The Optimal Quadrature Formula for the Approximate Calculation of Fourier Coefficients in the Space $\widetilde{W_2}^{(2,1)}$ of Periodic Functions

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Abstract In this work, the process of constructing the optimal quadrature formula in the Hilbert space $\widetilde{W}_2^{(2,1)}(0,1]$ of complex-valued periodic functions for the numerical calculation of Fourier coefficients is studied. Here a quadrature sum consists of a linear combination of the given function values on a uniform mesh. The error of a quadrature formula is estimated from above by the functional norm of the error based on the Cauchy-Schwarz inequality. To calculate the norm, the concept of an extremal function is used. Also, the optimal coefficients of the quadrature formula are found. Furthermore, the sharp upper bound of the error of the constructed optimal quadrature formula is found, and it is shown that the order of convergence of the optimal quadrature formula is $O\left(\left(\frac{1}{N+|\omega|}\right)^2\right)$.

1 Introduction

Numerical calculation of the strongly oscillating integrals is one of the more critical problems in numerical analysis because such integrals are widely used in science and

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technology. The following types of Fourier integrals are also examples of strongly oscillating integrals for sufficiently large $\omega \in \mathbb{R}$

$$I(\omega,\varphi) = \int_{\Omega} e^{i\omega g(x)}\varphi(x)dx, \tag{1}$$

where φ and g are non-oscillating functions, ω is oscillation frequency and Ω is some piecewise smooth region.

Numerical approximation of these integrals can be challenging, particularly when the oscillations are highly localized and rapidly alternating. However, there are several techniques that can be used to obtain accurate numerical approximations of these integrals. One such technique is the stationary phase method, which involves identifying the stationary points of the integrand and approximating the integral using the properties of the integrand near these points. Another technique is to use a quadrature method that is designed explicitly for oscillatory integrals, such as the Filon, Clenshaw-Curtis, modified Clenshaw-Curtis method, Levin type methods, Gauss-Laguerre quadrature and generalized quadrature (see [26] for full details, for instance, [17, 22] and references therein).

These methods involve different approaches to approximating oscillatory integrals, and which approach is best suited to a given problem will depend on the specific properties of the integrand and the desired accuracy of the approximation. Additionally, it's worth noting that an effective approximation of a strongly oscillatory integral may require a large number of samples.

Based on the properties of the functions φ and g, various methods have been developed for the numerical calculation of highly oscillating integrals (1). There are *the Filon method, the asymptotic expansion, Levin's collocation method*, *the steepest descent and optimal quadrature formulas* methods of approximate calculation of such integrals.

In recent years, scientists such as A. Iserles [13], S.P. Nørsett [14], S. Olver [21], G.V. Milovanović [18, 19, 20], Z. Xu [27], S. Zhang and E. Novak [28] have been engaged in the approximate solution of integral (1). Also, G.V. Milovanović, Kh.M. Shadimetov and A.R. Hayotov [2, 3] have conducted scientific research on constructing optimal quadrature formulas for calculating Fourier coefficients and integrals in different spaces, including in spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$. The results of constructing optimal quadrature formulas for the numerical calculation of Fourier integrals in the Sobolev space $\widetilde{L_2^{(m)}}$ of periodic functions and applying the constructed formulas to the reconstruction of computed tomography images were obtained in the researches of A.R. Hayotov, S. Jeon, Ch.-O. Lee and Kh. M. Shadimetov [5, 6, 7, 8].

In this work, the problem of constructing an optimal quadrature formula for the approximate calculation of Fourier coefficients in Hilbert space $\widetilde{W}_2^{(2,1)}(0,1]$ of periodic functions is studied.

2 Statement of the Problem

In this paper, we are concerned with obtaining optimal quadrature formulas. It is assumed that the integrand belongs to the Hilbert space $W_2^{(2,1)}$. Recall the definition of this Hilbert space, based on the work [1].

$W_2^{(2,1)}(0,1)$ is the Hilbert space of complex-valued functions and it is defined as follows

$$W_2{}^{(2,1)} = \{\varphi : [0,1] \to \mathbb{C} | \varphi' \text{ is abs. continuous and } \varphi'' \in L_2(0,1)\}.$$

The space $W_2^{(2,1)}$ is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{W_2^{(2,1)}} = \int_0^1 (\varphi''(x) + \varphi'(x))(\overline{\psi}''(x) + \overline{\psi}'(x))dx, \tag{2}$$

and the corresponding norm is

$$\|\varphi\|_{W_2^{(2,1)}} = \left(\langle \varphi, \varphi \rangle_{W_2^{(2,1)}}\right)^{1/2}.$$

The last equality is a semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = d_0 + d_1 e^{-x}$, where d_0 and d_1 are constants. Every element of the space $W_2^{(2,1)}(0,1)$ is a set of functions which differ from each other on a linear combination of any constant and e^{-x}. So, the space $W_2^{(2,1)}(0,1)$ is a factor space.

We denote by $\widetilde{W}_2^{(2,1)}(0,1]$ the subspace of $W_2^{(2,1)}(0,1)$ consisting of 1-periodic functions. Every element of the space $\widetilde{W}_2^{(2,1)}(0,1]$ satisfies the following condition of 1-periodicity

$$\varphi(x+\beta) = \varphi(x) \text{ for } x \in \mathbb{R} \text{ and } \beta \in \mathbb{Z}.$$

Now we consider the following quadrature formula

$$\int_0^1 e^{2\pi i w x} \varphi(x)dx \cong \sum_{k=1}^{N} C_k \varphi(hk), \tag{3}$$

where ω is a non-zero integer, $\varphi \in \widetilde{W_2}^{(2,1)}(0,1]$, C_k are coefficients to be determined and $h = \frac{1}{N}$ is a step of the mesh.
The difference

$$(\ell, \varphi) = \int_0^1 e^{2\pi i w x} \varphi(x)dx - \sum_{k=1}^{N} C_k \varphi(hk) \tag{4}$$

between the integral and the quadrature sum is called *the error* of the quadrature formula (3) and the corresponding *error functional* is

$$\ell(x) = e^{2\pi i w x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta), \tag{5}$$

where δ is the Dirac delta-function. The error functional $\ell(x)$ is called *the periodic error functional* of the quadrature formula (3) and it belongs to the conjugate the space $\widetilde{W}_2^{(2,1)*}(0,1]$.

Since the error functional ℓ is defined on the space $\widetilde{W}_2^{(2,1)}(0,1]$, the following equality is valid as in the work [25]

$$(\ell, 1) = 0\,. \tag{6}$$

The error (4) of the quadrature formula (3) is a linear functional in the space $\widetilde{W}_2^{(2,1)*}\,(0,1]$. The absolute value of the error (4) is estimated by the Cauchy-Schwarz inequality as follows

$$|(\ell,\varphi)| \leq \|\ell\|_{\widetilde{W_2}^{(2,1)*}} \cdot \|\varphi\|_{\widetilde{W_2}^{(2,1)}}\,,$$

where

$$\|\ell\|_{\widetilde{W_2}^{(2,1)*}} = \sup_{\varphi,\ \|\varphi\|_{\widetilde{W_2}^{(2,1)}} \neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|_{\widetilde{W_2}^{(2,1)}}}$$

is the norm of the error functional (5).

Hence, in order to get the minimum of the upper bound of the error for the quadrature formula (3) we solve the following.

Problem 1 Find the norm of the error functional (5) of the quadrature formula (3) in the space $\widetilde{W}_2^{(2,1)}\,(0,1]$.

Problem 2 Find the optimal coefficients C_k that give the minimum value to the norm of the error functional (5).

Problem 3 Calculate the norm of the error functional for the optimal quadrature formula (3) in the following form

$$\left\|\overset{\circ}{\ell}\right\|_{\widetilde{W}_2^{(2,1)*}} := \inf_{C_k} \|\ell\|_{\widetilde{W}_2^{(2,1)*}}\,.$$

Definition 1 The coefficients that give the minimum value to the norm $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$ are called *optimal coefficients* and are denoted by $\overset{\circ}{C}_k$. The quadrature formula (3) with these coefficients is said to be *the optimal quadrature formula*.

Similar problem was first proposed by S.L. Sobolev [25], later scientists such as Kh.M. Shadimetov [23, pp. 97–104], A.R. Hayotov [9], N.D. Boltaev [1] and S.S. Babaev [4] were engaged in solving these problems.

This paper is composed of the following parts in solving Problems 1-3.

- the first and second sections are devoted to the introduction and statement of the problem. Also, the work carried out on the approximation of Fourier integrals is studied in it;

- in the third section, we find the extremal function corresponding to the error functional (5) and calculate the norm of the error functional (5) using it;
- in the fourth section, the process of solving the system of linear equations for the coefficients of the optimal quadrature formula is presented;
- in the fifth section, the sharp upper bound of the error of the constructed optimal quadrature formula is found;
- and in the last section some numerical results are presented.

3 The Norm of the Error Functional (5)

To calculate the norm $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$, we use *the extremal function* ψ_ℓ for the error functional ℓ (see [25]) that satisfies the following equality:

$$(\ell, \psi_\ell) = \|\ell\|_{\widetilde{W_2}^{(2,1)*}} \cdot \|\psi_\ell\|_{\widetilde{W_2}^{(2,1)}} . \tag{7}$$

Since $\widetilde{W_2}^{(2,1)}$ (0, 1] is the Hilbert space, using the Riesz representation theorem, we obtain

$$(\ell, \varphi) = \langle \varphi, \psi_\ell \rangle_{\widetilde{W_2}^{(2,1)}} \quad \text{for all } \varphi \in W_2^{(2,1)}, \tag{8}$$

where $\langle \psi_\ell, \varphi \rangle_{\widetilde{W_2}^{(2,1)}}$ is the inner product of the functions ψ_ℓ and φ which is defined by expression (2). In addition, the equality $\|\ell\|_{\widetilde{W_2}^{(2,1)*}} = \|\psi_\ell\|_{\widetilde{W_2}^{(2,1)}}$ is fulfilled. So, taking into account (7), we derive

$$(\ell, \psi_\ell) = \|\ell\|^2_{\widetilde{W_2}^{(2,1)*}}. \tag{9}$$

Then from (8), integrating by parts, we obtain

$$\int_0^1 \ell(x)\varphi(x)dx = \int_0^1 \left(\overline{\psi_\ell}^{IV}(x) - \overline{\psi_\ell}^{II}(x)\right) \varphi(x)dx.$$

From the last equation, we have

$$\overline{\psi_\ell}^{IV}(x) - \overline{\psi_\ell}^{II}(x) = \ell(x). \tag{10}$$

For the solution of the last equation the following holds.

Theorem 1 *The solution to differential equation* (10) *is the extremal function* ψ_ℓ *corresponding to the error functional* ℓ *and it is expressed as*

$$\psi_\ell(x) = d_0 + e^{-2\pi i\omega x} \cdot \kappa(\omega) - \sum_{k=1}^{N} \overline{C_k} \sum_{\beta\neq 0} \kappa(\beta) e^{2\pi i\beta(x-hk)}, \tag{11}$$

where d_0 *is a complex number and*

$$\kappa(\omega) = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2}. \tag{12}$$

Proof We find the periodic solution of equation (10) using the following properties of the Fourier transforms (see, for instance, [1, 4])

$$F[\varphi] = \int_{-\infty}^{\infty} \varphi(x) e^{2\pi i p x} dx,$$

$$F^{-1}[\varphi] = \int_{-\infty}^{\infty} \varphi(p) e^{-2\pi i p x} dp,$$

$$F[\varphi^{(\alpha)}] = (-2\pi i p)^{\alpha} F[\varphi], \ (\alpha \in \mathbb{N}),$$

$$F^{-1}[F[\varphi(x)]] = \varphi(x).$$

We apply the Fourier transform to both sides of equation (10) and here

$$F[\overline{\psi_\ell}^{IV} - \overline{\psi_\ell}^{II}] = F[\ell].$$

Since the Fourier transform is a linear operator, we have

$$\left((2\pi i p)^4 - (2\pi i p)^2\right) F[\overline{\psi_\ell}] = F\left[e^{2\pi i w x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta)\right]$$

or

$$\left((2\pi i p)^4 - (2\pi i p)^2\right) F[\bar{\psi}_\ell] = F\left[e^{2\pi i \omega x}\right] - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} F\left[\delta(x - hk - \beta)\right]. \tag{13}$$

Now, using the following equalities

$$F\left[e^{2\pi i \omega x}\right] = \delta(p + \omega), \ \sum_{\beta=-\infty}^{\infty} e^{2\pi i p \beta} = \sum_{\beta=-\infty}^{\infty} \delta(p - \beta),$$

and

$$\sum_{\beta=-\infty}^{\infty} F\left[\delta(x - hk - \beta)\right] = \sum_{\beta=-\infty}^{\infty} e^{2\pi i p (hk+\beta)} = e^{2\pi i p h k} \sum_{\beta=-\infty}^{\infty} e^{2\pi i p \beta}$$
$$= e^{2\pi i p h k} \sum_{\beta=-\infty}^{\infty} \delta(p - \beta),$$

we can rewrite equation (13) as follows

$$\left((2\pi i p)^4 - (2\pi i p)^2\right) F[\bar{\psi}_\ell] = \delta(p + \omega) - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} e^{2\pi i p h k} \delta(p - \beta).$$

We consider that the coefficient on the left-hand side of the last equation is not equal to zero at $p \neq 0$. Consequently, we can multiply both sides of the last equation by $\kappa(p)$ which is defined by (12). This multiplication is not uniquely defined. From the last equation the function $F[\bar{\psi}_\ell]$ is defined up to the term of the form $\delta(p)$. Taking into account the above and the properties of Dirac's delta-function, we get

$$F[\bar{\psi}_\ell] = \delta(p+\omega)\kappa(p) - \sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i p h k}\delta(p-\beta)\kappa(p) + d_0\delta(p),$$

where $\kappa(p)$ is defined by formula (12) and d_0 is a constant.

Using the property $f(x)\delta(x-a) = f(a)\delta(x-a)$ of the Dirac delta-function, we have the following

$$F[\bar{\psi}_\ell] = \delta(p+\omega)\kappa(\omega) - \sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i \beta h k}\kappa(\beta)\delta(p-\beta) + d_0\delta(p).$$

Then, applying the inverse Fourier transform to both sides of the last equation, we have

$$\bar{\psi}_\ell(x) = e^{2\pi i \omega x}\kappa(\omega) - \sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i \beta h k}\kappa(\beta)e^{-2\pi i \beta x} + d_0.$$

Since $\overline{\overline{\psi}}_\ell$ is equal to ψ_ℓ, we obtain (11), that is, Theorem 1 is proved. □

The following result is true for the solution of Problem 1.

Theorem 2 *The square of the norm* $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$ *for* $\omega \in \mathbb{Z}\backslash\{0\}$ *has the following form*

$$\|\ell\|^2_{\widetilde{W_2}^{(2,1)*}} = \kappa(\omega) - \kappa(\omega)\left[\sum_{k=1}^{N} C_k e^{-2\pi i \omega h k} + \sum_{k'=1}^{N} \overline{C_{k'}} e^{2\pi i \omega h k'}\right]$$
$$+ \sum_{k=1}^{N}\sum_{k'=1}^{N} C_k \overline{C_{k'}} \sum_{\beta\neq 0} \kappa(\beta) e^{2\pi i \beta h (k-k')}, \tag{14}$$

where $\kappa(\cdot)$ *is defined by* (12).

Proof To prove Theorem 2, we calculate the norm $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$ and use equalities (9), (14) and (11), respectively. As a result, we have the following

$$\|\ell\|^2_{\widetilde{W_2}^{(2,1)*}} = (\ell, \psi_\ell) = \int_0^1 \ell(x)\psi_\ell(x)dx$$
$$= \int_0^1 \ell(x)\left(d_0 + e^{-2\pi i \omega x}\cdot\kappa(\omega) - \sum_{k'=1}^{N} \overline{C_{k'}} \sum_{\gamma\neq 0} \kappa(\gamma) e^{2\pi i \gamma (x - hk')}\right)dx$$

Taking into account the condition (6), from the above equation, we get the following

$$\|\ell\|^2_{\widetilde{W_2}^{(2,1)*}} = \int_0^1 \ell(x)\left(e^{-2\pi i\omega x}\cdot\kappa(\omega)-\sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\gamma\neq 0}\kappa(\gamma)\cdot e^{2\pi i\gamma(x-hk')}\right)dx.$$

Simplifying the last expression, we obtain equality (14).

So, Theorem 2 is completely proven, that is, Problem 2 is solved. □

4 Optimal Coefficients of the Optimal Quadrature Formula of the Form (3)

In the present section, we find the optimal coefficients of the quadrature formula (3) that give the minimum value to the norm for the error functional ℓ. To find the minimum of the norm $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$ with respect to the coefficients C_k under the condition (6), we use the Lagrange method for finding the conditional extremal. We consider the following function

$$\daleth\left(C_k,\overline{C}_{k'},\mu\right)=\|\ell\|^2_{\widetilde{W_2}^{(2,1)}}-\mu\cdot(\ell,1)\quad \text{for}\ \ k=1,2,\ldots,N,$$

where μ is a constant.

Equating the partial derivatives by all variables C_k, $\overline{C}_{k'}$ and μ of the function $\daleth\left(C_k,\overline{C}_{k'},\mu\right)$ to zero, we have

$$\frac{\partial\daleth}{\partial C_k}=-e^{-2\pi i\omega hk}\kappa(\omega)+\sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\kappa(\beta)+\mu=0$$

for each $k=1,2,\ldots,N,$ (15)

$$\frac{\partial\daleth}{\partial \overline{C}_{k'}}=-e^{2\pi i\omega hk'}\kappa(\omega)+\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\kappa(\beta)=0$$

for each $k'=1,2,\ldots,N,$ (16)

$$\frac{\partial\daleth}{\partial\mu}=\int_0^1 e^{2\pi i\omega x}dx-\sum_{k=1}^{N}C_k=0, \tag{17}$$

where $\kappa(\cdot)$ is defined by (12).

A solution of this system which we denote by $\{\overset{\circ}{C}_k,\ k=1,2,\ldots,N\}$ and $\overset{\circ}{\mu}$, is a minimum point for the function $\daleth(C_k,\overline{C}_{k'},\mu)$. The system of equations (15) with (17) is equivalent to the system (16), (17). Therefore, it is sufficient to solve the system (16), (17).

The following theorem is valid for the solution of the system (16), (17).

Theorem 3 *Among the quadrature formulas of the form* (3) *with* $\omega \in \mathbb{Z}\setminus\{0\}$ *and* $\omega h \notin \mathbb{Z}$ *in the space* $\widetilde{W}_2^{(2,1)}(0,1]$ *of complex-valued periodic functions, there is a unique quadrature formula with optimal coefficients having the representation*

$$\overset{\circ}{C}_k = \frac{2}{(2\pi\omega)^4 + (2\pi\omega)^2}\left[\frac{h}{1-\cos 2\pi\omega h} - \frac{e^{2h}-1}{e^{2h}-2e^h\cos 2\pi\omega h + 1}\right]^{-1} \cdot e^{2\pi i\omega hk}$$

for each $k = 1, 2, \ldots, N.$ (18)

Proof Let ω be a non-zero integer and $\omega h \notin \mathbb{Z}$. Then we search a solution of the system (16),(17) in the following form

$$\overset{\circ}{C}_k = C(\omega, h)\cdot e^{2\pi i\omega hk} \quad \text{for each} \quad k = 1, 2, \ldots, N, \tag{19}$$

where $C(\omega, h)$ is an unknown function.

Using form (19), we rewrite the system (16) as follows

$$-e^{2\pi i\omega hk'}\kappa(\omega) + C(\omega, h)\sum_{\beta\neq 0} e^{-2\pi i\beta hk'}\kappa(\beta)\sum_{k=1}^{N} e^{2\pi i(\beta+\omega)hk} = 0$$
$$\text{for each} \quad k' = 1, 2, \ldots, N,$$

where

$$\sum_{k=1}^{N} e^{2\pi i(\beta+\omega)hk} = \begin{cases} 0 & \text{if } (\beta+\omega)h \neq t \in \mathrm{Z},\\ N & \text{if } (\beta+\omega)h = t \in \mathrm{Z}.\end{cases}$$

Simplifying the left-hand side of the last equation, we derive the following equation

$$-e^{2\pi i\omega hk'}\kappa(\omega) + C(\omega, h)\cdot N\sum_{t=-\infty}^{\infty} e^{2\pi i\omega hk'}\kappa(t\cdot N-\omega) = 0 \quad \text{for each} \quad k' = 1, 2, ..., N.$$

Simplifying the last equation, we have

$$C(\omega, h) = h\cdot\kappa(\omega)\left(\sum_{t=-\infty}^{\infty}\kappa(t\cdot N-\omega)\right)^{-1}. \tag{20}$$

Calculating the infinity series in the right-hand side of the last expression as in [24, p. 66], we have the following result

$$\sum_{t=-\infty}^{\infty}\kappa(t\cdot N-\omega) = \frac{h^2}{1-\cos 2\pi\omega h} - \frac{e^{2h}-1}{e^{2h}-2e^h\cos 2\pi\omega h+1}\cdot\frac{h}{2}.$$

Considering the last equality and equalities (19), (20), we get the equality (18). Thereby, we have proved Theorem 3. □

Thus, we have solved Problem 2.

In the theory of quadrature formulas, it is important to find the upper bound their error while constructing the quadrature formulas. Therefore, we find the sharp upper bound of the error of the constructed optimal quadrature formula in the next section.

5 The Norm for the Error Functional of the Optimal Quadrature Formula of the Form (3)

In the present section, we solve Problem 3, that is, we calculate the norm of the error functional (5).

The following theorem holds for the norm of the error functional ℓ for the optimal quadrature formula of the form (3).

Theorem 4 *The norm for the error functional of the optimal quadrature formula* (3) *for* $\omega \in \mathbb{Z}\setminus\{0\}$ *and* $\omega h \notin \mathbb{Z}$ *has the following form*

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}\left[1-\frac{2N}{(2\pi\omega)^4+(2\pi\omega)^2}\times\left[\frac{h}{1-\cos 2\pi\omega h}-\frac{e^{2h}-1}{e^{2h}-2e^h\cos 2\pi\omega h+1}\right]^{-1}\right], \tag{21}$$

where N is the number of nodes.

Proof We use (14)

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \kappa(\omega)-\kappa_m(\omega)\sum_{k=1}^{N}\overset{\circ}{C}_k e^{-2\pi i\omega hk}-\kappa(\omega)\sum_{k'=1}^{N}\overset{\circ}{\overline{C}}_{k'}e^{2\pi i\omega hk'}$$
$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}\overset{\circ}{C}_k\overset{\circ}{\overline{C}}_{k'}\sum_{\beta\neq 0}\kappa_m(\beta)\cdot e^{2\pi i\beta h(k-k')} = \kappa(\omega)-\kappa(\omega)\sum_{k=1}^{N}\overset{\circ}{C}_k e^{-2\pi i\omega hk}$$
$$-\sum_{k'=1}^{N}\overset{\circ}{\overline{C}}_{k'}\left[\kappa(\omega)e^{2\pi i\omega hk'}-\sum_{k=1}^{N}\overset{\circ}{C}_k\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\kappa(\beta)\right].$$

Considering (16) and $\kappa(\omega)$ defined by (12), we can rewrite the last expression as

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}\left[1-\sum_{k=1}^{N}\overset{\circ}{C}_k e^{-2\pi i\omega hk}\right].$$

Taking into account the formula (18) in the last equality, we get expression (21).

Thus, Theorem 4 is proved. □

Problem 3 is solved.

Next, we present some numerical results for the error of the optimal quadrature formula (3).

6 Numerical Results

In the present section, we give several numerical results of the sharp upper bounds for the error (4) of the optimal quadrature formulas (3) in the space $\widetilde{W}_2^{(2,1)}(0,1]$.

According to the Cauchy-Schwarz inequality, in the space $\widetilde{W}_2^{(2,1)}(0,1]$, for the absolute value of the error (4), we obtain

$$\left|\left(\overset{\circ}{\ell},\varphi\right)\right| \leq \|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}} \cdot \|\varphi\|_{\widetilde{W}_2^{(2,1)}},$$

where $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ is defined by equality (21).

Using Theorem 4 for $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$,we obtain the numerical results which are presented in Table 1 when $h = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$ and $\omega = 1, 11, 101, 1001$, 10001. From the first column of this table, we can see that order of convergence of our optimal quadrature formula is $O(h^2)$ and from the first row of Table 1 it is clear that the quantity $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ converges as $O\left(|\omega|^{-2}\right)$. From the other columns and rows of Table 1, we conclude that the order of convergence of our optimal quadrature formula is $O\left(\left(\frac{1}{N+|\omega|}\right)^2\right)$ in the space $\widetilde{W}_2^{(2,1)}(0,1]$.

Table 1 In this graph, the values of the norm of the error function $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ are presented at different h and w

h	$\omega = 1$	$\omega = 11$	$\omega = 101$	$\omega = 1001$	$\omega = 10001$
1	2.5015e-2	2.09319e-4	2.483115e-6	2.5279710e-8	2.53252306e-10
10^{-1}	3.9029e-4	2.09312e-4	2.483115e-6	2.5279710e-8	2.53252306e-10
10^{-2}	3.7285e-6	3.94094e-6	2.483115e-6	2.5279710e-8	2.53252306e-10
10^{-3}	3.7268e-8	3.72890e-8	3.907038e-8	2.5279710e-8	2.53252306e-10
10^{-4}	3.7268e-10	3.72680e-10	3.728566e-10	3.9038127e-10	2.53252306e-10

Using the values of Table 1, we generate graphs of the norm for the error functional at $N = 10$, $N = 100$ and $N = 1000$, where N is the number of nodes, that is, $N = 1/h$.

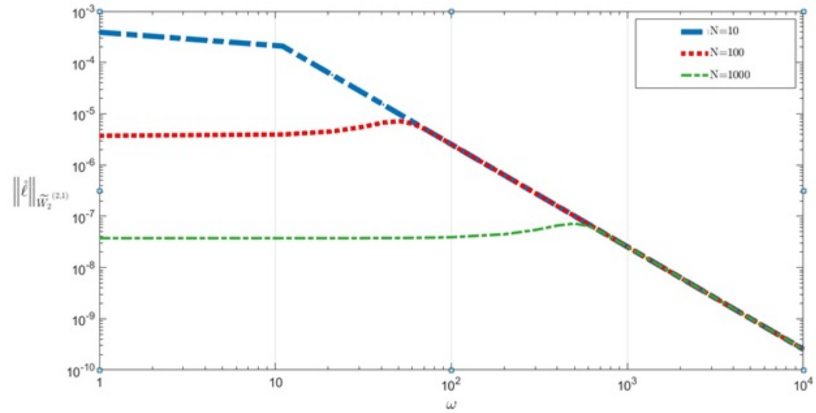

Fig. 1 Graph of the norm for the error functional depends on ω at $N = 10, N = 100$ and $N = 1000$.

Conclusion

Herein, the process of constructing the optimal quadrature formula for approximating strongly oscillating integrals in the Hilbert space of complex-valued periodic functions is studied. In this, firstly, to estimate the sharp upper bound of the absolute value of the error of the quadrature formula, the analytical form of the norm for the error functional is found, in which the extremal function corresponding to the error functional is initially found. In addition, the explicit formulas for optimal coefficients are found in the space $\widetilde{W_2}^{(2,1)}(0,1]$ of periodic functions. Finally, we presented some numerical results using the optimal coefficient. The numerical results show that the optimal quadrature formula approaches the considered integral by $O(h^2)$when $|\omega| < N$ and by $O\left(|\omega|^{-2}\right)$when $|\omega| \geq N$.

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