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THE NORM FOR THE ERROR FUNCTIONAL OF THE QUADRATURE FORMULA WITH DERIVATIVE IN THE SPACE $\widetilde{W}_2^{(2,1)}$ OF PERIODIC FUNCTIONS

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Abstract. *This paper is devoted to the process of constructing an optimal quadrature formula in the sense of Sard in the Hilbert space* $\widetilde{W}_2^{(2,1)}(0,1]$ *of periodic, complex-valued functions for numerical calculation of Fourier integrals. Here a quadrature sum consists of a linear combination of the given function value on a uniform mesh. The error of a quadrature formula is estimated from above by the functional norm of the error based on the Cauchy-Schwarz inequality. To calculate the norm, the concept of an extremal function is used. The extremal function corresponding to the error functional is found using the Riesz representation theorem. It is known from the theorem that the norm of the extremal function is equal to the norm of the error functional in the conjugate space. The norm of the error functionals is a multivariate quadratic function with respect to coefficients of the quadrature formula.*

Keywords: *optimal quadrature formula, Hilbert space, strongly oscillatory integrals, the error functional, Fourier transform.*

НОРМА ФУНКЦИОНАЛА ПОГРЕШНОСТИ КВАДРАТУРНОЙ ФОРМУЛЫ С ПРОИЗВОДНОЙ В ПРОСТРАНСТВЕ $\widetilde{W}_2^{(2,1)}$ ПЕРИОДИЧЕСКИХ ФУНКЦИЙ

Аннотация. *Работа посвящена процессу построения оптимальной квадратурной формулы по Сарду в гильбертовом пространстве* $\widetilde{W}_2^{(2,1)}(0,1]$ *периодических комплекснозначных функций для численного вычисления интегралов Фурье. Здесь квадратурная сумма представляет собой линейную комбинацию значений заданной функции на равномерной сетке. Погрешность квадратурной формулы оценивается сверху функциональной нормой погрешности на основе неравенства Коши-Шварца. Для вычисления нормы используется понятие экстремальной функции. Экстремальная функция, соответствующая функционалу погрешности, находится с помощью теоремы о представлении Рисса. Из теоремы известно, что норма экстремальной функции равна норме функционала погрешности в сопряженном пространстве. Норма функционалов погрешности представляет собой многомерную квадратичную функцию по коэффициенты квадратурной формулы.*

Ключевые слова: *оптимальная квадратурная формула, гильбертово пространство, сильно осциллирующие интегралы, функционал погрешности, преобразование Фурье.*

DAVRIY FUNKSIYALARNING $\widetilde{W}_2^{(2,1)}$ FAZOSIDA HOSILALI OPTIMAL KVADRATUR FORMULANING XATOLIK FUNKSIONALI NORMASI

Annotatsiya. *Ushbu maqola Furye integrallarini sonli hisoblash uchun davriy, kompleks qiymatli funksiyalarning* $\widetilde{W}_2^{(2,1)}(0,1]$ *Gilbert fazosida Sard ma'nosida optimal kvadratur formula qurish jarayoniga* *bag'ishlangan. Bu yerda kvadratur yig'indi teng qadamli to'rda berilgan funksiya qiymatining chiziqli kombinatsiyasidan iborat. Kvadratur formulaning xatosi yuqoridan Koshi-Shvars tengsizligiga asoslangan xatolik funksionali normasi bilan baholanadi. Xatolik funksionali normasini hisoblash uchun ekstremal funksiya tushunchasidan foydalaniladi. Xatolik funksionaliga mos keladigan ekstremal funksiya Riss teoremasi yordamida topiladi. Bu teoremadan ma'lumki, ekstremal funksiyaning normasi qo'shma fazodagi xatolik funksionali normasiga teng. Xatolik funksionalining normasi kvadratur formulaning koeffitsiyentlarga nisbatan ko'p o'zgaruvchili kvadratik funksiyadir.*

Kalit so'zlar: *optimal kvadratur formula, Gilbert fazosi, kuchli tebranuvchi integrallar, xatolik funksionali, Furye almashtirishi.*

1. Introduction

Numerical calculation of the strongly oscillating integrals is one of the more critical problems on numerical analysis because such integrals are widely used in science and technology. The following types of the Fourier integrals are also examples of strongly oscillating integrals for sufficiently large $\omega \in \mathbb{R}$

$$I(\omega,\varphi)=\int_{\Omega} e^{i\omega g(\mathbf{x})}\varphi(\mathbf{x})d\mathbf{x},$$

where φ and g are non-oscillating functions, ω is oscillation frequency and Ω is some piecewise smooth region.

Numerical approximation of these integrals can be challenging, particularly when the oscillations are highly localized and rapidly alternating. However, there are several techniques that can be used to obtain accurate numerical approximations of these integrals. One such technique is the stationary phase method, which involves identifying the stationary points of the integrand and approximating the integral using the properties of the integrand near these points. Another technique is to use a quadrature method that is designed explicitly for oscillatory integrals, such as the Filon, Clenshaw-Curtis, modified Clenshaw-Curtis method, Levin type methods, Gauss-Laguerre quadrature and generalized quadrature (see [20] for full details, for instance, [13,14] and references therein).

In the recent years, Kh.M. Shadimetov, G.V. Milovanović and A.R. Hayotov [1, 3, 4], N.D. Boltaev [2], S.S. Babaev [5] have conducted scientific research on constructing optimal quadrature formulas for calculating Fourier coefficients and integrals in different spaces, including in $L_2^{(m)}$ and $W_2^{(m,m-1)}$. The results of constructing optimal quadrature formulas for the numerical calculation of Fourier coefficients in the space $\widetilde{W}_2^{(m,m-1)}(0,1]$ of periodic functions and applying the constructed formulas to reconstruction of computed tomography images were obtained in the researches of A.R. Hayotov and U.N. Khayriev [8, 9, 10, 11, 12].

In this work, the problem of constructing an optimal quadrature formula with derivative in the Hilbert space $\widetilde{W}_2^{(2,1)}(0,1]$ of periodic functions is studied.

2. Statement of the problem

In this paper, we are concerned with obtaining optimal quadrature formulas. It is assumed here that the integrand belongs to the Hilbert space $W_2^{(2,1)}$. Recall the definition of this Hilbert space, based, for example, in the work [2].

$W_2^{(2,1)}(0,1)$ is the Hilbert space of complex-valued functions and it is defined as follows

$$W_2^{(2,1)}=\{\varphi:[0,1]\to\mathbb{C}\,|\,\varphi' \text{ is abs. continuous and } \varphi''\in L_2(0,1)\}.$$

The space is the Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_{W_2^{(2,1)}}=\int_0^1(\varphi''(x)+\varphi'(x))(\overline{\psi}''(x)+\overline{\psi}'(x))dx, \qquad (1)$$

and the corresponding norm is

$$\|\varphi\|_{W_2^{(2,1)}}=\left(\langle\varphi,\psi\rangle_{W_2^{(2,1)}}\right)^{1/2}.$$

This equality is a semi-norm and $\|\varphi\|=0$ if and only if $\varphi(x)=d_0+d_1e^{-x}$, where d_0 and d_1 is a constant. Every element of the space $W_2^{(2,1)}(0,1)$ is a set of functions which are differ from each other on linear combination of any constanta and e^{-x}. So, the space $W_2^{(2,1)}(0,1)$ is a factor space.

We denote by $\widetilde{W}_2^{(2,1)}(0,1]$ the subspace of $W_2^{(2,1)}(0,1)$ consisting of periodic, complex-valued functions. Every element of the space $\widetilde{W}_2^{(2,1)}(0,1]$ satisfies the following condition of 1-periodicity

$$\varphi(x+\beta)=\varphi(x) \text{ for } x\in\mathbb{R} \text{ and } \beta\in\mathbb{Z}.$$

We consider the following quadrature formula

$$\int_0^1 e^{2\pi iwx}\varphi(x)dx \cong \sum_{k=1}^{N} C_k\varphi(hk)+\sum_{k=1}^{N} A_k\varphi'(hk), \tag{2}$$

where ω is non-zero integer, $\varphi\in\widetilde{W}_2^{(2,1)}[0,1)$, C_k are coefficients given in the work [12]

$$C_k=\frac{2}{4\pi^2\omega^2+1}\cdot\frac{e^{2h}-2e^h\cos(2\pi\omega h)+1}{e^{2h}-1}\cdot e^{2\pi i\omega hk} \text{ for each } k=1,2,\dots,N,$$

and A_k are unknown coeffisients to be determined and $h=\dfrac{1}{N}$ is a step of the mesh.

The difference between the integral and the quadrature sum is called *the error* of the quadrature formula (2)

$$(\ell,\varphi)=\int_0^1 e^{2\pi iwx}\varphi(x)dx-\sum_{k=1}^{N}C_k\varphi(hk)-\sum_{k=1}^{N}A_k\varphi'(hk), \tag{3}$$

and the corresponding *error functional* is

$$\ell(x)=\left(e^{2\pi iwx}\varepsilon_{(0,1]}(x)-\sum_{k=1}^{N}C_k\delta(x-hk)+\sum_{k=1}^{N}A_k\delta'(x-hk)\right)*\phi_0(x), \tag{4}$$

where $\varepsilon_{(0,1]}(x)$ is the characteristic function of the interval $(0,1]$, δ is the Dirac delta-function and $\phi_0(x)=\sum_{\beta=-\infty}^{\infty}\delta(x-\beta)$. The error functional $\ell(x)$ is called *the periodic error functional* of the quadrature formula (2) and it belongs to the conjugate space $\widetilde{W}_2^{(2,1)*}(0,1]$.

Since the error functional ℓ is defined on the space $\widetilde{W}_2^{(2,1)}(0,1]$, the following equality is valid as in the work [19]

$$(\ell,1)=0. \tag{5}$$

The error (3) of the quadrature formula (2) is a linear functional in $\widetilde{W}_2^{(2,1)*}(0,1]$. The absolute value of the error (3) is estimated by the Cauchy-Schwarz inequality as follows

$$|(\ell,\varphi)|\le\|\ell\|_{\widetilde{W}_2^{(2,1)*}}\cdot\|\varphi\|_{\widetilde{W}_2^{(2,1)}},$$

where

$$\|\ell\|_{\widetilde{W}_2^{(2,1)*}}=\sup_{\varphi,\ \|\varphi\|_{\widetilde{W}_2^{(2,1)}}\neq 0}\frac{|(\ell,\varphi)|}{\|\varphi\|}$$

is the norm of the error functional (4).

Hence, in order to get the minimum of the upper bound of the error for the quadrature formula (2) we solve the following.

Problem 1. Find the norm of the error functional (4) of the quadrature formula (2) in the space $\widetilde{W}_2^{(2,1)}(0,1]$.

The coefficients that give $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$ the minimum value are called *optimal coefficients* and denote by $\overset{\circ}{A}_k$. the quadrature formula (2) with these coefficients is *the optimal quadrature formula in the sense of Sard*.

The solution to Problem 1 was first proposed by S.L. Sobolev [19], later the problem was solved in the space $W_2^{(m,m-1)}$ of functions in works [3, 10] and in the space $\widetilde{L_2}^{(m)}$ of periodic functions in works [15, 16] and in works [7, 17, 18] for optimal quadrature formulas with derivative.

Further, in the next sections in order to solve Problem 1 we do the following.

• First, we find the extremal function of the quadrature formula (2).

• Using the extremal function we calculate the norm of the error functional (4).

3. The norm of the error functional (4)

To calculate the norm $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$, we use *the extremal function* ψ_ℓ for the error functional ℓ (see [19]) that satisfies the following equality:

$$(\ell,\psi_\ell)=\|\ell\|_{\widetilde{W}_2^{(2,1)*}}\cdot\|\psi_\ell\|_{\widetilde{W}_2^{(2,1)}}. \tag{6}$$

Since $\widetilde{W}_2^{(2,1)}(0,1]$ is the Hilbert space, using the Riesz representation theorem, we obtain

$$(\ell,\varphi)=\langle\varphi,\psi_\ell\rangle_{\widetilde{W}_2^{(2,1)}}, \tag{7}$$

where $\langle\psi_\ell,\varphi\rangle_{\widetilde{W}_2^{(2,1)}}$ is the inner product of the functions ψ_ℓ and φ which is defined by expression (1). In addition, the equality $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}=\|\psi_\ell\|_{\widetilde{W}_2^{(2,1)}}$ is fulfilled. So, taking into account (6), we derive

$$(\ell,\psi_\ell)=\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}}. \tag{8}$$

Using the expression (7), we get the following

$$(\ell,\varphi)=\langle\varphi,\psi_\ell\rangle_{\widetilde{W}_2^{(2,1)}}, \tag{9}$$

where $\varphi(x)$ belongs to the space $\widetilde{W}_2^{(2,1)}(0,1]$. Then from (9), integrating by parts, we obtain

$$\int_0^1 \ell(x)\varphi(x)dx=\int_0^1\left(\overline{\psi_\ell}^{IV}(x)-\overline{\psi_\ell}^{II}(x)\right)\varphi(x)dx.$$

From the last equation we have

$$\overline{\psi_\ell}^{IV}(x)-\overline{\psi_\ell}^{II}(x)=\ell(x). \tag{10}$$

For the solution of the last equation the following holds.

Theorem 1. *The solution to differential equation (10) is the extremal function ψ_ℓ corresponding to the error functional ℓ, expressed as*

$$\psi_\ell(x)=d_0+e^{-2\pi i\omega x}\cdot\kappa(\omega)-\sum_{k=1}^N\overline{C_k}\sum_{\beta\neq0}\kappa(\beta)e^{2\pi i\beta(x-hk)}-\sum_{k=1}^N\overline{A_k}\sum_{\beta\neq0}\kappa(\beta)\cdot(2\pi i\beta)e^{2\pi i\beta(x-hk)}, \tag{11}$$

where d_0 is a complex number and

$$\kappa(\omega)=\frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}. \tag{12}$$

Proof. We find the periodic solution of equation (10) using the following properties of the Fourier transform (see, for instance [2, 5])

$$F[\varphi]=\int_{-\infty}^{\infty}\varphi(x)e^{2\pi ipx}dx,$$

$$F^{-1}[\varphi]=\int_{-\infty}^{\infty}\varphi(p)e^{-2\pi ipx}dp,$$

$$F[\varphi^{(\alpha)}] = (-2\pi ip)^{\alpha} F[\varphi], (\alpha \in \mathbb{N}),$$

$$F^{-1}[F[\varphi(x)]] = \varphi(x).$$

We apply the Fourier transform to both sides of equation (10)

$$F[\overline{\psi_{\ell}}^{IV} - \overline{\psi_{\ell}}^{II}] = F[\ell].$$

Since the Fourier transform is linear operator, we have

$$\left((2\pi ip)^4 - (2\pi ip)^2\right) F[\overline{\psi_{\ell}}] = F\left[\ell\right]. \tag{13}$$

To find the Fourier transform of the error functional ℓ, we simplify it

$$\ell(x) = \left(e^{2\pi iwx} \varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k \delta(x-hk) + \sum_{k=1}^{N} A_k \delta'(x-hk) \right) * \sum_{\beta=-\infty}^{\infty} \delta(x-\beta).$$

Taking into account the convulation of two continouns functions f and g which is

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(y) \cdot g(x-y) dy$$

and properties of the Dirac delta-function, we get the following

$$\ell(x) = e^{2\pi iwx} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x-hk-\beta) + \sum_{k=1}^{N} A_k \sum_{\beta=-\infty}^{\infty} \delta'(x-hk-\beta). \tag{14}$$

Using (14) we rewrite eqution (13)

$$\left((2\pi ip)^4 - (2\pi ip)^2\right) F[\overline{\psi_{\ell}}] = F\left[e^{2\pi iwx} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x-hk-\beta) + \sum_{k=1}^{N} A_k \sum_{\beta=-\infty}^{\infty} \delta'(x-hk-\beta) \right],$$

or

$$\left((2\pi ip)^4 - (2\pi ip)^2\right) F[\overline{\psi}_{\ell}] = F\left[e^{2\pi i\omega x}\right] - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} F\left[\delta(x-hk-\beta)\right]$$

$$+ \sum_{k=1}^{N} A_k \sum_{\beta=-\infty}^{\infty} F\left[\delta'(x-hk-\beta)\right]. \tag{15}$$

Now, using the following equalities

$$F\left[e^{2\pi i\omega x}\right] = \delta(p+\omega), \quad \sum_{\beta=-\infty}^{\infty} e^{2\pi ip\beta} = \sum_{\beta=-\infty}^{\infty} \delta(p-\beta),$$

$$\sum_{\beta=-\infty}^{\infty} F\left[\delta(x-hk-\beta)\right] = \sum_{\beta=-\infty}^{\infty} e^{2\pi ip(hk+\beta)} = e^{2\pi iphk} \sum_{\beta=-\infty}^{\infty} e^{2\pi ip\beta} = e^{2\pi iphk} \sum_{\beta=-\infty}^{\infty} \delta(p-\beta)$$

and $\sum_{\beta=-\infty}^{\infty} F\left[\delta'(x-hk-\beta)\right] = (-2\pi ip) e^{2\pi iphk} \sum_{\beta=-\infty}^{\infty} \delta(p-\beta),$

we can rewrite equation (15) as follows

$$\left((2\pi ip)^4 - (2\pi ip)^2\right) F[\overline{\psi}_{\ell}] = \delta(p+\omega) - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} e^{2\pi iphk} \delta(p-\beta)$$

$$- \sum_{k=1}^{N} A_k \sum_{\beta=-\infty}^{\infty} (2\pi ip) e^{2\pi iphk} \delta(p-\beta).$$

We consider that the coefficient on the left-hand side of the last equation is not equal to zero. Consequently, we can divide both sides of the last equation by $\kappa(p)$ which is defined by (12). This division is not uniquely defined. From the last equation the function $F[\overline{\psi}_{\ell}]$ is defined up to the term of the form $\delta(p)$. Taking into account the above said and the properties of the delta-function, we get

$$F[\bar{\psi}_\ell]=\delta(p+\omega)\kappa(p)-\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi iphk}\delta(p-\beta)\kappa(p)$$
$$-\sum_{k=1}^{N}A_k\sum_{\beta\neq 0}(2\pi ip)e^{2\pi iphk}\kappa(p)+d_0\delta(p),$$

where $\kappa(p)$ is defined by formula (12) and d_0 is a constant.

Using the property $f(x)\delta(x-a)=f(a)\delta(x-a)$ of delta-function, we have the following

$$F[\bar{\psi}_\ell]=\delta(p+\omega)\kappa(\omega)-\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi i\beta hk}\kappa(\beta)\delta(p-\beta)$$
$$-\sum_{k=1}^{N}A_k\sum_{\beta\neq 0}(2\pi i\beta)e^{2\pi i\beta hk}\kappa(\beta)\delta(p-\beta)+d_0\delta(p).$$

Then, applying the inverse Fourier transform to both sides of the last equation, we have

$$\bar{\psi}_\ell=e^{2\pi i\omega x}\kappa(\omega)-\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi i\beta hk}\kappa(\beta)e^{-2\pi i\beta x}-\sum_{k=1}^{N}A_k\sum_{\beta\neq 0}(2\pi i\beta)e^{2\pi i\beta hk}\kappa(\beta)e^{-2\pi i\beta x}+d_0.$$

Since $\bar{\bar{\psi}}_\ell$ is equal to ψ_ℓ, we obtain (11), that is, Theorem 1 is proved.

The following result is true for the solution to Problem 1.

Theorem 2. *On the space* $\widetilde{W}_2^{(2,1)*}(0,1]$ *of functionals, the square of the norm* $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$ *for* $\omega\in\mathbb{Z}\setminus\{0\}$ *has the following form*

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}}=\|\ell_0\|^2_{\widetilde{W}_2^{(2,1)*}}+2\pi i\omega\cdot\kappa(\omega)\left[\sum_{k=1}^{N}A_ke^{-2\pi i\omega hk}+\sum_{k'=1}^{N}\overline{A_{k'}}e^{2\pi i\omega hk'}\right]$$
$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}\sum_{\beta\neq 0}2\pi i\beta\cdot\kappa(\beta)e^{2\pi i\beta h(k-k')}\left[A_k\overline{C_{k'}}+C_k\overline{A_{k'}}\right]$$
$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}A_k\overline{A_{k'}}\sum_{\beta\neq 0}(2\pi i\beta)^2\cdot\kappa(\beta)e^{2\pi i\beta h(k-k')},$$

where

$$\|\ell_0\|^2_{\widetilde{W}_2^{(2,1)*}}=\kappa(\omega)-\kappa(\omega)\left[\sum_{k=1}^{N}C_ke^{-2\pi i\omega hk}+\sum_{k'=1}^{N}\overline{C_{k'}}e^{2\pi i\omega hk'}\right]+\sum_{k=1}^{N}\sum_{k'=1}^{N}C_k\overline{C_{k'}}\sum_{\beta\neq 0}\kappa(\beta)e^{2\pi i\beta h(k-k')}$$

and $\kappa(\cdot)$ *is defined by (12).*

Proof. To prove Theorem 2, we calculate the norm $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$ and use equalities (8), (14) and (11), respectively. As a result we have the following

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}}=(\ell,\psi_\ell)=\int_0^1\ell(x)\psi_\ell(x)dx$$
$$=\int_0^1\ell(x)\left(d_0+e^{-2\pi i\omega x}\cdot\kappa(\omega)-\sum_{k=1}^{N}\overline{C_k}\sum_{\beta\neq 0}\kappa(\beta)e^{2\pi i\beta(x-hk)}-\sum_{k=1}^{N}\overline{A_k}\sum_{\beta\neq 0}\kappa(\beta)\cdot(2\pi i\beta)e^{2\pi i\beta(x-hk)}\right)dx.$$

As such, given the algorithm to calculate the norm $\|\ell_0\|_{\widetilde{W}_2^{(2,1)*}}$ in works [9, 10].

Taking into account condition (5), simplifying the last expression we obtaing the result of the last theorem. So, Theorem 2 is completely proved.

Conclusion.

Here, the problem of constructing an optimal quadrature formula with derivative in the sense of Sard for approximating strongly oscillatory integrals in the Hilbert space of complex-valued, periodic functions is studied. Furthermore, to estimate the sharp upper bound of the absolute value of the error of the quadrature formula, the analytical form of the norm for the error functional is found, in which the extremal function corresponding to the error functional is initially found.

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