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Optimal quadrature formulas for approximating strongly oscillating integrals in the Hilbert space $\widetilde{W}_2^{(m,m-1)}$ of periodic functions

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ABSTRACT

The present paper is dedicated to a variational method for the construction of optimal quadrature formulas in the sense of Sard in the Hilbert space $\widetilde{W}_2^{(m,m-1)}$ of complex-valued and periodic functions. In this, the coefficients of the optimal quadrature formula are found separately in the case ωh is integer and non-integer cases. In addition, using the constructed optimal quadrature formula, the numerical results of exponentially weighted integrals of certain functions in the case $m = 2$ is presented. The numerical results show that the order of convergence of the optimal quadrature formula is $O\left(\left(\frac{1}{N+|\omega|}\right)^2\right)$ in the space $\widetilde{W}_2^{(2,1)}$.

1. Introduction

As a result of many scientific and practical studies carried out on a global scale, solving problems of image analysis, modulation and demodulation of signals for communication systems, computed tomography in industry and medicine is reduced to the calculation of some integrals of strongly oscillating functions. Standard methods of numerical integration for the approximate calculation of strongly oscillatory integrals require a large amount of computational work, and their direct application in practice may not give effective results. Therefore, the development of special methods for the approximate calculation of such integrals, the creation of new methods for the approximate calculation of these integrals in various classes of functions, and the estimation of their errors are considered one of the important problems of Computational Mathematics.

It is known that many problems of science and technology are brought to the calculation of certain integrals of strongly oscillating functions, especially, Fourier coefficients and integrals. Fourier integrals of the following form are examples of strongly oscillating integrals at sufficiently large values of $|\omega| \gg 1$

$$I(\omega,\varphi) = \int_a^b e^{2\pi i\omega x}\varphi(x)dx. \tag{1}$$

Since it is not always possible to calculate integral (1) using analytical methods, their numerical integration is required. Several special methods have been developed for the approximate calculation of strongly oscillatory integrals. For example, the Filon

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method, asymptotic expansion method, Levin's collocation method, methods of the steepest descent, and optimal quadrature and cubature formulas methods. One of the first methods of the approximate calculation of such integrals is Filon's method which is based on piecewise approximation of the appropriate function with parabolic arcs in the interval of integration(see [1]). Later, Filon-type, Clanshaw–Curtis–Filon-type, modified Clanshaw–Curtis, Levin-type, Gauss–Laguerre quadrature formulas and generalized quadrature formulas were developed for numerical calculation of integrals with different types of strongly oscillating functions (see, for example [2–4]). In recent years, scientists such as H. Wang, L. Zhang, D. Huybrechs [5], A. Asheim [6], J. Gao [7], S. Xiang, G. He and Y.J. Cho [8] carried out research in these methods. Scientists such as A. Iserles, S.P. Nørsett [2], S.-I.S. Zaman, S.I.U. Nasib [9] and S. Olver [10,11] conducted scientific research on the asymptotic expansion method.

There are the Sobolev method, spline functions, and ϕ-functions methods for constructing optimal quadrature and cubature formulas for approximate calculation of definite integrals of given functions in certain Banach spaces. Initially, S.L. Sobolev was engaged in the theory of constructing optimal quadrature and cubature formulas using the concept of a discrete analogue of a linear differential operator in the space $L_2^{(m)}(\Omega)$ (see [12]). The algorithm for construction of optimal quadrature and cubature formulas proposed by Academician S.L. Sobolev was developed by Kh.M. Shadimetov [13,14] and A.R. Hayotov [15]. In the work [13], lattice cubature formulas for weighted integrals in the Sobolev space of periodic functions were constructed. In particular, I. Babuška's optimal quadrature formula is derived from the work [16] which is constructed for the approximate calculation of Fourier coefficients when the weight function is $\exp(i\sigma x)$. In the work [17], E. Novak, M. Ullrich and H. Woźniakowski studied the approximate calculations of one-variable oscillating integrals in the space H^s of standard Sobolev of periodic and non-periodic functions.

It should be noted that in recent years, in the Hilbert spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$, Kh.M. Shadimetov, G.V. Milovanović, A.R. Hayotov and N.D. Boltaev [18], A.R. Hayotov, C.-O. Lee and S. Jeon [12,19,20] and S.S. Babaev [21,22] and carried out scientific research on the construction of optimal quadrature formulas for approximate calculation for strongly oscillatory integrals and their practical applications. As a result, they have achieved high-resolution reconstruction of Computed Tomography images in the industrial and medical fields under laboratory conditions.

Since the analytical representation of the coefficients of the optimal quadrature formulas constructed in the above works is relatively complicated, the problem of constructing simpler formulas is relevant. Such formulas are relatively simpler in the space of periodic functions. Therefore, looking at these issues in the space of periodic functions is one of the urgent problems of today.

We know that the sharp upper bound of the error of the constructed optimal quadrature formulas in work [18] in the Hilbert space $W_2^{(m,m-1)}$ is smaller than the sharp upper bound of the error of the optimal quadrature formulas were constructed by N.D. Boltaev, A.R. Hayotov and Kh.M. Shadimetov [23] in the Sobolev space $L_2^{(m)}$ for the approximate calculation of integral (1). Similarly, we can see in work [24] that the constructed optimal quadrature formula in the Hilbert space $\widetilde{W_2}^{(1,0)}$ of periodic functions approaches the integral (1) faster than the optimal quadrature formula in the Sobolev space $\widetilde{L_2^{(1)}}$ of periodic functions.

The goal of this work is to construct new optimal quadrature formulas in the Hilbert space $\widetilde{W_2}^{(m,m-1)}$ of periodic functions to approximate integral (1) which is a better approximation than the optimal quadrature formulas that were constructed in the Sobolev space $\widetilde{L_2^{(m)}}$ of periodic functions in work [25].

For the approximate calculation of integral (1) in the case $\omega = 0$, A.R. Hayotov and U.N. Khayriev constructed optimal quadrature formulas in spaces $\widetilde{W_2}^{(1,0)}$ and $\widetilde{W_2}^{(m,m-1)}$ with $m \geq 2$ of periodic functions in works [26,27], and in the case $\omega \in \mathbb{Z}$, in the space $\widetilde{W_2}^{(1,0)}$ optimal quadrature formula was obtained by U.N. Khayriev [28]. In this paper, we consider the problem of constructing optimal quadrature formulas in the Hilbert space $\widetilde{W_2}^{(m,m-1)}$ when $\omega \in \mathbb{Z} \setminus \{0\}$.

Let us consider the Hilbert space $W_2^{(m,m-1)}[0,1]$ of complex-valued functions φ defined on the interval $[0,1]$, which possess an absolute continuous $(m-1)^{st}$ derivative on the interval $[0,1]$ and whose mth order derivative (generalized) is square integrable [18, 22], with the inner product

$$\langle \varphi, \psi \rangle_{W_2^{(m,m-1)}} = \int_0^1 (\varphi^{(m)}(x) + \varphi^{(m-1)}(x))(\overline{\psi}^{(m)}(x) + \overline{\psi}^{(m-1)}(x))dx,$$

where $\overline{\psi}$ is the complex conjugate to the function ψ. The norm of the element φ in this space is respectively defined by the following formula

$$\|\varphi\|_{W_2^{(m,m-1)}} = \left\{ \langle \varphi, \varphi \rangle_{W_2^{(m,m-1)}} \right\}^{1/2}.$$

This equality is a semi-norm and $\|\varphi\|_{W_2^{(m,m-1)}} = 0$ if and only if $\varphi(x) = P_{m-2}(x) + de^{-x}$, where $P_{m-2}(x)$ is a polynomial of degree $(m-2)$ and d is a constant. Every element of the space $W_2^{(m,m-1)}[0,1]$ is a class of functions that are differ from each other by linear combination of any polynomial of degree $(m-2)$ and e^{-x}. The space $W_2^{(m,m-1)}[0,1]$ is a quotient space. We denote by $\widetilde{W_2}^{(m,m-1)}(0,1]$ the subspace of the space $W_2^{(m,m-1)}[0,1]$ consisting of complex-valued and 1-periodic functions $\varphi(x)$. Notice that every element of the space $\widetilde{W_2}^{(m,m-1)}(0,1]$ satisfies the following condition of 1-periodicity

$$\varphi(x+\beta) = \varphi(x), \quad x \in \mathbb{R}, \ \beta \in \mathbb{Z}$$

and is a class of functions φ which differ from each other by a constant term.

This paper is devoted to the problem of constructing optimal quadrature formulas in the Hilbert space $\widetilde{W_2}^{(m,m-1)}(0,1]$ with $m \geq 2$ for the approximate calculation of the integral (1) when $\omega \in \mathbb{Z} \setminus \{0\}$.

2. Statement of the problem

We consider a quadrature formula of the form

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)dx \cong \sum_{k=1}^{N} C_k\varphi(hk), \tag{2}$$

where $\omega \in \mathbb{Z}\backslash\{0\}$ is a parameter, φ belongs to the space $\widetilde{W}_2^{(m,m-1)}(0,1]$ with $m \geq 2$ and C_k $(k = 1, 2, \dots, N)$ are coefficients of the quadrature formula, they are complex numbers, $i^2 = -1$ and $N \in \mathbb{N}$, $h = 1/N$.

The difference between the quadrature sum and the integral is called *the error* of the quadrature formula (2)

$$\begin{aligned}(\ell, \varphi) &= \int_0^1 \ell(x)\varphi(x)dx = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx - \sum_{k=1}^{N} C_k\varphi(hk)\\ &= \int_0^1 \left(e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta)\right)\varphi(x)dx,\end{aligned} \tag{3}$$

that is, the value of the functional ℓ on a function φ is equal to the error of the quadrature formula (2). Here $\delta(x)$ is Dirac's delta-function. This difference defines a linear functional

$$\ell(x) = e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta) \tag{4}$$

which is called *the error functional* of the quadrature formula (2), and belongs to the conjugate space $\widetilde{W}_2^{(m,m-1)*}(0,1]$. The error functional ℓ is 1-periodic functional.

Since the error functional ℓ is defined on the space $\widetilde{W}_2^{(m,m-1)}(0,1]$, we assume that the following equality holds as in the work [29]

$$(\ell, 1) = 0. \tag{5}$$

This condition means the quadrature formula (2) exacts for any constant, and it can be written as follows

$$\sum_{k=1}^{N} C_k = \int_0^1 e^{2\pi i\omega x}dx.$$

In addition, equality (5) expresses boundedness of the error functional ℓ [29].

Using the Cauchy–Schwarz inequality, we obtain the following upper bound for the absolute value of the error (3)

$$|(\ell, \varphi)| \leq \|\ell\|_{\widetilde{W}_2^{(m,m-1)*}} \cdot \|\varphi\|_{\widetilde{W}_2^{(m,m-1)}},$$

where $\|\ell\|_{W_2^{(m,m-1)*}} = \sup\limits_{\|\varphi\|\neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|}$.

It can be seen that the norm of the error function ℓ depends on the coefficients of C_k. The problem of finding the minimum of the norm of the error functional ℓ by coefficients C_k when the nodes are fixed is called *Sard's problem.* The obtained formula is called *the optimal quadrature formula in the sense of Sard.* This problem was first investigated by A. Sard [30] in the space $L_2^{(m)}$ that is the Sobolev space of functions which $(m-1)$-st derivative is absolutely continuous and mth derivative is square integrable for some m.

Thus, to construct the optimal quadrature formula in the sense of Sard in the space $\widetilde{W}_2^{(m,m-1)}$ we should solve the following problems.

Problem 1. Find the analytical representation of the norm of the quadrature formula error functional (4) of the form (2).

Problem 2. Find the optimal coefficients $\overset{\circ}{C}_k$ which give the minimum for the norm of the error functional (4).

Problem 3. Calculate the norm for the error functional of the optimal quadrature formulas (2)

The main purpose of this paper is to construct the optimal quadrature formulas of the form (2) in the space $\widetilde{W}_2^{(m,m-1)}$ when $m \geq 2$. That is, to find analytical representation of optimal coefficients and to calculate the norm for the error functional ℓ of the optimal quadrature formula (2).

In order to solve the above problem for all values of arbitrary $m \geq 2$ and $\omega \in \mathbb{Z}\backslash\{0\}$, we have to consider these problems separately in the following cases:

- $\omega \in \mathbb{Z} \setminus \{0\}$ and $\omega h \notin \mathbb{Z}$,
- $\omega h \in \mathbb{Z}$.

This paper consists of the following parts:

- in Section 3, using representation theorem, the extremal function for the error functional (4) is found and with its help the norm of the error functional ℓ is calculated, that is, Problem 1 is solved;
- in Section 4, in order to find the minimum of the norm for the error functional (4) by coefficients C_k the system of linear equations is obtained. In addition, the optimal coefficients of the quadrature formulas (2) are found. Thus, Problem 2 is solved;
- Section 5 is devoted to calculation of the norm for the optimal error functional $\overset{\circ}{\ell}$, that is, Problem 3 is solved;
- Section 6 presents some numerical results.

3. The extremal function and the norm of the error functional (4)

In this section we solve Problem 1, that is, we find an analytic form of the norm for the error functional (4).

Using *the extremal function* ψ_ℓ (see [29,31]) that satisfies the following equality, we find an analytical representation of the norm for the error functional ℓ

$$\left(\ell,\psi_\ell\right) = \|\ell\|_{\widetilde{W}_2^{(m,m-1)*}} \cdot \|\psi_\ell\|_{\widetilde{W}_2^{(m,m-1)}}. \tag{6}$$

To find the extremal function ψ_ℓ, using the Riesz representation theorem [32, Theorem 2.5.8], we obtain the following

$$(\ell,\varphi) = \langle \varphi,\psi_\ell\rangle_{\widetilde{W}_2^{(m,m-1)}} \quad \text{and} \quad \|\ell\|_{\widetilde{W}_2^{(m,m-1)*}} = \|\psi_\ell\|_{\widetilde{W}_2^{(m,m-1)}}. \tag{7}$$

From equality (7), we get

$$\left(\ell,\psi_\ell\right) = \|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}}. \tag{8}$$

Integrating by parts the right-hand side of the first equation in (7), keeping in mind that $\varphi,\psi_\ell \in \widetilde{W}_2^{(m,m-1)}(0,1]$, we have the following differential equation

$$\overline{\psi_\ell}^{(2\ m)}(x) - \overline{\psi_\ell}^{(2m-2)}(x) = (-1)^m \ell(x). \tag{9}$$

The following theorem is true.

Theorem 1. *The generalized solution of Eq. (9) is the extremal function ψ_ℓ of the error functional (4) in the space $\widetilde{W}_2^{(m,m-1)}(0,1]$ with $m \geq 2$ and it is expressed as*

$$\psi_\ell(x) = e^{-2\pi i\omega x}\cdot \kappa_m(\omega) - \sum_{k=1}^{N}\overline{C}_k \sum_{\beta\neq 0} e^{2\pi i\beta(x-hk)}\cdot \kappa_m(\beta) + \overline{d_0},\ \ \omega \in \mathbb{Z}\setminus\{0\}, \tag{10}$$

where $\overline{d_0}$ is the complex conjugate to $d_0$which is a complex constant term and

$$\kappa_m(\omega) = \frac{1}{(2\pi\omega)^{2m} + (2\pi\omega)^{2m-2}}. \tag{11}$$

Proof. We consider the Fourier and inverse Fourier transforms, respectively, as follows
$F[\varphi] = \int_{-\infty}^{\infty}\varphi(x)e^{2\pi ipx}dx$ and $F^{-1}[\varphi] = \int_{-\infty}^{\infty}\varphi(p)e^{-2\pi ipx}dp$.
We use the following properties of the Fourier transform

$$F[\varphi^{(\alpha)}] = (-2\pi ip)^\alpha F[\varphi],\ (\alpha \in \mathrm{N}),$$

$$F[\varphi * g] = F[\varphi]\cdot F[g],$$

$$F[\varphi \cdot g] = F[\varphi] * F[g].$$

Now, using the definition of the Fourier transform, we directly obtain the following

$$F\left[e^{2\pi i\omega x}\right] = \delta(p+\omega),\ \sum_{\beta=-\infty}^{\infty} e^{2\pi ip\beta} = \sum_{\beta=-\infty}^{\infty}\delta(p-\beta)$$

and

$$\sum_{\beta=-\infty}^{\infty} F\left[\delta(x-hk-\beta)\right] = \sum_{\beta=-\infty}^{\infty} e^{2\pi ip(hk+\beta)} = e^{2\pi iphk}\sum_{\beta=-\infty}^{\infty}\delta(p-\beta).$$

Applying the Fourier transform to both sides of Eq. (9) we have

$$\left((2\pi ip)^2 - 1\right) F[\bar{\psi}_\ell] = -F\left[e^{2\pi i\omega x}\right] + \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} F\left[\delta(x-hk-\beta)\right]. \tag{12}$$

We consider that the coefficient on the left-hand side of Eq. (12) is not equal to zero. Consequently, we can divide both sides of Eq. (12) by $\kappa_m(\omega)$. This division is not uniquely defined. From (12) the function $F[\bar{\psi}_\ell]$ is defined up to the term of the form $\delta(p)$. Taking into account the above said and the properties of the delta-function, we get

$$F[\bar{\psi}_\ell] = \delta(p+\omega)\kappa_m(\omega) - \sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i\beta hk}\delta(p-\beta)\kappa_m(\beta) + d_0\delta(p).$$

Now, applying the inverse Fourier transform to both sides of the last equation, we get

$$\overline{\psi}_\ell(x) = e^{2\pi i\omega x}\cdot\kappa_m(\omega) - \sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i\beta hk} e^{-2\pi i\beta x}\cdot\kappa_m(\beta) + d_0.$$

Hence, taking the complex conjugate, we obtain the following

$$\psi_\ell(x) = e^{-2\pi i\omega x}\cdot\kappa_m(\omega) - \sum_{k=1}^{N} \overline{C}_k \sum_{\beta\neq 0} e^{2\pi i\beta(x-hk)}\cdot\kappa_m(\beta) + \overline{d_0}.$$

Thus, Theorem 1 is completely proved.

Now, we calculate the square of the norm $\|\ell\|_{\widetilde{W}_2^{(m,m-1)*}}$ for $m\geq 2$ and $\omega\in\mathbb{Z}\setminus\{0\}$. For this purpose, using successive equalities (8), (4) and (10), we have

$$\|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}} = \left(\ell,\psi_\ell\right) = \int_0^1 \ell(x)\psi_\ell(x)dx$$

$$= \int_0^1 \ell(x)\left(e^{-2\pi i\omega x}\cdot\kappa_m(\omega) - \sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\gamma\neq 0} e^{2\pi i\gamma(x-hk')}\cdot\kappa_m(\gamma) + \overline{d_0}\right)dx.$$

Considering condition (5), we get the following

$$\|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}} = \int_0^1 \left(e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\right)$$
$$\times\left(e^{-2\pi i\omega x}\cdot\kappa_m(\omega) - \sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\gamma\neq 0} e^{2\pi i\gamma(x-hk')}\cdot\kappa_m(\gamma)\right)dx.$$

Simplifying the last expression, we obtain the following analytic representation of the norm for the error functional ℓ

$$\|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}} = \kappa_m(\omega) - \kappa_m(\omega)\sum_{k'=1}^{N}\overline{C}_{k'}e^{2\pi i\omega hk'} - \kappa_m(\omega)\sum_{k=1}^{N} C_k e^{-2\pi i\omega hk}$$
$$+ \sum_{k=1}^{N}\sum_{k'=1}^{N} C_k\overline{C}_{k'}\sum_{\beta\neq 0} e^{2\pi i\beta h(k-k')}\kappa_m(\beta) \text{ for } m\geq 2 \text{ with } \omega\in\mathbb{Z}\setminus\{0\}, \tag{13}$$

where $\kappa_m(\omega)$ is defined by (11).

It can be easily showed that $\|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}}$ is a real non-negative quantity [18,23].

Thus, Problem 1 is solved.

Using equality (13), we find the optimal coefficients of the quadrature formula (2) in the next section, that is, we solve Problem 2.

4. Calculation of coefficients of the optimal quadrature formula of the form (2)

We find the optimal coefficients of the quadrature formula (2) which give the minimum value to the quantity $\|\ell\|_{\widetilde{W}_2^{(m,m-1)*}}$ for $m\geq 2$ or the norm of the error functional ℓ is a multivariable quadratic function of the coefficients C_k. To find the minimum of the norm $\|\ell\|_{\widetilde{W}_2^{(m,m-1)*}}$ with respect to the coefficients C_k under condition (5), we use the Lagrange method for finding the conditional extremum. We consider the following function

$$L\left(C_k,\overline{C}_{k'},\mu\right) = \|\ell\|^2_{\widetilde{W}_2^{(m,m-1)}} - \mu\cdot(\ell,1) \quad\text{for}\quad k=1,2,\ldots,N,$$

where μ is a constant.

Equating the partial derivatives by all variables C_k, $\overline{C}_{k'}$ and μ of the function $L(C_k,\overline{C}_{k'},\mu)$ to with respect zero, we have

$$\frac{\partial L}{\partial C_k} = -e^{-2\pi i\omega hk}\kappa_m(\omega) + \sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\beta\neq 0} e^{2\pi i\beta h(k-k')}\kappa_m(\beta) + \mu = 0$$
$$\text{for each } k=1,2,\ldots,N, \tag{14}$$

$$\frac{\partial L}{\partial \overline{C}_{k'}} = -e^{2\pi i\omega hk'}\kappa_m(\omega) + \sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i\beta h(k-k')}\kappa_m(\beta) = 0$$

$$\text{for each } k' = 1,2,\dots,N, \tag{15}$$

$$\frac{\partial L}{\partial \mu} = \int_0^1 e^{2\pi i\omega x}dx - \sum_{k=1}^{N} C_k = 0, \tag{16}$$

where $\kappa_m(\omega)$ is defined by (11).

It can be proved that the system of Eqs. (14)–(16) is a system of the discrete Wiener–Hopf type that it has a unique solution as in the work [13]. A solution of this system which we denote by $\{\overset{\circ}{C}_k,\ k = 1,2,\dots,N\}$ and $\overset{\circ}{\mu}$, is a minimum point for the function $L(C_k,\overline{C}_{k'},\mu)$. Since the system of Eqs. (14) with (16) is equivalent to the system (15),(16), we find an analytic solution for system (15),(16) instead of system (14),(16).

The following theorem is valid for the solution of the system (15),(16).

Theorem 2. *Among the quadrature formulas in the sense of Sard of the form (2) with $\omega \in \mathbb{Z}\setminus\{0\}$ and $\omega h \notin \mathbb{Z}$ in the space $\widetilde{W}_2^{(m,m-1)}(0,1]$ with $m \geq 2$ of 1-periodic, complex-valued functions, there is a unique quadrature formula with coefficients having the representation*

$$\overset{\circ}{C}_k = \frac{2K_{\omega,m}}{(2\pi\omega)^{2m} + (2\pi\omega)^{2m-2}} \cdot e^{2\pi i\omega hk} \quad for\ each\ k = 1,2,\dots,N, \tag{17}$$

where

$$K_{\omega,m} = (-1)^{m-1}\cdot\left[\frac{e^{2h}-1}{e^{2h}+1-2e^{h}\cos(2\pi\omega h)} + \sum_{n=1}^{m-1}\frac{2h^{2n-1}\cdot\lambda E_{2n-2}(\lambda)}{(2n-1)!\cdot(1-\lambda)^{2n}}\right]^{-1}, \tag{18}$$

and $E_{2n-2}(\lambda)$is the Euler–Frobenius polynomial of degree $(2n-2)$and $\lambda = e^{2\pi i\omega h}$.

Proof. Let ω be a non-zero integer and $\omega h \notin \mathbb{Z}$. Then we search a solution of system (15)–(16) in the following form

$$\overset{\circ}{C}_k = C(\omega,h,m)\cdot e^{2\pi i\omega hk} \quad \text{for each } k = 1,2,\dots,N \quad \text{and } m \geq 2, \tag{19}$$

where $C(\omega,h,m)$ is an unknown function.

Using designation (19), we rewrite the system (15) as follows

$$-e^{2\pi i\omega hk'}\kappa_m(\omega) + C(\omega,h,m)\sum_{\beta\neq 0} e^{-2\pi i\beta hk'}\kappa_m(\beta)\sum_{k=1}^{N} e^{2\pi i(\beta+\omega)hk} = 0,$$

$$k' = 1,2,\dots,N,$$

where

$$\sum_{k=1}^{N} e^{2\pi i(\beta+\omega)hk} = \begin{cases} 0 & \text{if}\,(\beta+\omega)h \neq t \in \mathbb{Z}, \\ N & \text{if}\,(\beta+\omega)h = t \in \mathbb{Z}. \end{cases}$$

Herein, since $\beta \neq 0$, it follows that $\omega h \neq t \in \mathbb{Z}$. This condition shows that we should consider Problem 2 in the cases $\omega h \notin \mathbb{Z}$ and $\omega h \in \mathbb{Z}$.

Simplifying the left-hand side of the last equation, we make the following equation

$$-e^{2\pi i\omega hk'}\kappa_m(\omega) + C(\omega,h,m)\cdot N\sum_{t=-\infty}^{\infty} e^{2\pi i\omega hk'}\kappa_m(t\cdot N-\omega) = 0.$$

from this equation, we get

$$C(\omega,h,m) = h\cdot\kappa_m(\omega)\left(\sum_{t=-\infty}^{\infty}\kappa_m(t\cdot N-\omega)\right)^{-1}. \tag{20}$$

Considering (19) and (20), we have

$$\overset{\circ}{C}_k = h\cdot\kappa_m(\omega)\left(\sum_{t=-\infty}^{\infty}\kappa_m(t\cdot N-\omega)\right)^{-1}\cdot e^{2\pi i\omega hk}. \tag{21}$$

To find optimal coefficients $\overset{\circ}{C}_k$, we should calculate the series in (21). For this, using equality (11), we rewrite this series as follows

$$\sum_{t=-\infty}^{\infty}\kappa_m(t\cdot N-\omega) = \sum_{t=-\infty}^{\infty}\frac{1}{(2\pi\,(t\cdot N-\omega))^{2m} + (2\pi\,(t\cdot N-\omega))^{2m-2}}$$

$$= \left(\frac{h}{2\pi}\right)^{2m} \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^{2m-2}\left((t-\omega h)^2 + \left(\frac{h}{2\pi}\right)^2\right)}$$

$$= \left(\frac{h}{2\pi}\right)^{2m} \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^{2m-2}\left(t-\omega h - \frac{hi}{2\pi}\right)\left(t-\omega h + \frac{hi}{2\pi}\right)}. \tag{22}$$

Denoting $p = t - \omega h$, we rewrite (22) as follows

$$\sum_{t=-\infty}^{\infty} \kappa_m(p \cdot N) = \left(\frac{h}{2\pi}\right)^{2m} \cdot \sum_{t=-\infty}^{\infty} \frac{1}{p^{2m-2}\left(p - \frac{hi}{2\pi}\right)\left(p + \frac{hi}{2\pi}\right)}. \tag{23}$$

To calculate the above sum, we decompose the following fraction into multipliers

$$\frac{1}{p^{2m-2}\left(p - \frac{hi}{2\pi}\right)\left(p + \frac{hi}{2\pi}\right)} = \frac{A_1}{p^{2m-2}} + \frac{A_2}{p^{2m-3}} + \frac{A_3}{p^{2m-4}} + \cdots +$$

$$+ \frac{A_{2m-3}}{p^2} + \frac{A_{2m-2}}{p} + \frac{B_1}{p - \frac{hi}{2\pi}} + \frac{B_2}{p + \frac{hi}{2\pi}}, \tag{24}$$

where $A_k,\ B_1$ and $B_2\ \ (k = 1, 2, \ldots, 2m-2)$ are unknown coefficients.

For finding coefficients $A_k,\ B_1$ and B_2, giving the least common denominator of fractionals in the right-hand side of (24), and using identity as follows, we equalize their numerators

$$\begin{aligned} 1 \ =\ & A_1\left(p^2 + \frac{h^2}{4\pi^2}\right) + A_2 p\left(p^2 + \frac{h^2}{4\pi^2}\right) + A_3 p^2\left(p^2 + \frac{h^2}{4\pi^2}\right) + \cdots + A_{2m-3} \\ & \times\ p^{2m-4}\left(p^2 + \frac{h^2}{4\pi^2}\right) + A_{2m-2}p^{2m-3}\left(p^2 + \frac{h^2}{4\pi^2}\right) + B_1 p^{2m-2}\left(p + \frac{hi}{2\pi}\right) \\ & +\ B_2 p^{2m-2}\left(p - \frac{hi}{2\pi}\right). \end{aligned}$$

From this identity, we obtain the following result

$$A_n = \begin{cases} (-1)^{\frac{n+3}{2}} \cdot \left(\frac{2\pi}{h}\right)^{n+1}, & \text{if } n \text{ is odd number}, \\ 0, & \text{if } n \text{ is even number}, \end{cases}$$

$$B_1 = (-1)^m \frac{i}{2} \cdot \left(\frac{2\pi}{h}\right)^{2m-1},$$

$$B_2 = (-1)^{m+1} \frac{i}{2} \cdot \left(\frac{2\pi}{h}\right)^{2m-1}.$$

Using value of $A_k,\ B_1$ and $B_2\ \ (k = 1, 2, \ldots, 2m-2)$, we can write (24) as follows

$$\begin{aligned} \frac{1}{p^{2m-2}\left(p - \frac{hi}{2\pi}\right)\left(p + \frac{hi}{2\pi}\right)} &= \left(\frac{2\pi}{h}\right)^2 \cdot p^{2-2m} - \left(\frac{2\pi}{h}\right)^4 \cdot p^{4-2m} + \left(\frac{2\pi}{h}\right)^6 \\ &\times p^{6-2m} - \cdots + (-1)^m \cdot \left(\frac{2\pi}{h}\right)^{2m-2} \cdot p^{-2} + (-1)^m \frac{i}{2} \cdot \left(\frac{2\pi}{h}\right)^{2m-1} \\ &\times \left(p - \frac{hi}{2\pi}\right)^{-1} + (-1)^{m+1} \frac{i}{2} \cdot \left(\frac{2\pi}{h}\right)^{2m-1} \cdot \left(p + \frac{hi}{2\pi}\right)^{-1}. \end{aligned}$$

Considering the right-hand side of the last expression, we calculate the sum in (23)

$$\begin{aligned} \sum_{t=-\infty}^{\infty} \kappa_m(t \cdot N - \omega) &= \left[\left(\frac{h}{2\pi}\right)^{2m-2} \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^{2m-2}} - \left(\frac{h}{2\pi}\right)^{2m-4}\right. \\ &\times \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^{2m-4}} + \left(\frac{h}{2\pi}\right)^{2m-6} \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^{2m-6}} - \cdots + (-1)^{m-1} \\ &\times \left.\left(\frac{h}{2\pi}\right)^4 \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^4} + (-1)^m \left(\frac{h}{2\pi}\right)^2 \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^2}\right] + (-1)^m \\ &\times \frac{hi}{4\pi}\left[\sum_{t=-\infty}^{\infty} \frac{1}{t - \omega h - \frac{hi}{2\pi}} - \sum_{t=-\infty}^{\infty} \frac{1}{t - \omega h + \frac{hi}{2\pi}}\right] \\ &= \sum_{n=1}^{m-1} (-1)^{m-1} s_2 + (-1)^m \frac{hi}{4\pi} s_1, \end{aligned} \tag{25}$$

where

$$s_1 = \frac{hi}{\pi} \sum_{t=-\infty}^{\infty} \frac{1}{\left(t - \omega h - \frac{hi}{2\pi}\right)\left(t - \omega h + \frac{hi}{2\pi}\right)}, \tag{26}$$

and

$$s_2 = (-1)^n \left(\frac{1}{2\pi}\right)^{2n} \sum_{t=-\infty}^{\infty} \frac{1}{\left(h^{-1}t - \omega\right)^{2n}}. \tag{27}$$

Now, we should calculate s_1 and s_2. Firstly, we calculate the sum s_1. For this, we rewrite (26) in the following form

$$s_1 = \frac{hi}{\pi} \sum_{t=-\infty}^{\infty} f(t), \tag{28}$$

where

$$f(t) = \frac{1}{\left(t - \omega h - \frac{hi}{2\pi}\right)\left(t - \omega h + \frac{hi}{2\pi}\right)}.$$

We consider the following function

$$f(z) = \frac{1}{\left(z - \omega h - \frac{hi}{2\pi}\right)\left(z - \omega h + \frac{hi}{2\pi}\right)},$$

$z_1 = \omega h + \frac{hi}{2\pi}$ and $z_2 = \omega h - \frac{hi}{2\pi}$ are the poles of the function $f(z)$.

To find s_1, we calculate the sum of the series in Eq. (28). For this, using the following formula in the book [33]

$$\sum_{t=-\infty}^{\infty} f(t) = -\sum_{z_1,z_2} \operatorname{res}\left(\pi \cot(\pi z) \cdot f(z)\right), \tag{29}$$

we have

$$\sum_{t=-\infty}^{\infty} f(t) = \frac{2\pi^2}{h} \cdot \frac{e^{2h} - 1}{e^{2h} + 1 - 2e^h \cos(2\pi\omega h)}. \tag{30}$$

Taking into account that the last expression and (28), we get

$$s_1 = 2i\pi \cdot \frac{e^{2h} - 1}{e^{2h} + 1 - 2e^h \cos(2\pi\omega h)}. \tag{31}$$

Now, in the work [13, Chapter 2], using equality (59)

$$F[h\widehat{G}_k](p) = \frac{(-1)^k}{(2\pi)^{2k}} \sum_{\beta=-\infty}^{\infty} \frac{1}{\left(p - h^{-1}\beta\right)^{2k}},$$

and formula (66)

$$F[h\widehat{G}_k](p) = \frac{h^{2k}\lambda E_{2k-2}(\lambda)}{(2k-1)! \cdot (1-\lambda)^{2k}},$$

we have the value of s_2 which is defined by (27)

$$s_2 = \frac{h^{2n}\lambda E_{2n-2}(\lambda)}{(2n-1)!(1-\lambda)^{2n}}. \tag{32}$$

Thus, considering expressions (25), (31) and (32), we have

$$\sum_{t=-\infty}^{\infty} \kappa_m(t \cdot N - \omega) = (-1)^{m-1} \left[\frac{h}{2} \cdot \frac{e^{2h} - 1}{e^{2h} + 1 - 2e^h \cos(2\pi\omega h)} + \sum_{n=1}^{m-1} \frac{h^{2n}\lambda E_{2n-2}(\lambda)}{(2n-1)!(1-\lambda)^{2n}}\right].$$

Taking into account that (21), we obtain the result of Theorem 2.

Thus, Theorem 2 is completely proved, that is, Problem 2 is solved.

The following corollaries follow from Theorem 2.

Corollary 1. *When $\omega \in \mathbb{Z}$and $\omega h \notin \mathbb{Z}$, the coefficients of the optimal quadrature formula in the sense of Sard of the form (2) in the space $\widetilde{W_2}^{(1,0)}(0,1]$have the form*

$$\overset{\circ}{C}_k = \frac{2K_{\omega,1}}{4\pi^2\omega^2 + 1} \cdot e^{2\pi i\omega hk} \qquad for\ \ each\ \ k = 1, 2, \dots, N,$$

where

$$K_{\omega,1} = \frac{e^{2h} - 2e^{h}\cos(2\pi\omega h) + 1}{e^{2h} - 1}.$$

Remark 1. Corollary 1 shows that formula (17) gives the result of Theorem 1 in work [26] at $m = 1$.

Corollary 2. *When $\omega \in \mathbb{Z}\setminus\{0\}$and $\omega h \notin \mathbb{Z}$, the coefficients of the optimal quadrature formula (2) for in the space $\widetilde{W_2}^{(2,1)}(0,1]$are expressed as follows*

$$\overset{\circ}{C}_k = \frac{K_{\omega,2}}{8\pi^4\omega^4 + 2\pi^2\omega^2} \cdot e^{2\pi i\omega hk} \quad for\ each \quad k = 1, 2, \ldots, N, \tag{33}$$

where

$$K_{\omega,2} = -\left[\frac{e^{2h} - 1}{e^{2h} + 1 - 2e^{h}\cos(2\pi\omega h)} + \frac{h}{\cos(2\pi\omega h) - 1}\right]^{-1}.$$

We recall that formulas for the optimal coefficients $\overset{\circ}{C}_k = \overset{\circ}{C}{}^R_k + i\cdot \overset{\circ}{C}{}^I_k$ consists of two parts: real C^R_k and imaginary C^I_k in Theorem 2. Therefore from formulas (18) and (19), we obtain the following results.

Corollary 3. *When $\omega \in \mathbb{Z}\setminus\{0\}$ and $\omega h \notin \mathbb{Z}$, optimal coefficients of the quadrature formula of the form*

$$\int_0^1 \cos 2\pi\omega x \cdot \varphi(x)\mathrm{d}x \cong \sum_{k=1}^{N} C^R_k \varphi(hk)$$

have the following form

$$\overset{\circ}{C}{}^R_k = \frac{2K_{\omega,m}}{(2\pi\omega)^{2m} + (2\pi\omega)^{2m-2}} \cdot \cos(2\pi\omega hk) \quad for\ each \quad k = 1, 2, \ldots, N,$$

where $K_{\omega,m}$is defined by (18) and $\varphi \in \widetilde{W_2}^{(m,m-1)}(0,1]$.

Corollary 4. *When $\omega \in \mathbb{Z}\setminus\{0\}$ and $\omega h \notin \mathbb{Z}$, optimal coefficients of the following quadrature formula*

$$\int_0^1 \sin 2\pi\omega x \cdot \varphi(x)\mathrm{d}x \cong \sum_{k=1}^{N} C^I_k \varphi(hk)$$

have the form

$$\overset{\circ}{C}{}^I_k = \frac{2K_{\omega,m}}{(2\pi\omega)^{2m} + (2\pi\omega)^{2m-2}} \cdot \sin(2\pi\omega hk) \quad for\ each \quad k = 1, 2, \ldots, N,$$

where $K_{\omega,m}$is defined by (18) and $\varphi \in \widetilde{W_2}^{(m,m-1)}(0,1]$.

The following theorem is true.

Theorem 3. *In the space $\widetilde{W_2}^{(m,m-1)}(0,1]$with $m \geq 2$,when $\omega h \in \mathbb{Z}$and $\omega \neq 0$, the following equality holds for coefficients of the optimal quadrature formula in the sense of Sard of the form (2)*

$$\overset{\circ}{C}_k = 0 \quad for\ each \quad k = 1, 2, \ldots, N. \tag{34}$$

Proof. Considering $e^{2\pi i\omega hk} = 1$ when $\omega h \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathrm{N}$, we rewrite expression (13) as follows

$$\begin{aligned}\|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}} &= \kappa_m(\omega) - \kappa_m(\omega)\sum_{k'=1}^{N}\overline{C}_{k'} - \kappa_m(\omega)\sum_{k=1}^{N} C_k \\ &+ \sum_{k=1}^{N}\sum_{k'=1}^{N} C_k\overline{C}_{k'}\sum_{\beta\neq 0} e^{2\pi i\beta h(k-k')}\kappa_m(\beta)\end{aligned} \tag{35}$$

where $\kappa_m(\omega)$ is defined by (11).

From condition (5), we get $\sum_{k=1}^{N} C_k = 0$ when $\omega \neq 0$.

Taking into account the above equalities, from (35) we obtain the following

$$\|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}} = \kappa_m(\omega) + \sum_{k=1}^{N}\sum_{k'=1}^{N} C_k\overline{C}_{k'}\sum_{\beta\neq 0} e^{2\pi i\beta h(k-k')}\kappa_m(\beta),$$

$$\text{for} \quad m \geq 2 \quad \text{and} \quad \omega h \in \mathbb{Z}\setminus\{0\}. \tag{36}$$

To find the conditional extremum of the norm $\|\ell\|_{\widetilde{W}_2^{(m,m-1)*}}$, we use the Lagrange method. We consider the following function

$$L_2(C_k, \overline{C}_{k'}, \mu_2) = \|\ell\|^2_{\widetilde{W}_2^{(m,m-1)*}} - \mu_2 \cdot (\ell, 1),$$

where μ_2 is a complex number.

Equating all partial derivatives of the function $L_2(C_k, \overline{C}_{k'}, \mu_2)$ by C_k, $\overline{C}_{k'}$ and μ_2, we have

$$\frac{\partial L_2}{\partial C_k} = \sum_{k'=1}^{N} \overline{C}_{k'} \sum_{\beta \neq 0} e^{2\pi i \beta h(k-k')} \kappa_m(\beta) + \mu_2 = 0$$
$$\text{for } k = 1, 2, \dots, N, \tag{37}$$
$$\frac{\partial L_2}{\partial \overline{C}_{k'}} = \sum_{k=1}^{N} C_k \sum_{\beta \neq 0} e^{2\pi i \beta h(k-k')} \kappa_m(\beta) = 0$$
$$\text{for } \ k' = 1, 2, \dots, N, \tag{38}$$
$$\frac{\partial L_2}{\partial \mu_2} = \int_0^1 e^{2\pi i \omega x} dx - \sum_{k=1}^{N} C_k = 0, \tag{39}$$

where $\kappa_m(\omega)$ is defined by (11).

We search a solution of the system (38),(39) in the following form when $m \geq 2$ and $\omega h \in \mathbb{Z} \setminus \{0\}$

$$\overset{\circ}{C}_k = C_2(\omega, h, m) \quad \text{for} \quad k = 1, 2, \dots, N, \tag{40}$$

where the unknown function $C_2(\omega, h, m)$ depends on ω, h and m.

Putting (40) to system (38), we get

$$\sum_{k=1}^{N} C_2(\omega, h, m) \sum_{\beta \neq 0} \kappa_m(\beta) e^{2\pi i \beta h(k-k')} = 0, k' = 1, 2, \dots, N.$$

Since the series $\sum_{\beta \neq 0} \kappa_m(\beta) e^{2\pi i \beta h(k-k')}$ is convergent, we can write the last equation as follows

$$C_2(\omega, h, m) \sum_{\beta \neq 0} \kappa_m(\beta) e^{-2\pi i \beta h k'} \sum_{k=1}^{N} e^{2\pi i \beta h k} = 0, \ \ k' = 1, 2, \dots, N. \tag{41}$$

Considering the following

$$\sum_{k=1}^{N} e^{2\pi i \beta h k} = \begin{cases} 0 & \text{if } \beta h \neq \gamma \in \mathbb{Z}, \\ N & \text{if } \beta h = \gamma \in \mathbb{Z}, \end{cases}$$

and simplifying the left-hand side of Eq. (41), we make the following equation

$$N \cdot C_2(\omega, h, m) \sum_{\gamma \neq 0} e^{-2\pi i \gamma k'} \kappa_m(\gamma N) = 0 \ \text{ for } \ k' = 1, 2, \dots, N.$$

Since $e^{-2\pi i \gamma k'} = 1$, we have

$$N \cdot C_2(\omega, h, m) \sum_{\gamma \neq 0} \kappa_m(\gamma N) = 0. \tag{42}$$

Now, we show that the sum on the right-hand side of Eq. (42) is not equal to zero. Since $\kappa_m(\gamma N)$ is even function, we write the sum in the last expression as follows

$$\sum_{\gamma \neq 0} \kappa_m(\gamma N) = 2 \cdot \sum_{\gamma=1}^{\infty} \kappa_m(\gamma N). \tag{43}$$

It is not difficult to show that this series is not equal to zero. However, since we use its value in further calculations, we present the following lemma.

Lemma 1. *For $m \geq 2$, the following equality holds*

$$\sum_{\gamma=1}^{\infty} \kappa_m(\gamma N) = \frac{(-1)^{m-1}}{2} \sum_{n=2m-1}^{\infty} \frac{B_n}{n!} \cdot h^n,$$

where B_n is the Bernoulli number.

The proof of Lemma 1 is similar to formula (31) in the work [26].

Now, Using (43) and Lemma 1, we get

$$\sum_{\gamma\neq 0}\kappa_m(\gamma N)=(-1)^{m-1}\sum_{n=2m-1}^{\infty}\frac{B_n}{n!}\cdot h^n\neq 0 \quad \text{for} \quad m\geq 2.$$

It can be seen from the last equality and Eq. (42) that $C_2(\omega,h,m)=0$. From (40) we obtain that $\overset{\circ}{C}_k=0$ for each $k=1,2,\dots,N$. Thus, Theorem 3 is proved.

5. The norm for the error functional of the optimal quadrature formula (2)

In the present section, we calculate the norm of the error functional in the following form

$$\left\|\overset{\circ}{\ell}\right\|_{\widetilde{W}_2^{(m,m-1)*}}:=\inf_{C_k}\|\ell\|_{\widetilde{W}_2^{(m,m-1)*}} \tag{44}$$

of the optimal quadrature formulas (2), that is, we solve Problem 3.

To solve Problem 3, we consider the following cases

1. $m\geq 2$ and $\omega\in\mathbb{Z}\setminus\{0\}$ with $\omega h\notin\mathbb{Z}$;
2. $m\geq 2$ and $\omega h\in\mathbb{Z}\setminus\{0\}$;

The following theorem holds the norm (44) for $m\geq 2$ and $\omega\in\mathbb{Z}\setminus\{0\}$ with $\omega h\notin\mathbb{Z}$.

Theorem 4. *On the space $\widetilde{W}_2^{(m,m-1)*}(0,1]$ of functionals with $m\geq 2$, the norm (44) for the error functional of the optimal quadrature formula (2) for $\omega\in\mathbb{Z}\setminus\{0\}$ and $\omega h\notin\mathbb{Z}$ has the following form*

$$\begin{aligned}\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}}&=\frac{1}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}\\&\times\left[1-\frac{2}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}\cdot\frac{1}{hK^{-1}_{\omega,m}}\right],\end{aligned} \tag{45}$$

where

$$hK^{-1}_{\omega,m}=(-1)^{m-1}\cdot\left[\frac{h(e^{2h}-1)}{e^{2h}+1-2e^h\cos(2\pi\omega h)}+\sum_{n=1}^{m-1}\frac{2h^{2n}\cdot\lambda E_{2n-2}(\lambda)}{(2n-1)!\cdot(1-\lambda)^{2n}}\right],$$

$E_{2n-2}(\lambda)$ is the Euler–Frobenius polynomial of degree $(2n-2)$ and $\lambda=e^{2\pi i\omega h}$.

Proof. Using expression (13)

$$\begin{aligned}\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}}&=\kappa_m(\omega)-\kappa_m(\omega)\sum_{k'=1}^{N}\overline{\overset{\circ}{C}}_{k'}e^{2\pi i\omega hk'}-\kappa_m(\omega)\sum_{k=1}^{N}\overset{\circ}{C}_ke^{-2\pi i\omega hk}\\&+\sum_{k=1}^{N}\sum_{k'=1}^{N}\overset{\circ}{C}_k\overline{\overset{\circ}{C}}_{k'}\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\times\kappa_m(\beta)=\kappa_m(\omega)-\sum_{k'=1}^{N}\overline{\overset{\circ}{C}}_{k'}\\&\times\left[\kappa_m(\omega)e^{2\pi i\omega hk'}-\sum_{k=1}^{N}\overset{\circ}{C}_k\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\kappa_m(\beta)\right]\\&-\kappa_m(\omega)\sum_{k=1}^{N}\overset{\circ}{C}_ke^{-2\pi i\omega hk}\end{aligned}$$

and taking into account that (15) and $\kappa_m(\omega)$ is defined by (11), we write the last expression

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}}=\frac{1}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}-\frac{1}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}\sum_{k=1}^{N}\overset{\circ}{C}_ke^{-2\pi i\omega hk}.$$

Using (17), we have

$$\begin{aligned}\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}}&=\frac{1}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}-\frac{1}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}\\&\times\sum_{k=1}^{N}\frac{2K_{\omega,m}}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}=\frac{1}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}\\&\times\left[1-\frac{2}{(2\pi\omega)^{2m}+(2\pi\omega)^{2m-2}}\cdot\frac{1}{hK^{-1}_{\omega,m}}\right].\end{aligned}$$

Thus, Theorem 4 is proved.

We have the following result as follows.

Corollary 5. *Considering formula (45) and the following equality*

$$\lim_{h\to 0} hK^{-1}_{\omega,m} = \frac{2}{(2\pi\omega)^{2m} + (2\pi\omega)^{2m-2}},$$

we have

1. $\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} \to 0$ *as* $h \to 0$ *for* $\omega \in \mathbb{R} \setminus \{0\}$ *is fixed;*
2. $\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} \to 0$ *as* $\omega \to \infty$ *for* h *is fixed.*

In the case $m = 2$, the following result follows from Theorem 4.

Corollary 6. *On the space* $\widetilde{W}_2^{(2,1)*}(0,1]$ *the norm (44) for* $\omega \in \mathbb{Z}\setminus\{0\}$ $\omega h \notin \mathbb{Z}$ *has the following form*

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2}\left[1 - \frac{2}{(2\pi\omega)^4 + (2\pi\omega)^2}\cdot\frac{1}{hK^{-1}_{\omega,2}}\right],$$

where

$$hK^{-1}_{\omega,2} = -\left[\frac{h(e^{2h}-1)}{e^{2h}+1-2e^h\cos(2\pi\omega h)} + \frac{h^2}{\cos(2\pi\omega h)-1}\right].$$

The following theorem holds for the case $m \geq 2$ and $\omega h \in \mathbb{Z}\setminus\{0\}$.

Theorem 5. *On the space* $\widetilde{W}_2^{(m,m-1)*}(0,1]$ *with* $m \geq 2$*, the norm for the error functional (44) of the optimal quadrature formulas (2) when* $\omega h \in \mathbb{Z}\setminus\{0\}$ *has the following form*

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} = \frac{1}{(2\pi\omega)^{2m} + (2\pi\omega)^{2m-2}}. \tag{46}$$

Proof. Taking into account that (13) when $\omega h \in \mathbb{Z}\setminus\{0\}$ and $C_k = \overset{\circ}{C}_k$, we have

$$\begin{aligned}\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} &= \kappa_m(\omega) + \sum_{k=1}^{N}\sum_{k'=1}^{N}\overset{\circ}{C}_k\overline{\overset{\circ}{C}}_{k'}\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\kappa_m(\beta)\\ &= \kappa_m(\omega) + \sum_{k'=1}^{N}\overline{\overset{\circ}{C}}_{k'}\left[\sum_{k=1}^{N}\overset{\circ}{C}_k\sum_{\beta\neq 0}e^{2\pi i\beta h(k-k')}\kappa_m(\beta)\right].\end{aligned}$$

Considering Theorem 3 or the system of Eqs. (38), we can write the last expression as follows

$$\left\|\overset{\circ}{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} = \kappa_m(\omega).$$

Thus, Theorem 5 is proved, and at the same time, Problem 3 is completely solved.

Remark 2. It can be seen from Theorem 5 that the worst case error of the optimal quadrature formulas (2) does not depend on the number of nodes N in the case $\omega h \in \mathbb{Z}\setminus\{0\}$. In this case, the function values do not give any useful information and therefore the zero-algorithm is optimal, i.e., Theorem 3 is fulfilled. This case was already shown by Zhang et al. [34, Theorem 7 and Remark 9].

6. Numerical results

In this section, we give several numerical results of the sharp upper bounds for the error (3) of the optimal quadrature formula (2) in the space $\widetilde{W}_2^{(2,1)}(0,1]$.

According to the Cauchy–Schwarz inequality, in the space $\widetilde{W}_2^{(2,1)}(0,1]$ for the absolute value of the error (3) of the optimal quadrature formula (2) we obtain

$$\left|\left(\overset{\circ}{\ell},\varphi\right)\right| \leq \|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}\cdot\|\varphi\|_{\widetilde{W}_2^{(2,1)}},$$

where $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ is the norm for the optimal error functional ℓ which corresponds to the optimal quadrature formulas (2).

Using Corollary 6 for $m = 2$, we obtain the numerical results which are presented in Table 1 when $h = 0.2,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$ and $\omega = 1,\ 11,\ 101,\ 1001,\ 10001$. From the first column of this table we can see that order of convergence of our optimal quadrature formula is $O(h^2)$ and from the first row of Table 1 it is clear that the quantity $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ converges as $O\left(|\omega|^{-2}\right)$.

Table 1
The numerical results for $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ at $h = 0.2,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$ and $\omega = 1,\ 11,\ 101,\ 1001,\ 10001$.

h	$\omega = 1$	$\omega = 11$	$\omega = 101$	$\omega = 1001$	$\omega = 10001$
0.2	2.5015e−2	2.0931e−4	2.4831e−6	2.5279e−8	2.5325e−10
10^{-1}	3.9029e−4	2.0931e−4	2.4831e−6	2.5279e−8	2.5325e−10
10^{-2}	3.7285e−6	3.9409e−6	2.4831e−6	2.5279e−8	2.5325e−10
10^{-3}	3.7268e−8	3.7289e−8	3.9070e−8	2.5279e−8	2.5325e−10
10^{-4}	3.7268e−10	3.7268e−10	3.7285e−10	3.9038e−10	2.5325e−10

Table 2
The values of $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ for $\omega = 10001$ when $h = 10^{-k}$ and for $h = 10^{-4}$ when $\omega = 10^k,\ k = 2, 3, \dots, 8$.

	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$	$h = 10^{-5}$	$h = 10^{-6}$	$h = 10^{-7}$	$h = 10^{-8}$
$\omega = 10001$	2.53e−10	2.53e−10	2.53e−10	3.90e−12	3.73e−14	3.73e−16	3.73e−18
	$\omega = 10^2$	$\omega = 10^3$	$\omega = 10^4$	$\omega = 10^5$	$\omega = 10^6$	$\omega = 10^7$	$\omega = 10^8$
$h = 10^{-4}$	3.73e−10	3.9e−10	2.53e−10	2.53e−12	2.53e−14	2.53e−16	2.53e−18

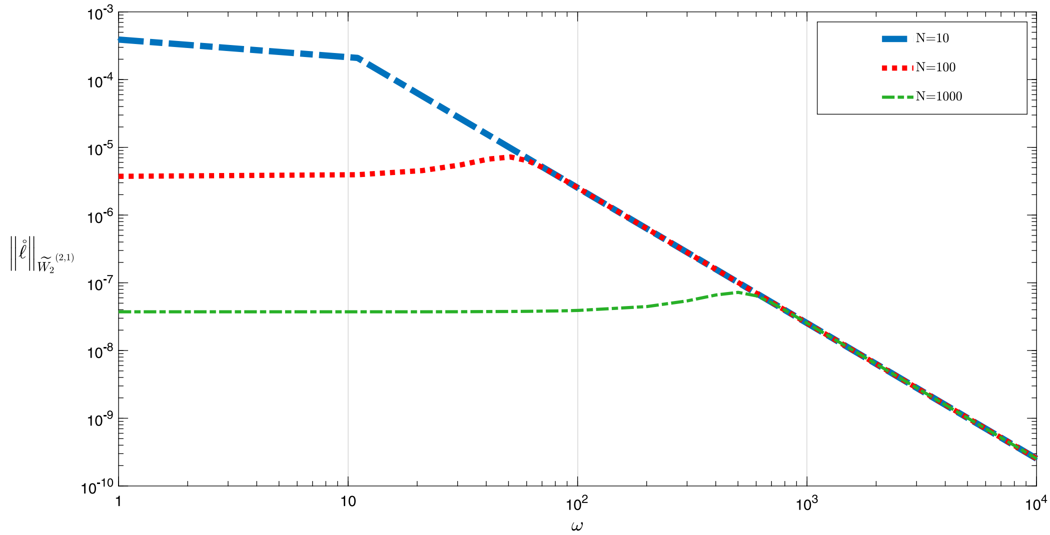

Fig. 1. Graph of the norm for the error functional depends on ω at $N = 10, N = 100$ and $N = 1000$.

Using Corollary 6 to better understand the data in Table 1, we give the following values of $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}}$ for fixed ω as $h \to 0$ and for fixed h as $\omega \to \infty$.

The numerical results show that $\|\overset{\circ}{\ell}\|_{\widetilde{W}_2^{(2,1)*}} \to 0$ for fixed ω as $h \to 0$ and for fixed h as $\omega \to \infty$. From Table 2, we conclude that the order of convergence of the optimal quadrature formula (2) is $O(h^2)$ when $|\omega| < N$ and is $O(|\omega|^{-2})$ when $|\omega| \geq N$, that is, the order of convergence of our optimal quadrature formula is $O\left(\left(\frac{1}{N+|\omega|}\right)^2\right)$ in the space $\widetilde{W}_2^{(2,1)}(0,1]$.

Using the values of Table 1, we generate the graph of the norm for the error functional in Fig. 1 at $N = 10$, $N = 100$ and $N = 1000$, where N is the number of nodes, i.e., $N = 1/h$.

Now we consider the Bernoulli polynomials of 4th degree $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$ as an integrand function. Then for the error (3) of the optimal quadrature formula (2) we have

$$
\begin{aligned}
R_h(\omega) &= \left|\left(\overset{\circ}{\ell}, x^4 - 2x^3 + x^2 - \frac{1}{30}\right)\right| = \left|\int_0^1 e^{2\pi i\omega x}\left(x^4 - 2x^3 + x^2 - \frac{1}{30}\right)dx\right. \\
&\quad \left. - \sum_{k=1}^{N} \overset{\circ}{C}_k \left((hk)^4 - 2\cdot(hk)^3 + (hk)^2 - \frac{1}{30}\right)\right| \leq \left\|x^4 - 2x^3 + x^2 - \frac{1}{30}\right\|_{\widetilde{W}_2^{(2,1)}} \\
&\quad \times \left\|\overset{\circ}{\ell}\right\|_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{105}\sqrt{9030}\cdot\left\|\overset{\circ}{\ell}\right\|_{\widetilde{W}_2^{(2,1)*}} = U_h(\omega).
\end{aligned}
$$

Table 3
The numerical values of $R_h(\omega)$ and $U_h(\omega)$ in the space $\widetilde{W}_2^{(2,1)}(0,1]$ for some selected values of h and ω.

h	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$	
	$R_h(\omega)$	$U_h(\omega)$	$R_h(\omega)$	$U_h(\omega)$	$R_h(\omega)$	$U_h(\omega)$	$R_h(\omega)$	$U_h(\omega)$
0.2	5.0e−6	1.6e−2	2.6e−8	1.9e−4	3.7e−12	2.2e−6	3.8e−16	2.3e−8
10^{-1}	9.2e−8	3.5e−4	2.6e−8	1.9e−4	3.7e−12	2.2e−6	3.8e−16	2.3e−8
10^{-2}	8.5e−12	3.4e−6	7.7e−14	3.6e−6	3.7e−12	2.2e−6	3.8e−16	2.3e−8
10^{-3}	8.4e−16	3.4e−8	7.0e−18	3.4e−8	9.0e−20	3.5e−8	3.8e−16	2.3e−8

Table 4
R_W and R_L are the absolute errors of the optimal quadrature formulas in the spaces $\widetilde{W}_2^{(2,1)}(0,1]$ and $\widetilde{L_2^{(2)}}(0,1]$, respectively.

N	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$	
	R_W	R_L	R_W	R_L	R_W	R_L	R_W	R_L
2	2.47e−2	1.55e−1	2.53e−4	1.59e−3	2.53e−6	1.59e−5	2.53e−8	1.59e−7
10	8.43e−4	5.30e−3	2.53e−4	1.59e−3	2.53e−6	1.59e−5	2.53e−8	1.59e−7
100	8.33e−6	5.24e−5	8.44e−6	5.30e−5	2.53e−6	1.59e−5	2.53e−8	1.59e−7
1000	8.33e−8	5.24e−7	8.33e−8	5.24e−7	8.44e−8	5.30e−7	2.53e−8	1.59e−7

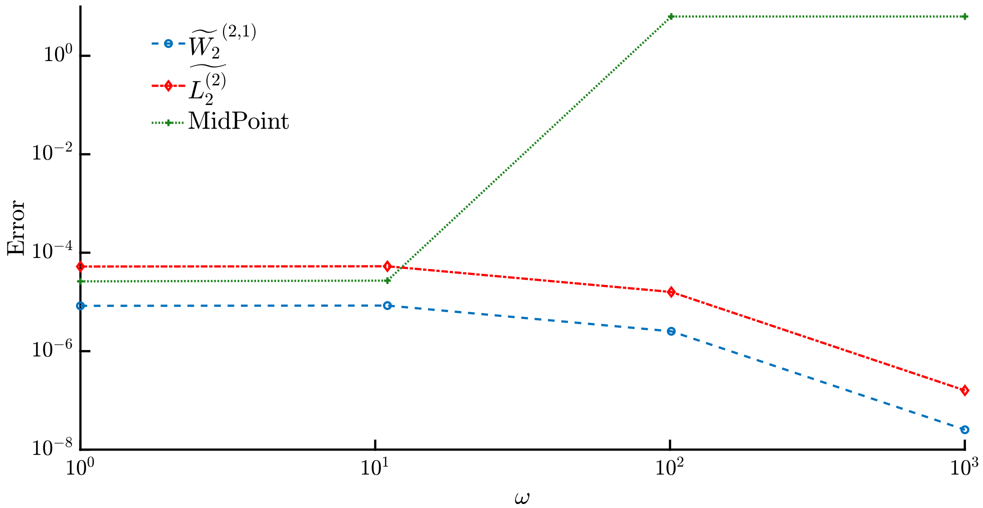

Fig. 2. Graph of errors of the midpoint formula and optimal quadrature formulas in spaces $\widetilde{W}_2^{(2,1)}$ and $\widetilde{L_2^{(2)}}$ at $N = 100$.

We calculate the left sides of the last inequality using the formula of optimal coefficients $\mathring{C}_k$ presented in Corollary 4, and to calculate the right-hand side of this inequality, we use the norm which is given in Corollary 6 when $h = 0.2,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$ and $\omega = 1,\ 11,\ 101,\ 1001,\ 10001$. We obtain the numerical results given in the first row of Table 3.

These numerical results show that the assertions given in Corollary 5 are correct, i.e., the order of convergence of the optimal quadrature formula (3) is m.

We denote the absolute value of the difference between the quadrature sum and the integral as follows

$$R = \left| \int_0^1 e^{2\pi i \omega x} \varphi(x) dx - \sum_{k=1}^{N} C_k \varphi(hk) \right|, \tag{47}$$

where $\varphi(x) = \frac{e^{1-x}+e^{x}}{2(1-e)}$ and $\omega \in \mathbb{Z}\backslash\{0\}$.

Table 4 shows that the order of approximation of the optimal quadrature formula (2) is 2 in the space $\widetilde{W}_2^{(2,1)}(0,1]$. From this table one can see that the absolute value of the error for the optimal quadrature formula (2) in the space $\widetilde{W}_2^{(2,1)}(0,1]$ is smaller than the absolute value of the error for the optimal quadrature formula in the space $\widetilde{L_2^{(2)}}(0,1]$ (see [25]).

Using the values of the above table, the following figure compares the errors of the optimal quadrature formulas in the spaces $\widetilde{W}_2^{(2,1)}(0,1]$ and $\widetilde{L_2^{(2)}}(0,1]$ with the error of the midpoint formula. We can see from Fig. 2 that when $|\omega| < N$, all three formulas converge in the same order. However, when $|\omega| \geq N$ the midpoint formula does not give an effective result. Vice versa, we can see that the errors (47) of the optimal quadrature formulas constructed in spaces $\widetilde{W}_2^{(2,1)}(0,1]$ and $\widetilde{L_2^{(2)}}(0,1]$ approach zero as $\omega \to \infty$.

Table 5
R_W is the absolute errors of the optimal quadrature formula in the space $\widetilde{W}_2^{(2,1)}(0,1]$ and R_F is the absolute errors of Filon-type method with nodes $\{0, \frac{1}{2}, 1\}$ for (48).

	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$		$\omega = 10001$	
	R_W	R_F	R_W	R_F	R_W	R_F	R_W	R_F	R_W	R_F
$N = 2$	1.0e−1	1.0e−2	1.3e−5	6.5e−4	1.8e−7	7.8e−6	1.9e−9	7.9e−8	1.9e−9	7.9e−8

Now we consider the following integral to compare the optimal quadrature formula in the space $\widetilde{W}_2^{(2,1)}$ and Filon-type methods (see Table 5).

$$\int_0^1 \frac{e^{2\pi i\omega x}}{1+25x^2}dx. \tag{48}$$

Conclusion

In this work, optimal quadrature formulas are constructed and the explicit formulas for optimal coefficients are found in the space $\widetilde{W}_2^{(m,m-1)}(0,1]$ with $m \geq 2$ of periodic functions. Using the extremal function in the space $\widetilde{W}_2^{(m,m-1)}(0,1]$, we find analytic form of the norm for the error functional ℓ. We have taken the system of linear equations of the coefficients for finding optimal coefficients of the quadrature formulas. In addition, we have found the analytical forms of coefficients of the optimal quadrature formulas, and we have calculated the norm for the error functional ℓ of the optimal quadrature formulas (2) in the following cases:

- for $m \geq 2$ and $\omega \in \mathbb{Z}\setminus\{0\}$ with $\omega h \notin \mathbb{Z}$;
- for $m \geq 2$ and $\omega h \in \mathbb{Z}\setminus\{0\}$.

Data availability

No data was used for the research described in the article.

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All authors approved the version of the manuscript to be published.

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