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Construction of an Optimal Quadrature Formula in the Hilbert Space of Periodic Functions

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Abstract—This paper is devoted to constructing a new optimal quadrature formula in the Gilbert space of real-valued, periodic functions. Here, the norm of the error functional is calculated to obtain the upper bound for the absolute error of the considered quadrature formula. For this the extremal function of the quadrature formula is used. As well, optimal coefficients of the quadrature formula that give the minimum value to the norm of the error functional are found, and the norm of the error functional for the optimal quadrature formula is calculated. It is shown that the value of the norm of the found error functional is less than the value of the norm of the error functional for the constructed optimal quadrature formula in the space $\widetilde{L_2}^{(1)}$.

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1. INTRODUCTION, STATEMENT OF THE PROBLEM

It is known that fractional integration and fractional differentiation are as old as integer integration and differentiation. For a long time they developed slowly. Though, in the last years, there has been a growing interest in fractional calculus because of its applications in science and technology. Fractional derivatives have provided excellent tools for characterizing memory and hereditary properties of various materials and processes. There are several analytical methods which are used to solve very special (mostly linear) fractional differential and integral equations, such as the Fourier transform, the Laplace transform, the Mellin transform, and the Green function methods.

It should be noted that the existence and uniqueness of solutions of boundary and initial problems posed for differential and integro-differential equations of fractional or integer orders are mainly investigated analytically. In particular, in [1] authors considered an inverse boundary value problem for a mixed type partial differential equation with Hilfer operator of fractional integro-differentiation in a positive rectangular domain and with spectral parameter in a negative rectangular domain. There, using the Fourier series method, the solutions of direct and inverse boundary value problems were constructed in the form of a Fourier series. In the work [2], in three-dimensional domain the single-value solvability of a mixed problem for a Hilfer type nonlinear partial differential equation of the even order with small positive parameters in mixed derivatives was considered. The regular solution of the fractional differential equation was studied in the case $0 < \alpha < 1$.

Note also that, finding explicit analytical solutions to the corresponding problems is very difficult. Therefore, developing efficient and reliable numerical methods for solving general fractional differential and integral equations is useful in application. There are many recent works (for example, see the books [3, 4] and references therein) which are mainly devoted to study of numerical methods for fractional

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integrals, fractional derivatives, and fractional differential equations. For instance, in the work [5] a mathematical model and numerical method for simulation of the continuous casting process in a variable in time domain are presented. The mathematical model of the process is a Stefan problem with prescribed convection and non-linear Robin boundary condition. Considered differential equation is approximated by a finite difference scheme. The work [6] is devoted to consideration of a homogeneous Dirichlet initial-boundary value problem for a quasilinear parabolic equation with Caputo fractional time derivative. There, the equation is approximated by two finite-difference schemes: implicit and fractional step scheme. In [7] a mean field game model in the interpretation of optimal control is investigated theoretically and numerically. In the work [8] the author considered a positive semi-definite eigenvalue problem for second-order self-adjoint elliptic differential operator defined on a bounded domain in the plane with smooth boundary and Dirichlet boundary condition. There, the original differential eigenvalue problem is approximated by the finite element method with numerical integration and Lagrange curved triangular finite elements of arbitrary order.

Generally, formulas of numerical integrations play the main role in numerical solving of fractional and integer order differential and integral equations. Because the method of numerical integration is one of the methods for approximate solutions of these equations. The solutions of these equations can be expressed by the following weighted integrals

$$I(p,\varphi)=\int\limits_a^b p(x)\varphi(x)dx,$$

where $p(x)$ is a given weight function, as usual it is integrable, $\varphi(x)$ is a sufficiently smooth function from a Banach space B. Depending on the weight function $p(x)$ we can get the following type of integrals:

$a)$ regular integrals for $p(x)=1$;

$b)$ weakly singular integrals for $p(x)=(x-t)^{-\alpha}$, $0<\alpha<1$;

$c)$ singular integrals for $p(x)=(x-t)^{-1}$;

$d)$ oscillating integrals for $p(x)=\exp(2\pi i\omega x)$, where ω is a sufficiently large real number.

As usual, in the theory of quadrature formulas, the integral $I(p,\varphi)$ is approximated, for example, by the following quadrature sum

$$S(p,\varphi)=\sum_{\beta=0}^{N}C_\beta\varphi(x_\beta).$$

It should be noted that the quadrature sum $S(p,\varphi)$ can has more general form. Then, we get the quadrature formula of the form

$$I(p,\varphi)\cong S(p,\varphi), \tag{1}$$

where C_β are coefficients and x_β $(\in[a,b])$ are nodes of the formula.

In order to construct the certain quadrature formula it is necessary to find the coefficients C_β and the nodes x_β. Depending on the methods of finding the coefficients and nodes there are several approaches of construction of quadrature formulas:

classical quadrature formulas, for example, Rectangular, Trapezoidal, Simpson, Newton-Cotes, Gauss quadrature formulas (see, textbook [9]);

Monte-Carlo methods which are based on the methods of the probability theory;

optimal quadrature formulas in the sense of Sard (was started in the work [10] by A. Sard),

the best quadrature formulas which were firstly constructed by S. M. Nikolskii (see, for instance, [11]),

optimal cubature formulas were firstly considered and obtained by S. L. Sobolev, see, for instance, [12].

It should be noted that the process of construction of optimal formulas are based on the methods of functional analysis.

We recall that the difference between the integral $I(p,\varphi)$ and the quadrature sum $S(p,\varphi)$, i.e.,

$$(\ell,\varphi)=I(p,\varphi)-S(p,\varphi),$$

is called *the error* for the quadrature formula (1) and it defines the linear functional ℓ on the functions φ of the Banach space B. The functional ℓ is said to be *the error functional* of the quadrature formula (1).

The absolute value of the error for the quadrature formula (1) is estimated as follows

$$|(\ell,\varphi)| \leq ||\ell||_{B^*}\, ||\varphi||_B,$$

where B^* is the conjugate space to the Banach space B. In addition, the norm $||\ell||_{B*}$ of the error functional ℓ is bounded and it depends on coefficients C_β and nodes x_β of the formula (1).

Let us remember that the problem of finding the minimum of the norm of the error functional ℓ by coefficients C_β and nodes x_β, is called *the Nikolskii problem* or *the best quadrature formula problem*, and the obtained formula is called *the optimal quadrature formula in the sense of Nikolskii* or *the best quadrature formula*. As we said above this problem was first considered by S. M. Nikolskii. Minimization of the norm of the error functional ℓ by coefficients C_β when the nodes x_β are prescribed is *called Sard's problem*. And the obtained formula is called *the optimal quadrature formula in the sense of Sard*. Since this problem was first studied by A. Sard.

Since in the present paper we consider construction of a optimal quadrature formula we first give a brief review on optimal formulas.

In various Hilbert and Banach spaces of periodic and non-periodic functions, optimal quadrature formulas of the form (1) with weight function $p(x) = 1$ have been constructed by many researchers. The results for this case can be found, for instance, in the books [11, 13] and the literature in them. In particular, some recent results on optimal quadrature formulas are obtained in the works [14, 15].

Numerical integration formulas for weakly singular integrals with weight function $p(x) = (x-t)^{-\alpha}$, where $0 < \alpha < 1$, are presented, for instance, in [3, 4]. Recently, in [16], for the functions from the Sobolev space $L_2^{(m)}$ optimal quadrature formulas were constructed, for numerical approximation of the Abel generalized integral equations.

There are special quadrature formulas for numerical calculations of Cauchy type singular integrals. Some of resent results on numerical calculation of singular integrals with Cauchy kernel $p(x) = (x-t)^{-1}$ can be found, for example, in the works [17–19].

It is known that the Fourier transforms are widely used in science and technology, particularly, in the problems of Computed Tomography (see, for instance [20, 21]). Since in practice we have finite discrete values of a function, we have to approximately calculate the Fourier transforms. Therefore, one has to consider the problem of approximate calculation of the integral with weight function $p(x) = \exp(2\pi i\omega x)$. Particularly, the works [22, 23] are devoted to approximate calculation of Fourier integrals in the Hilbert space $W_2^{(m,m-1)}$ of non-periodic functions. Recently, in the works [20, 21]) authors constructed optimal quadrature formulas for numerical calculation of Fourier integrals in the Sobolev space $L_2^{(m)}$, where $L_2^{(m)}$ is the Hilbert space of functions which are square integrable with mth order derivative. Compared with the optimal quadrature formulas in non-periodic case constructed in [20], the approximation formula for the periodic case constructed in the work [21] is much simpler, therefore it is easy to implement and costs less computation even though both provide the similar performances.

Therefore, construction of new optimal quadrature formulas, which are simple in implementation, in various Hilbert spaces of periodic functions is very important.

We note that results of this paper is the continuation of the results of the paper [24]. Here we solve the problem of construction of optimal quadrature formula in the Hilbert space $\widetilde{W}_2^{(1,0)}(0,1]$ of periodic functions.

We recall the definition of this Hilbert space. The space $\widetilde{W}_2^{(1,0)}[0,1]$ is the Hilbert space of periodic, real-valued functions $\varphi(x)$, $0 < x \leq 1$, that are absolute continuous and square integrable with first order derivative. The inner product of two functions φ and ψ in this space is defined as follows

$$\langle\varphi,\psi\rangle_{\widetilde{W}_2^{(1,0)}} = \int_0^1 (\varphi'(x)+\varphi(x))(\psi'(x)+\psi(x))dx \tag{2}$$

and the corresponding norm of the function φ has the form

$$||\varphi||_{\widetilde{W}_2^{(1,0)}} = \left(\int_0^1 \left(\varphi'(x)+\varphi(x)\right)^2 dx\right)^{1/2}.$$

We note every function φ in this space satisfies the following periodicity condition

$$\varphi(x+\beta)=\varphi(x) \quad \text{for} \quad x\in\mathbb{R} \quad \text{and} \quad \beta\in\mathbb{Z}.$$

Here, we consider a quadrature formula of the following form

$$\int_0^1 \varphi(x)dx \cong \sum_{k=1}^{N} C_k\varphi(hk), \tag{3}$$

where $\varphi(x)\in \widetilde{W}_2^{(1,0)}$, C_k are the coefficients of the quadrature formula and $N\in\mathbb{N}$, $h=\frac{1}{N}$.

The error of the quadrature formula (3) is given as follows

$$\int_0^1 \varphi(x)dx - \sum_{k=1}^{N} C_k\varphi(hk) = \int_0^1 \left[\left(\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k\delta(x-hk)\right) * \phi_0(x)\right]\varphi(x)dx = (\ell,\varphi), \tag{4}$$

where $\varepsilon_{(0,1]}(x)$ is the characteristic function of the interval $(0,1]$, δ is the Dirac delta-function, $\phi_0(x) = \sum_{\beta=-\infty}^{\infty}\delta(x-\beta)$, * is the convolution operation and

$$\ell(x) = \left(\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k\delta(x-hk)\right) * \phi_0(x) \tag{5}$$

is *the periodic error functional* of the quadrature formula (3). Henceforth, we say the error functional instead of the periodic error functional.

The error (4) of the quadrature formula (3) is a linear functional in $\widetilde{W}_2^{(1,0)*}(0,1]$, where $\widetilde{W}_2^{(1,0)*}(0,1]$ is the conjugate space for the space $\widetilde{W}_2^{(1,0)}(0,1]$. The absolute value of the error (4) is estimated by the Cauchy–Schwarz inequality as follows

$$|(\ell,\varphi)| \le ||\ell||_{\widetilde{W}_2^{(1,0)*}}\, ||\varphi||_{\widetilde{W}_2^{(1,0)}},$$

where

$$||\ell||_{\widetilde{W}_2^{(1,0)*}} = \sup_{||\varphi||_{\widetilde{W}_2^{(1,0)}}=1} |(\ell,\varphi)| \tag{6}$$

is the norm of the error functional (5).

The problem of constructing the optimal quadrature formula (3) is as follows.

Problem 1. Find the coefficients $\mathring{C}_k$ that give the minimum value to the quantity $||\ell||_{\widetilde{W}_2^{(1,0)*}}$, and calculate the following quantity

$$\left|\left|\mathring{\ell}\right|\right|_{\widetilde{W}_2^{(1,0)*}} = \inf_{C_k} ||\ell||_{\widetilde{W}_2^{(1,0)*}}.$$

We note that the coefficients $\mathring{C}_k$ which are the solution for Problem 1 are called *the optimal coefficients* and the quadrature formula (3) with these coefficients is said to be *the optimal quadrature formula in the sense of Sard* [10] in the Hilbert space $\widetilde{W}_2^{(1,0)}$.

Further, in next sections we solve Problem 1.

The rest of the paper is organized as follows. In Section 2 we present the main results as Theorems 1 and 2. The section 3 is devoted to calculation the norm of the error functional and to obtain the system of linear equations for optimal coefficients which give the minimum value to the norm of the error function. In section 4 this system is solved and explicit expressions of the coefficients (which are optimal) for the optimal quadrature formula (3) are found. Finally, in Section 5 we calculate the quantity $\left|\left|\mathring{\ell}\right|\right|_{\widetilde{W}_2^{(1,0)*}}$ which is the sharp upper bound for the error of the optimal quadrature formula (3).

2. MAIN RESULTS

To calculate the norm (6), we use *the extremal function* ψ_ℓ for the error functional ℓ (see [12]) that satisfies the following equality

$$(\ell, \psi_\ell) = ||\ell||_{\widetilde{W_2}^{(1,0)*}} ||\psi_\ell||_{\widetilde{W_2}^{(1,0)}}. \tag{7}$$

Since $\widetilde{W_2}^{(1,0)}$ is the Hilbert space by the Riesz theorem for the error functional ℓ for any φ from $\widetilde{W}_2^{(1,0)}$ there exists an element $\psi_\ell \in \widetilde{W}_2^{(1,0)}$ that satisfies the equality

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle_{\widetilde{W_2}^{(1,0)}}, \tag{8}$$

where $\langle \psi_\ell, \varphi \rangle_{\widetilde{W_2}^{(1,0)}}$ is the inner product of the functions ψ_ℓ and φ defined by the formula (2) for $m = 1$. In addition, the equality $||\ell||_{\widetilde{W_2}^{(1,0)*}} = ||\psi_\ell||_{\widetilde{W_2}^{(1,0)}}$ is fulfilled. So, taking into account the equality (7), we derive

$$(\ell, \psi_\ell) = ||\ell||^2_{\widetilde{W_2}^{(1,0)*}}. \tag{9}$$

Integrating by parts the right-hand side of (8), keeping in mind periodicity of functions, for ψ_ℓ we have

$$\psi_\ell''(x) - \psi_\ell(x) = -\ell(x). \tag{10}$$

The solution of differential equation (10) is given in the work [25, Theorem 1], and it is equal to the following

$$\psi_\ell(x) = 1 + \sum_{k=1}^{N} C_k G_1(x - hk), \tag{11}$$

where

$$G_1(x) = -\sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i \beta x}}{(2\pi\beta)^2 + 1}, \tag{12}$$

$i^2 = -1$ and it is known that $\int_0^1 G_1(x - y)dy = -1$.

Now, we give the main results of this work.

Theorem 1. *If* $\varphi \in \widetilde{W}_2^{(1,0)}$*, then the following formulas are valid for the optimal coefficients of the quadrature formula (3) with the error functional (5)*

$$\mathring{C}_k = \frac{2\left(e^h - 1\right)}{e^h + 1}, \qquad k = 1, 2, ..., N.$$

Theorem 2. *In the space* $\widetilde{W}_2^{(1,0)}$ *for the norm of the error functional (5) of the optimal quadrature formula, the following holds*

$$\left|\left|\mathring{\ell}\right|\right|^2_{\widetilde{W}_2^{(1,0)}} = 1 - \frac{2\left(e^h - 1\right)}{h\left(e^h + 1\right)}. \tag{13}$$

Remark. It should be noted that from (13) we obtain

$$||\mathring{\ell}||^2_{\widetilde{W}_2^{(1,0)}} = \frac{1}{12}h^2 - \frac{1}{120}h^4 + O(h^6).$$

This is less than the sharp error bound

$$||\mathring{\ell}||^2_{\widetilde{L}_2^{(1)}} = \frac{1}{12}h^2$$

of the optimal quadrature of the form (3) in the space $L_2^{(1)}(0, 1]$ (see [9, Theorem 4.5, page 205]).

In order to prove Theorem 1 we calculate the norm of the error functional (5) in the next section.

3. THE NORM FOR THE ERROR FUNCTIONAL OF THE QUADRATURE FORMULA

Initially, to calculate the norm of the error functional ℓ, simplifying the error functional of the form (5), we can rewrite it in the following form

$$\ell(x)=\sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(x)*\delta(x-\beta)-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk)*\delta(x-\beta)$$

$$=\sum_{\beta=-\infty}^{\infty}\int_{-\infty}^{\infty}\varepsilon_{(0,1]}(y)\delta(x-\beta-y)dy-\sum_{k=1}^{N}\sum_{\beta=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(y-hk)\delta(x-\beta-y)dy$$

$$=\sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(x-\beta)-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-\beta-hk)=1-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-\beta-hk). \tag{14}$$

Using equalities (9), (11) and (14) we have

$$||\ell||^2_{\widetilde{W_2}^{(1,0)*}}=\int_0^1\left(1-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\right)\left(1+\sum_{k'=1}^{N}C_{k'}G_1(x-hk')\right)dx.$$

Hence we get the following for square of the norm of the error functional

$$||\ell||^2_{\widetilde{W_2}^{(1,0)*}}=1+\sum_{k=1}^{N}C_k\int_0^1 G_1(x-hk)dx-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\int_{-\infty}^{\infty}\varepsilon_{(0,1]}(x)\delta(x-hk-\beta)dx$$

$$-\sum_{k=1}^{N}\sum_{k'=1}^{N}C_kC_{k'}\sum_{\beta=-\infty}^{\infty}\int_{-\infty}^{\infty}\varepsilon_{(0,1]}(x)G_1(x-hk')\delta(x-hk-\beta)dx$$

$$=1-\sum_{k=1}^{N}C_k-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(hk+\beta)-\sum_{k=1}^{N}\sum_{k'=1}^{N}C_kC_{k'}$$

$$\times\sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(hk+\beta)G_1(hk+\beta-hk').$$

Taking into account

$$\int_0^1 G_1(x-hk)dx=-1,\quad \text{for}\quad k=1,2,...,N$$

and using equality (12) for the square of the norm of the error functional of the quadrature formula, we obtain the following analytical expression

$$||\ell||^2_{\widetilde{W_2}^{(1,0)*}}=\sum_{k=1}^{N}\sum_{k'=1}^{N}C_kC_{k'}\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi i\beta h(k-k')}}{(2\pi\beta)^2+1}-2\sum_{k=1}^{N}C_k+1. \tag{15}$$

For finding the minimum of the norm of the error functional ℓ, we consider the following function

$$L(C_1,C_2,...,C_N)=||\ell||^2_{\widetilde{W_2}^{(1,0)*}}.$$

To solve Problem 1, taking the partial derivatives of the function L with respect to C_k ($k=\overline{1,N}$) we get

$$\frac{\partial L}{\partial C_k}=0,\quad \text{for}\quad k=1,2,...,N.$$

They give the following system of the linear equations with respect to C_k:

$$\sum_{k'=1}^{N} C_{k'} \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta h(k-k')}}{(2\pi\beta)^2+1} = 1, \quad \text{for} \quad k = 1, ..., N. \tag{16}$$

By virtue of equations (16), we see that the extremal function $\psi_\ell(x)$, defined by equality (11), vanishes at the nodes of the quadrature formula (3).

The solution of the system (16) gives the minimum to the square of the norm (15) for the error functional (5) in certain values of $C_k = \mathring{C}_k$ $(k = 1, 2, ..., N)$, $\mathring{C}_k$ are called *the optimal coefficients*.

4. THE OPTIMAL COEFFICIENTS OF THE QUADRATURE FORMULA (3)

In this section we prove Theorem 1. To do this, we seek the solution of the system (16) in the form

$$\mathring{C}_k = C(h), \quad \text{for} \quad k = 1, 2, ..., N, \tag{17}$$

where $C(h)$ is an unknown function of h.

Putting (17) into (16), we obtain

$$\sum_{k'=1}^{N} C(h) \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta h(k-k')}}{(2\pi\beta)^2+1} = 1, \quad \text{for} \quad k = 1, 2, ..., N.$$

Since the infinite series in (16) is convergent, we can rewrite the last system as follows

$$C(h) \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta hk}}{(2\pi\beta)^2+1} \sum_{k'=1}^{N} e^{2\pi i\beta hk'} = 1. \tag{18}$$

It is obvious that

$$\sum_{k'=1}^{N} e^{2\pi i\beta hk'} = \frac{e^{2\pi i\beta h}(1-e^{2\pi i\beta})}{1-e^{2\pi i\beta h}} = \begin{cases} 0, & \text{if} \quad \beta \neq \gamma N, \\ N, & \text{if} \quad \beta = \gamma N. \end{cases} \tag{19}$$

where $\beta, \gamma \in \mathbb{Z}$ and $N = \frac{1}{h}$.

Taking into account (19), we write equation (18) as follows

$$C(h) \sum_{\gamma=-\infty}^{\infty} \frac{e^{-2\pi i\gamma Nhk}}{(2\pi\gamma N)^2+1} N = 1.$$

From the last equation and taking into account $e^{-2\pi i\gamma Nhk} = 1$ $(k = 1, 2, ..., N$ and $\gamma \in \mathbb{Z})$ we obtain the following

$$C(h) = h\left(\sum_{\gamma=-\infty}^{\infty} \frac{1}{(2\pi\gamma N)^2+1}\right)^{-1} = \frac{(2\pi)^2}{h}\left(\sum_{\gamma=-\infty}^{\infty} f(\gamma)\right)^{-1}, \tag{20}$$

where

$$f(\gamma) = \frac{1}{\left(\gamma - \frac{ih}{2\pi}\right)\left(\gamma + \frac{ih}{2\pi}\right)}. \tag{21}$$

To calculate the sum the series in (20) we use the following well-known formula from the residual theory (see [25], p. 296)

$$\sum_{\gamma=-\infty}^{\infty} f(\gamma) = -\sum_{z_1, z_2, \ldots, z_n} \operatorname{res}(\pi\ \cot(\pi z)\, f(z)), \tag{22}$$

where $z_1, z_2, ..., z_n$ are poles of the function $f(z)$.

From the expression (21) we have $f(z)=\frac{1}{(z-\frac{ih}{2\pi})(z+\frac{ih}{2\pi})}$. It is understandable that $z_1=\frac{ih}{2\pi}$ and $z_2=-\frac{ih}{2\pi}$ are the poles of the first order of the function $f(z)$. After taking into account the formula (22) we have

$$\sum_{\gamma=-\infty}^{\infty} f(\gamma)=-\sum_{z_1,z_2}\mathrm{res}(\pi\ \cot(\pi z)\, f(z)). \tag{23}$$

Since

$$\mathop{\mathrm{res}}_{z=z_1}(\pi\ \cot(\pi z)\, f_1(z))=\lim_{z\to z_1}\frac{\pi\ \cot(\pi z)}{z+\frac{ih}{2\pi}}=\frac{\pi^2}{i\,h}\cot\frac{ih}{2}$$

and

$$\mathop{\mathrm{res}}_{z=z_2}(\pi\ \cot(\pi z)\, f(z))=\lim_{z\to z_2}\frac{\pi\cot(\pi z)}{z-\frac{ih}{2\pi}}=\frac{\pi^2}{i\,h}\cot\frac{i\,h}{2}$$

from (23) we get

$$\sum_{\gamma=-\infty}^{\infty} f(\gamma)=-\frac{2\pi^2}{i\,h}\cot\frac{i\,h}{2}.$$

Using the well-known formula cot $z=\frac{e^{z\,i}+e^{-z\,i}}{e^{z\,i}-e^{-z\,i}}\,i$, after some simplifications for the above series, we obtain the following result

$$\sum_{\gamma=-\infty}^{\infty} f(\gamma)=\frac{2\pi^2}{h}\,\frac{e^h+1}{e^h-1}. \tag{24}$$

Therefore, from (20) and (24) we get

$$C(h)=\frac{2(e^h-1)}{e^h+1}. \tag{25}$$

From (17) and (25) the assertion of Theorem 1 follows, i.e., for the optimal coefficients of the quadrature formula (3) we have the form

$$\mathring{C}_k=\frac{2(e^h-1)}{e^h+1},\quad \text{for}\quad k=1,2,...,N. \tag{26}$$

And so, Theorem 1 is proved.

5. CALCULATION OF THE NORM FOR THE ERROR FUNCTIONAL OF THE OPTIMAL QUADRATURE FORMULA (3)

In this section we prove Theorem 2. To do this, let's simplify $||\ell||^2$ which is defined by expression (15)

$$\left|\left|\mathring{\ell}\right|\right|^2_{\widetilde{W_2}^{(1,0)*}}=\sum_{k=1}^{N}\sum_{k'=1}^{N}\mathring{C}_k\mathring{C}_{k'}\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi\, i\beta h(k-k')}}{(2\pi\beta)^2+1}-2\sum_{k=1}^{N}\mathring{C}_k+1$$
$$=\sum_{k=1}^{N}\mathring{C}_k\left[\sum_{k'=1}^{N}\mathring{C}_{k'}\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi\, i\,\beta h(k-k')}}{(2\pi\beta)^2+1}-1\right]-\sum_{k=1}^{N}\mathring{C}_k+1.$$

Hence, taking into account (16) for the square of $||\mathring{\ell}||$, we have

$$\left|\left|\mathring{\ell}\right|\right|^2_{\widetilde{W_2}^{(1,0)*}}=1-\sum_{k=1}^{N}\mathring{C}_k.$$

From equality (26), we finally obtain for the square of the norm for the error functional

$$\left|\left|\mathring{\ell}\right|\right|^2_{\widetilde{W_2}^{(1,0)*}} = 1 - \frac{2(e^h - 1)}{h(e^h + 1)}.$$

Thus, Theorem 2 is completely proved.

6. CONCLUSIONS

In the present paper, the optimal quadrature formula in the sense of Sard is constructed in the space $\widetilde{W}_2^{(1,0)}(0,1]$ of periodic, real-valued functions to an approximation of the Fourier integrals with $\omega = 0$. Here, we found analytical forms for coefficients of the constructed optimal quadrature formula. In addition, we calculated the norm of the error functional for the optimal quadrature formula and obtained that this norm is less than the norm of the error functional for the optimal quadrature formula in the space $\widetilde{L_2}^{(1)}(0,1]$ of periodic, real valued functions.

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