rangle = \int_t^1 (\varphi^{(m)}(x) + \varphi^{(m-1)}(x))(\psi^{(m)}(x) + \psi^{(m-1)}(x))dx.$$

And norm of the function in this space is determined as

$$\|\varphi\|_{W_2^{(m,m-1)}} = \sqrt{\int_t^1 \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x)\right)^2 dx}.$$

The error of the quadrature formula is a linear functional in $W_2^{(m,m-1)*}[t,1]$, where $W_2^{(m,m-1)*}[t,1]$ is the conjugate space to the space $W_2^{(m,m-1)}[t,1]$.

It is natural to evaluate the quality of the quadrature formula (3) using the maximum error of this formula on the unit ball of the Hilbert space $W_2^{(m,m-1)}[t,1]$, that is, using the norm of the functional

$$\|\ell\|_{W_2^{(m,m-1)*}} = \sup_{\|\varphi\|_{W_2^{(m,m-1)}}=1} |(\ell,\varphi)|\,.$$

Obviously, the norm of the error functional depends on C_β coefficients and x_β nodes.

If

$$\|\mathring{\ell}\|_{W_2^{(m,m-1)*}} = \inf_{C_\beta, x_\beta} \|\ell\|_{W_2^{(m,m-1)*}}$$

then the functional $\mathring{\ell}$ is said to correspond to *the optimal quadrature formula* in $W_2^{(m,m-1)}[t,1]$.

Problem above in such a general formulation is quite difficult. Minimizing the norm of the error functional with respect to the coefficients C_β is a linear problem, and with respect to the nodes x_β it is actually non-linear, complicated problem, therefore, for simplicity, we consider this problem with fixed nodes x_β .

The main problem, in this work, is as follows.

Problem 1. Find the coefficients $\mathring{C}_\beta$ that give minimum value to $\|\ell\|_{W_2^{(m,m-1)*}}$, and calculate

$$\left\|\mathring{\ell}\right\|_{W_2^{(m,m-1)*}} = \inf_{C_\beta} \|\ell\|_{W_2^{(m,m-1)*}} .$$

For solving this problem, firstly, we must find the norm of the error functional. For this we need an extremal function of the error functional ℓ. General form of the extremal function in $W_2^{(m,m-1)}[t,1]$ space was found in works [2, 4, 5] and [10]. In particular, we get the following

$$\|\ell\|^2_{W_2^{(m,m-1)*}} = (\ell, \psi_\ell) = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)dx, \tag{4}$$

here ψ_ℓ is the extremal function and it is defined as follows

$$\psi_\ell(x) = (-1)^m \ell(x) * G_m(x) + P_{m-2}(x) + de^{-x}, \tag{5}$$

herein

$$G_m(x) = \frac{sgn(x)}{2}\left(\frac{e^x - e^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!}\right).$$

Using (5) from relation (4) we get the following

$$\|\ell\|^2_{W_2^{(m,m-1)*}} = (\ell, \psi_\ell) = \int_{-\infty}^{\infty} \ell(x)\left((-1)^m \ell(x) * G_m(x) + P_{m-2}(x) + de^{-x}\right) dx. \tag{6}$$

It should be noted that since the error functional $\ell(x)$ is defined on the space $W_2^{(m,m-1)}$, it satisfies the following conditions

$$\left(\ell, e^{-x}\right) = 0, \tag{7}$$

$$\left(\ell, x^k\right) = 0, \quad k = 0, 1, \ldots, m-2. \tag{8}$$

The equalities (7) and (8) mean that the quadrature formula (3) is exact for the function e^{-x} and for any polynomial of degree $\leqslant m-2$.

Then, from equality (6), taking into account the expressions (7) and (8), we have the following

$$\|\ell\|^2_{W_2^{(m,m-1)*}} = (-1)^m \int_{-\infty}^{\infty} \ell(x)\left(\ell(x) * G_m(x)\right) dx =$$

$$= (-1)^m \int\limits_{-\infty}^{\infty} \left(\frac{\varepsilon_{[t,1]}(x)}{(x-t)^{1-\alpha}} - \sum_{\beta=0}^{N} C_\beta \delta(x - x_\beta)\right)\left(\int\limits_t^1 \frac{G_m(x-y)}{(y-t)^{1-\alpha}}dy - \sum_{\gamma=0}^{N} C_\gamma G_m(x - x_\gamma)\right) dx.$$

From here we get

$$\begin{aligned}\|\ell\|^2_{W_2^{(m,m-1)*}} &= (-1)^m \Bigg(\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma G_m(h\beta - h\gamma) \\ &\quad -2\sum_{\beta=0}^{N} C_\beta \int_t^1 \frac{G_m(x-x_\beta)dx}{(x-t)^{1-\alpha}} + \int_t^1\int_t^1 \frac{G_m(x-y)dxdy}{(x-t)^{1-\alpha}(y-t)^{1-\alpha}}\Bigg).\end{aligned} \tag{9}$$

The expression (9) is a multivariate function with respect to coefficients C_β. Given the conditions (7) and (8), we consider the Lagrange function to find the minimum of the expression (9)

$$\Psi(\mathbf{C},\lambda)=\|\ell\|^2-2(-1)^m\sum_{k=0}^{m-2}\lambda_k(\ell,x^k)-2(-1)^m\lambda_{m-1}(\ell,e^{-x}),$$

here $\mathbf{C}=(C_0,C_1,\dots,C_N)$, and $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_{m-1})$.

Also, we have

$$(\ell,e^{-x})=\int_t^1\frac{e^{-x}dx}{(x-t)^{1-\alpha}}-\sum_{\beta=0}^{N}C_\beta e^{-x_\beta}=0,$$

$$(\ell,x^k)=\int_t^1\frac{x^k dx}{(x-t)^{1-\alpha}}-\sum_{\beta=0}^{N}C_\beta x_\beta^k=0,\quad k=0,1,\dots,m-2.$$

In that case, equating to 0 the partial derivatives of the function Ψ by $\mathbf{C}$ and λ, we get the following system of linear equations

$$\sum_{\gamma=0}^{N}C_\gamma G_m(h\beta-h\gamma)+\sum_{k=0}^{m-2}\lambda_k x_\beta^k+\lambda_{m-1}e^{-x_\beta}=f_m(t),\quad \beta=0,1,\dots,N, \tag{10}$$

$$\sum_{\gamma=0}^{N}C_\gamma e^{-x_\gamma}=g_1(t), \tag{11}$$

$$\sum_{\gamma=0}^{N}C_\gamma x_\gamma^k=g_2(t),\quad k=0,1\dots,m-2. \tag{12}$$

Here,

$$f_m(t)=\int_t^1\frac{G_m(x-x_\beta)}{(x-t)^{1-\alpha}}dx,$$

$$g_1(t)=\int_t^1\frac{e^{-x}}{(x-t)^{1-\alpha}}dx,$$

and

$$g_2(t)=\int_t^1\frac{x^k}{(x-t)^{1-\alpha}}dx,\quad k=0,1,\dots,m-2.$$

For calculating above integrals we use Taylor series for following functions

$$e^{-x}=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!},$$

$$e^{x-t}=\sum_{k=0}^{\infty}\frac{(x-t)^k}{k!},$$

and

$$e^{t-x}=\sum_{k=0}^{\infty}\frac{(-1)^k(x-t)^k}{k!}.$$

Thus, we calculate the above integrals and we get the following expressions

$$f_m(t)=\int_t^1\frac{G_m(x-x_\beta)}{(x-t)^{1-\alpha}}dx=$$

$$=\int_t^1\frac{sgn(x-x_\beta)}{2(x-t)^{1-\alpha}}\left(\frac{e^{x-x_\beta}-e^{-x+x_\beta}}{2}-\sum_{k=1}^{m-1}\frac{(x-x_\beta)^{2k-1}}{(2k-1)!}\right)dx$$

$$= -\frac{1}{2}\int_t^{h\beta+t}\left(\frac{e^{-h\beta-t}}{2}\frac{e^{x}}{(x-t)^{1-\alpha}} - \frac{e^{h\beta+t}}{2}\frac{e^{-x}}{(x-t)^{1-\alpha}} - \sum_{k=1}^{m-1}\frac{(x-t-h\beta)^{2k-1}}{(x-t)^{1-\alpha}(2k-1)!}\right)dx$$

$$+\frac{1}{2}\int_{t+h\beta}^{1}\left(\frac{e^{-h\beta-t}}{2}\frac{e^{x}}{(x-t)^{1-\alpha}} - \frac{e^{h\beta+t}}{2}\frac{e^{-x}}{(x-t)^{1-\alpha}} - \sum_{k=1}^{m-1}\frac{(x-t-h\beta)^{2k-1}}{(x-t)^{1-\alpha}(2k-1)!}\right)dx$$

$$= \frac{e^{h\beta}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(h\beta)^{\alpha+k}}{k!(\alpha+k)} - \frac{e^{-h\beta}}{2}\sum_{k=0}^{\infty}\frac{(h\beta)^{\alpha+k}}{k!(\alpha+k)} -$$
$$-\frac{e^{-h\beta}}{4}\sum_{k=0}^{\infty}\frac{(1-t)^{\alpha+k}}{k!(\alpha+k)} - \frac{e^{h\beta}}{4}\sum_{k=0}^{\infty}\frac{(-1)^k(1-t)^{\alpha+k}}{k!(\alpha+k)}$$

$$+\sum_{k=1}^{m-1}\sum_{i=0}^{2k-1}\frac{\binom{2k-1}{i}(-1)^i(h\beta)^{2k+\alpha-1}}{(2k-1)!(2k+\alpha-1-i)} - \sum_{k=1}^{m-1}\sum_{i=0}^{2k-1}\frac{\binom{2k-1}{i}(-1)^i(h\beta)^i(1-t)^{2k+\alpha-1-i}}{2(2k-1)!(2k+\alpha-1-i)}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k e^{h\beta} - e^{-h\beta}}{4k!(k+\alpha)}\left(2(h\beta)^{\alpha+k} - (1-t)^{\alpha+k}\right)$$
$$+\sum_{k=1}^{m-1}\frac{1}{(2k-1)!}\sum_{i=0}^{2k-1}\binom{2k-1}{i}\frac{(-h\beta)^i}{2k+\alpha-i-1}\left((h\beta)^{2k+\alpha-i-1} - \frac{(1-t)^{2k+\alpha-i-1}}{2}\right),$$

Likewise,

$$g_1(t) = \int_t^1 \frac{e^{-x}}{(x-t)^{1-\alpha}}dx = \int_t^1 \frac{e^{t-x}e^{-t}}{(x-t)^{1-\alpha}}dx = \int_t^1 \frac{\sum_{k=0}^{\infty}\frac{e^{-t}(-1)^k(x-t)^k}{k!}}{(x-t)^{1-\alpha}}dx$$
$$= \sum_{k=0}^{\infty}\frac{e^{-t}(-1)^k}{k!}\int_t^1 (x-t)^{k+\alpha-1}dx = \sum_{k=0}^{\infty}\frac{e^{-t}(1-t)^{k+\alpha}(-1)^k}{k!(k+\alpha)},$$

and,

$$g_2(t) = \int_t^1 \frac{x^k}{(x-t)^{1-\alpha}}dx = \int_t^1 \frac{(x-t+t)^k}{(x-t)^{1-\alpha}}dx$$
$$= \int_t^1 \sum_{i=0}^{k}\binom{k}{i}t^i(x-t)^{k-i-1+\alpha}dx = \sum_{i=0}^{k}\frac{\binom{k}{i}t^i(1-t)^{k-i+\alpha}}{k-i+\alpha}.$$

In this way, we have redefined the system of linear equations (10)-(12).

2 Existence and uniqueness of the solution of the system of equations (10)-(12)

System (10)-(12) has a unique solution. The proof of the uniqueness of the solution of the system (10)-(12) is similar to that in [11, 12]. For completeness, we present it.

Denote the solution of system (10) by $\mathbf{C} = (C_0, C_1, \ldots, C_N)$, and $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1})$. It is a stationary point of the function Φ.

Now in (9) we make the replacement $C_\beta = \bar{C}_\beta + C_{1\beta}$, then we immediately obtain that the minimization of (9) under the conditions (7) and (8) is equivalent to the minimization of the expression

$$\|\ell\|^2 = (-1)^m \left(\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \bar{C}_\beta \bar{C}_\gamma G_m(x_\beta - x_\gamma) + \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} (2\bar{C}_\beta C_{1\gamma} + C_{1\beta} C_{1\gamma}) G_m(x_\beta - x_\gamma) \right.$$
$$\left. -2 \sum_{\beta=0}^{N} (\bar{C}_\beta + C_{1\beta}) \int_t^1 \frac{G_m(x - x_\beta) dx}{(x-t)^{1-\alpha}} + \int_t^1 \int_t^1 \frac{G_m(x-y) dx dy}{(x-t)^{1-\alpha} (y-t)^{1-\alpha}} \right). \quad (13)$$

$$\sum_{\gamma=0}^{N} \bar{C}_\gamma G_m(x_\beta - x_\gamma) + \sum_{k=0}^{m-2} \lambda_k x_\beta^k + \lambda_{m-1} e^{-x_\beta} = F_m(t), \quad \beta = 0, 1, \ldots, N, \quad (14)$$

$$\sum_{\gamma=0}^{N} \bar{C}_\gamma e^{-x_\gamma} = 0, \quad (15)$$

$$\sum_{\gamma=0}^{N} \bar{C}_\gamma x_\gamma^k = 0, \quad k = 0, 1 \ldots, m-2. \quad (16)$$

where $F_m(t) = f_m(t) - \sum_{\gamma=0}^{N} C_{1\gamma} G_m(x_\beta - x_\gamma)$, and $C_{1\gamma}$ is a particular solution of the system of equations (11) - (12).

Therefore, it suffices to prove that the system (14)-(16) has a unique solution in $\bar{\mathbf{C}} = (\bar{C_0}, \bar{C_1}, \ldots, \bar{C_N})$, and $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1})$ and this solution gives a minimum to the expression $\|\ell\|^2$.

From the theory of conditional extremum, a sufficient condition is known under which the solution of the system (14)-(16) gives a conditional minimum to the value of $\|\ell\|^2$ on the manifold (15), (16). It consists in the positive definiteness of the quadratic form

$$\Phi(\bar{\mathbf{C}}) = \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \frac{\partial^2 \Psi}{\partial \bar{C}_\beta \partial \bar{C}_\gamma} \bar{C}_\beta \bar{C}_\gamma \quad (17)$$

on the set of vectors $\bar{\mathbf{C}} = (\bar{C_0}, \bar{C_1}, \ldots, \bar{C_N})$ under the condition

$$S\bar{\mathbf{C}} = 0, \quad (18)$$

where S is the following matrix of equations (15), (16)

$$S = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{m-2} & x_1^{m-2} & \cdots & x_N^{m-2} \\ e^{-x_0} & e^{-x_1} & \cdots & e^{-x_N} \end{pmatrix}.$$

We show that in the case under consideration this condition is satisfied.

Theorem 1. *For any vector $\bar{\mathbf{C}} \in \mathbb{R}^{N+}$ lying in the subspace $S\bar{\mathbf{C}} = 0$, the function $\Phi(\bar{\mathbf{C}})$ is strictly positive.*

Proof. Using the definition of the function $\Psi(\mathbf{C}, \lambda)$ and equations (13), (15), (16) from (17) we obtain

$$\Phi(\bar{\mathbf{C}}) = 2(-1)^m \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} G_m(x_\beta - x_\gamma) \bar{C}_\beta \bar{C}_\gamma. \quad (19)$$

Consider a linear combination of delta functions

$$\delta_{\bar{C}}(x) = \sqrt{2}\sum_{\beta=0}^{N} \bar{C}_\beta \delta(x - x_\beta). \tag{20}$$

By virtue of condition (18), this functional belongs to the space $W_2^{(m,m-1)*}$. Hence, it has an extremal function $u_{\bar{C}}(x) \in W_2^{(m,m-1)}$, which is a solution of the equation

$$\left(\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}\right) u_{\bar{C}}(x) = (-1)^m \delta_{\bar{C}}(x). \tag{21}$$

As $u_{\bar{C}}(x)$ we can take a linear combination of shifts of the fundamental solution:

$$u_{\bar{C}}(x) = \sqrt{2}(-1)^m \sum_{\beta=0}^{N} \bar{C}_\beta G_m(x - x_\beta). \tag{22}$$

The square of its norm in the space $W_2^{(m,m-1)}$ coincides with $\Phi(\bar{C})$:

$$\|u_{\bar{C}}\|^2_{W_2^{(m,m-1)}} = (\delta_{\bar{C}}, u_{\bar{C}}) = 2(-1)^m \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} \bar{C}_\beta \bar{C}_\gamma G_m(x_\beta - x_\gamma). \tag{23}$$

Hence it is obvious that for nonzero $\bar{C}$ the function $\Phi(\bar{C})$ is strictly positive.

Theorem 1 is proved.

If the nodes $x_0, x_1, \ldots, x_N$ are chosen so that the matrix S has a right inverse matrix, then system (10) - (12) has a unique solution.

Theorem 2. *The real matrix S has a right inverse matrix, so the matrix Q is not generated in the system (14) - (16).*

Proof. Denote by M the matrix of the quadratic form (19). If a homogeneous system of linear equations has only a trivial solution, then the corresponding inhomogeneous system has a unique solution. Consider a homogeneous system corresponding to system (14) - (16) in the following matrix form:

$$Q\begin{pmatrix}\bar{C}\\ \lambda\end{pmatrix} = \begin{pmatrix} M & S^* \\ S & 0\end{pmatrix}\begin{pmatrix}\bar{C}\\ \lambda\end{pmatrix} = 0. \tag{24}$$

Let us check that the unique solution of (24) is the identical zero. Let $\bar{C}, \lambda$ be a solution to (24). Consider the function $\delta_{\bar{C}}(x)$, which is defined by equality (20). As an extremal function for the function $\delta_{\bar{C}}(x)$, we take the following function:

$$u_{\bar{C}}(x) = \sqrt{2}(-1)^m \sum_{\beta=0}^{N} \bar{C}_\beta G_m(x - x_\beta) + \sum_{k=0}^{m-2} \lambda_k x^k + \lambda_{m-1} e^{-x}.$$

This is possible because $u_{\bar{C}}(x)$ belongs to the space $W_2^{(m,m-1)}$ and is a solution to equation (21). The first $N+1$ equations of system (24) mean that $u_{\bar{C}}(x)$ takes zero values at all nodes x_β. Then, with respect to the norm in $W_2^{(m,m-1)*}$ of the functional $\delta_{\bar{C}}(x)$, we have

$$\|\delta_{\bar{C}}(x)\|^2_{W_2^{(m,m-1)*}} = (\delta_{\bar{C}}, u_{\bar{C}}) = \sqrt{2}\sum_{\beta=0}^{N} \bar{C}_\beta u_{\bar{C}}(x_\beta) = 0,$$

which is possible only for $\bar{C} = 0$. Taking this into account, from the first $N+1$ equations of system (24) we get

$$S^*\lambda = 0. \tag{25}$$

By assumption, the matrix S has a right inverse, but then S^* has a left inverse. From here and from (25) we conclude that the solution λ is also equal to zero.

Theorem 2 is proved.

It follows from (9) that for a fixed step value h, the square of the norm of the error functional, being a quadratic function of the coefficients, i.e. from the function C_β, has a minimum, and, moreover, a unique one, for some specific value of $C_\beta = \mathring{C}_\beta$.

A quadrature formula with coefficients $\mathring{C}_\beta$ ($\beta = \overline{0,N}$ corresponding to this minimum is called *optimal*, and $\mathring{C}_\beta$ ($\beta = \overline{0,N}$ are called *optimal coefficients* of the quadrature formula.

Conclusion

In this work, the problem of construction of an optimal quadrature formula with high accuracy was discussed for the approximate calculation of the right Riemann–Liouville integral of the form (2). For this, the error of the quadrature formula in the form (3) was analyzed, and the problem of finding the analytical form of the coefficients of the quadrature formula that minimizes the norm of its error functional was studied. After that, the system of linear equations for coefficients of optimal quadrature formulas of the form (3) was obtained for the interval $[t, 1]$ when $0 < \alpha < 1$. Finally, the existence and uniqueness of the solution of the system of equations were proved.

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ОПТИМАЛЬНАЯ КВАДРАТУРНАЯ ФОРМУЛА ДЛЯ АППРОКСИМАЦИИ ПРАВОГО ИНТЕГРАЛА РИМАНА-ЛИУВИЛЛЯ*

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В настоящей статье обсуждена задача построения оптимальной квадратурной формулы в смысле Сарду для численного интегрирования правого интеграла Римана-Лиувилля в гильбертовом пространстве вещественнозначных функций. Сначала находится норма функционала ошибки с помощью экстремальной функции квадратурной формулы. Так как функционал ошибки определен на гильбертовом пространстве, квадратурная формула, которую мы строим, является точной для нулей этого пространства, т. е. выполняются условия, при которых влияние функционала ошибки на эти функции равно нулю. Затем строится функция Лагранжа для нахождения условного экстремума нормы функционала ошибки. Тем самым получается система линейных уравнений для коэффициентов оптимальной квадратурной формулы. Исследуется существование и единственность решения полученной системы.

Ключевые слова: оптимальная квадратурная формула, экстремальная функция, функционал ошибки, оптимальный коэффициент, функция Лагранжа.

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