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An Optimal Quadrature Formula for Approximating Fourier Integrals in a Hilbert Space

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Abstract. This paper presents the construction process of the optimal quadrature formula for the approximate calculation of the integrals $\int_a^b e^{2\pi i\omega x}\varphi(x)dx$ with real ω in the Hilbert space of complex-valued periodic functions. Here, initially, in order to obtain a sharp upper bound for the error of the quadrature formula, the norm of the error functional is calculated. For this, the extremal function of the error functional for the quadrature formula is used. Then, by minimizing the norm of the error functional with respect to the coefficients, an optimal quadrature formula is obtained. Using the explicit form of the optimal coefficients, the norm of the error functional of the optimal quadrature formula is calculated. Finally, using this optimal quadrature formula the approximation formula for Fourier integrals $\int_a^b e^{2\pi i\omega x}\varphi(x)dx$ with $\omega \in \mathbb{R}$ is obtained in the interval $[a,b]$.

INTRODUCTION. STATEMENT OF THE PROBLEM

The Fourier integrals are broadly used in science and technology, specifically, in the problems of Computed Tomography (CT). It is widely known that when complete continuous X-ray data are accessible, CT images can be reconstructed exactly using the filtered back-projection formula (see, for instance [1]). This formula consequentially uses the Radon transform, the Fourier transforms, and the back-projection formula. Fourier transforms play an important role in the filtered back-projection method of CT image reconstruction [2, 3]. Since in practice we have finite discrete values of the Radon transform, we have to approximately calculate the Fourier transforms. That reason, one has to consider the problem of approximate work out of the integral

$$I_\varphi(\omega) = \int_a^b e^{2\pi i\omega x}\varphi(x)dx, \tag{1}$$

where $i^2 = -1$ and $\omega \in \mathbb{R}$. Such type of integrals for sufficiently large ω is called integrals with *highly* or *strongly oscillating integrands.* In any case, it is unsolvable to get the sharp values of such integrals. They are mainly computed by special effective numerical methods. Originally, the formula for the numerical integration of the integral (1) was given by Filon L.N.G. in the work [4].

We consider the Hilbert space $W_2^{(1,0)}[a,b]$ of complex-valued functions f defined on the interval $[a,b]$, which possess an absolute continuous on $[a,b]$ and whose first order derivative is square integrable [5, 6, 7, 8, 9, 10]. The space $W_2^{(1,0)}[a,b]$ equipped with the inner product

$$\langle\varphi,\psi\rangle_{W_2^{(1,0)}} = \int_a^b \left(\varphi'(x)+\varphi(x)\right)\left(\bar{\psi}'(x)+\bar{\psi}(x)\right)dx, \tag{2}$$

where $\bar{\psi}$ is the complex conjugate to the function ψ. We note that every element of the space $W_2^{(1,0)}[a,b]$ is a class of the functions which differ from each other by de^{-x} ($d = const.$). The semi-norm of function φ is correspondingly defined by the formula

$$\|\varphi\|_{W_2^{(1,0)}[a,b]} = \langle\varphi,\varphi\rangle^{1/2}.$$

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Moreover, we denote by $\widetilde{W_2}^{(1,0)}(a,b]$ the subspace of periodic complex-valued functions from the space $W_2^{(1,0)}[a,b]$ with period $T=b-a$. Notice that every element of the space $\widetilde{W_2}^{(1,0)}(a,b]$ satisfies the following condition of T-periodicity

$$\varphi(x+T\beta)=\varphi(x), \text{ for } x\in\mathbb{R},\ \beta\in\mathbb{Z}.$$

It should be noted that in [11] and [5], based on the Sobolev method, the problem of constructing optimal quadrature formulas in the sense of Sard [12] for the numerical calculation of the integral (1) with $\omega\in\mathbb{Z}$ was solved in the Hilbert spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$, respectively. Recently, In the work [5] in the Hilbert space $W_2^{(m,m-1)}$ of differentiable functions the optimal quadrature formulas were constructed for numerical computation of the integral (1) for any $m\in\mathbb{N}$ and $\omega\in\mathbb{R}$.

Coefficients for the optimal quadrature formulas in numerical calculation of $I_\varphi(\omega)$ in the space $L_2^{(m)}$ of differentiable functions were obtained in the works [11, 13]. In particular, in the space $L_2^{(2)}$ the exponentially weighted optimal quadrature formula with derivative was constructed in [14].

We notice that optimal quadrature formula for approximate integration of the Fourier coefficients (1) with $\omega\in\mathbb{Z}$ in the Hilbert space $\widetilde{W_2}^{(1,0)}$ of periodic functions was constructed in the work [6].

The main goal of this paper is to construct an approximation formula for the numerical calculation of the integral (1) with $\omega\in\mathbb{R}$ using the obtained optimal quadrature formula in the Hilbert space $\widetilde{W_2}^{(1,0)}(0,1]$ complex-valued periodic functions in the work [6] (Theorem 1).

In the above works, optimal quadrature formulas were constructed based on the Sobolev method for the approximate calculation of exponentially weighted integrals. Also, the optimal quadrature formulas for the approximate calculation of singular weighted integrals were obtained based on the Sobolev method (see [15, 16, 17, 18]), and in the space $W_2^{(m,0)}$ optimal quadrature formulas exactly for exponential-trigonometric functions were constructed [19]. In addition, optimal interpolation formulas were constructed using the discrete analogue of the differential operator found by this method in works [20, 21, 22, 23, 24, 25].

The rest of the paper is organized as follows. Section 2 presents the process of constructing the optimal quadrature formula for the exponential weighted integrals in the Hilbert space $\widetilde{W_2}^{(1,0)}(0,1]$. In particular, the optimal quadrature formula is given for Fourier coefficients $\int_0^1 e^{2\pi i\omega x}\varphi(x)dx$ with integer ω. In Section 3, Using this optimal quadrature formula, the approximation formula for numerical calculation of the Fourier integrals (1) with $\omega\in\mathbb{R}$ is obtained.

OPTIMAL QUADRATURE FORMULA FOR EXPONENTIAL WEIGHTED INTEGRALS

For elements φ of the space $\widetilde{W_2}^{(1,0)}(0,1]$ we examine the following quadrature formula

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)\cong\sum_{k=1}^{N}C_k\varphi(hk), \tag{3}$$

where $\omega\in\mathbb{Z}\setminus\{0\}$, C_k are coefficients of the quadrature formula (3), $h=1/N$, $N=1,2,...$, and the difference

$$\begin{aligned}(\ell,\varphi)&=\int_0^1 e^{2\pi i\omega x}\varphi(x)-\sum_{k=1}^{N}C_k\varphi(hk)\\&=\int_0^1\ell(x)\varphi(x)dx\end{aligned} \tag{4}$$

is called *the error* of the quadrature formula (3), and

$$\ell(x)=e^{2\pi i\omega x}-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta) \tag{5}$$

is *the periodic error functional* of the error (4) for the quadrature formula (3), where δ is Dirac's delta-function.

The error (4) of the quadrature formula (3) is a linear functional in $\widetilde{W_2}^{(1,0)*}(0,1]$ which is the conjugate space to the space $\widetilde{W_2}^{(1,0)}(0,1]$.

We note that in this section for completeness we give construction of the optimal quadrature formula (3) in the space $\widetilde{W_2}^{(1,0)}(0,1]$ of 1-periodic complex-valued functions based on the results of the work [6].

Succeeding, in this section, for construction of the optimal quadrature formula (3) in the space $\widetilde{W_2}^{(1,0)}(0,1]$, firstly, an extremal function of the error functional (4) is found, then, using the extremal function the norm of the error functional ℓ is calculated, next, a system of linear equations of coefficients giving the minimum to this norm is obtained, and uniqueness of the solution for this system is discussed.

The following theorem was proved in the work [6].

Theorem 1. *Among all quadrature formulas with the error functional ℓ of the form* (3) *in the space* $\widetilde{W_2}^{(1,0)}(0,1]$ *of 1-periodic complex-valued functions, there is a unique optimal quadrature formula with coefficients which has the following form:*

$$\mathring{C}_k = 2\cdot\kappa(\omega)\cdot\frac{e^{2h}-2e^{h}\cos(2\pi\omega h)+1}{e^{2h}-1}\cdot e^{2\pi i\omega hk}, \quad \text{for } k=1,2,...,N, \tag{6}$$

where $\omega\in\mathbb{Z}\setminus\{0\}$ *and*

$$\kappa(\omega)=\left(4\pi^2\omega^2+1\right)^{-1}. \tag{7}$$

An extremal function and the norm of the error functional for the quadrature formula (3)

In order to find the analytic form for the norm of the error functional ℓ we use its *extremal function* (see [26, 27]) that satisfies the following equality

$$(\ell,\psi_\ell)=\|\ell\|_{\widetilde{W_2}^{(1,0)*}}\cdot\|\psi_\ell\|_{\widetilde{W_2}^{(1,0)}}. \tag{8}$$

According to the Cauchy-Schwarz inequality, we have the following estimation for the error of the quadrature formula

$$|(\ell,\varphi)|\le\|\ell\|_{\widetilde{W_2}^{(1,0)*}}\cdot\|\varphi\|_{\widetilde{W_2}^{(1,0)}}.$$

The problem of constructing the optimal quadrature formula for the approximate calculation of the integral (1) is to calculate the quantity

$$\left\|\mathring{\ell}\right\|_{\widetilde{W_2}^{(1,0)*}}=\inf_{C_k}\sup_{\varphi,\|\varphi\|_{\widetilde{W_2}^{(1,0)}}\neq0}\frac{|(\ell,\varphi)|}{\|\varphi\|_{\widetilde{W_2}^{(1,0)}}}. \tag{9}$$

The coefficients giving the minimum are called *the optimal coefficients* and they are denoted by $\mathring{C}_k$. The quadrature formula with coefficients $\mathring{C}_k$ is called *the optimal quadrature formula*.
Thus, we come to the following problem

Problem 1. Find the coefficients $\mathring{C}_k$ that give the minimum and $\|\mathring{\ell}\|$ that depends on $\mathring{C}_k$.

It should be noted that, in the work [6], the first part of Problem 1 was solved, and the problem is completely solved in this paper.

Since $\widetilde{W_2}^{(1,0)}(0,1]$ is the Hilbert space with the inner product (2) then by the Riesz theorem on general form of a linear continuous functional on Hilbert spaces, for the error functional (5) there exist a unique function $\psi_\ell \in \widetilde{W_2}^{(1,0)}(0,1]$ such that for any φ from the space $\widetilde{W_2}^{(1,0)}(0,1]$ the following equality is fulfilled

$$(\ell,\varphi) = \langle \psi_\ell, \varphi\rangle_{\widetilde{W_2}^{(1,0)}} \tag{10}$$

and

$$\|\ell\|_{\widetilde{W_2}^{(1,0)*}} = \|\psi_\ell\|_{\widetilde{W_2}^{(1,0)}},$$

where ψ_ℓ is the Riesz element corresponding to the functional ℓ.

Thus, taking into account equality (8) we get

$$(\ell,\psi_\ell) = \|\ell\|^2_{\widetilde{W_2}^{(1,0)*}}. \tag{11}$$

Integrating by parts the right-hand side of equality (10), keeping in mind 1-periodicity of functions φ and ψ_ℓ, for ψ_ℓ we get the following equation

$$\bar{\psi}_\ell''(x) - \bar{\psi}_\ell(x) = -\ell(x). \tag{12}$$

For the periodic solution of equation (12), we get the following result.

Lemma 1. *The periodic extremal function ψ_ℓ of the error functional* (4)*, satisfying equation* (12)*, is determined by the formula*

$$\psi_\ell(x) = \kappa(\omega)e^{-2\pi i\omega x} - \sum_{k=1}^{N}\overline{C}_k \sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta(x-hk)}, \tag{13}$$

where $\kappa(\omega)$ is defined by equality (7)*, $\overline{C}_k$ are the complex conjugate for C_k.*

The proof of Lemma 1 was given in the work [6].

Now, using equalities (5), (11) and (13), we calculate the norm of the error functional (4)

$$\begin{aligned}\|\ell\|^2_{\widetilde{W_2}^{(1,0)*}} &= (\ell,\psi_\ell) = \int_0^1 \ell(x)\cdot\psi_\ell(x)dx \\ &= \int_0^1 \left(e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\right) \\ &\times\left(\kappa(\omega)e^{-2\pi i\omega x} - \sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\gamma=-\infty}^{\infty}\kappa(\gamma)e^{2\pi i\gamma(x-hk)}\right)dx\end{aligned}$$

Simplifying the above expression, we obtain the following analytical form for the norm of the error functional

$$\begin{aligned}\|\ell\|^2_{\widetilde{W_2}^{(1,0)*}} &= \kappa(\omega) - \kappa(\omega)\sum_{k'=1}^{N}\overline{C}_{k'}e^{2\pi i\omega hk'} - \kappa(\omega)\sum_{k=1}^{N}C_k e^{-2\pi i\omega hk} \\ &+ \sum_{k=1}^{N}\sum_{k'=1}^{N}C_k\overline{C}_{k'}\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta h(k-k')}.\end{aligned} \tag{14}$$

MINIMIZATION OF THE NORM OF THE ERROR FUNCTIONAL ℓ

Now we seek the minimum of expression (14) by C_k and $\overline{C}_k$. For this, we consider the following function

$$L(C_k,\overline{C}_{k'}) = \|\ell\|^2_{\widetilde{W}_2^{(1,0)*}}$$

that depends on C_k and $\overline{C}_k$, $k=1,2,\ldots,N$.

Equating all partial derivatives of the function $L(C_k,\overline{C}_{k'})$ by C_k and $\overline{C}_{k'}$, $k=1,2,\ldots,N$ to zero, we have the following system of the linear equations

$$\sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta h(k-k')} = \kappa(\omega)e^{-2\pi i\omega hk}, \text{ for } k=1,2,\ldots,N, \tag{15}$$

$$\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta h(k-k')} = \kappa(\omega)e^{2\pi i\omega hk'}, \quad \text{for } k'=1,2,\ldots,N. \tag{16}$$

It is not difficult to see that system of equations (15) are equivalent to the system (16). So, we are content with solving the system of equations (16).

It can be proved that the system of equations (16) has a unique solution, as in the work [28, 29]. The solution of this system which we denote by $\mathring{C}_k$, $k=1,2,...,N$ are the stationary points for the function $L(C_k,\overline{C}_{k'})$. The aim of the paper is to find the analytic solution for the system (16).

AN OPTIMAL QUADRATURE FORMULA FOR FOURIER COEFFICIENTS ON THE INTERVAL $(0,1]$

Now we present the proof of Theorem 1. For this, it is enough to solve the system (16) with respect to coefficients C_k. A solution of the system we search in the form

$$\mathring{C}_k = C_\omega(h)\cdot e^{2\pi i\omega hk}, \tag{17}$$

where $C_\omega(h)$ is an unknown function with respect to ω. Taking into account formula (17), we can rewrite the system of equations (16) as follows

$$-\kappa(\omega)e^{2\pi i\omega hk'} + \sum_{k=1}^{N}C_\omega(h)e^{2\pi i\omega hk}\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta h(k-k')} = 0,$$
$$\text{for } k'=1,2,...,N. \tag{18}$$

We introduce the following denotation

$$Z = C_\omega(h)\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{-2\pi i\beta hk'}\sum_{k=1}^{N}e^{2\pi i\omega hk}\cdot e^{2\pi i\beta hk}. \tag{19}$$

It is easy to see that

$$\sum_{k=1}^{N}e^{2\pi i(\omega+\beta)hk} = \begin{cases} 0, & \text{if } (\omega+\beta)h\notin\mathbb{Z},\\ h^{-1}, & \text{if } (\omega+\beta)h\in\mathbb{Z}.\end{cases}$$

So denoting $t=(\omega+\beta)h$, and we rewrite formula (19) as follows

$$Z = \frac{1}{4\pi^2 h}\cdot C_\omega(h)e^{2\pi i\omega hk'}\sum_{t=-\infty}^{\infty}\frac{1}{(t\cdot N-\omega)^2-\left(\frac{i}{2\pi}\right)^2}. \tag{20}$$

From equation (18) and equality (20) it follows that

$$\frac{1}{4\pi^2 h}\cdot C_\omega(h)e^{2\pi i\omega hk'}\sum_{t=-\infty}^{\infty}\frac{1}{(t\cdot N-\omega)^2-\left(\frac{i}{2\pi}\right)^2}=\kappa(\omega)e^{2\pi i\omega hk'},$$

for $k'=1,2,...,N$.

From the last equations we obtain the solution of the system (16) as follows

$$C_\omega(h)=4\pi^2 h\cdot\kappa(\omega)\left(\sum_{t=-\infty}^{\infty}f(t)\right)^{-1}, \tag{21}$$

where

$$f(t)=\frac{h^2}{\left(t-\omega h-\frac{ih}{2\pi}\right)\left(t-\omega h+\frac{ih}{2\pi}\right)}. \tag{22}$$

To simplify expression (21), we calculate the sum of the series in it. For this, using the following well-known formula from the residual theory (see [30], p. 296), we have

$$\sum_{t=-\infty}^{\infty}f(t)=2\pi^2 h\cdot\frac{e^{2h}-1}{e^{2h}-2e^h\cos(2\pi\omega h)+1}. \tag{23}$$

Using expressions (21) and (23) we get

$$C_\omega(h)=2\cdot\kappa(\omega)\cdot\frac{e^{2h}-2e^h\cos(2\pi\omega h)+1}{e^{2h}-1}. \tag{24}$$

We get the statement of Theorem 1 by introducing the value of the last expression into expression (17).

Now, we give main result of the paper, that is, we solve Problem 1 completely.

Theorem 2. *In the space* $\widetilde{W_2}^{(1,0)}(0,1]$ *for the norm of the error functional* ℓ *of the optimal quadrature formula* (3)*, the following holds*

$$\left\|\overset{o}{\ell}\right\|^2_{\widetilde{W}_2^{(1,0)*}}=\kappa(\omega)-2\cdot\kappa^2(\omega)\cdot\frac{e^{2h}+1-2e^h\cos(2\pi\omega h)}{h\left(e^{2h}-1\right)}, \tag{25}$$

where $\kappa(\omega)$ *is defined by equality* (7)*.*

Proof. For this, taking into account the condition of Problem 1, we can rewrite expression (14) as follows

$$\begin{aligned}\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(1,0)*}}&=\kappa(\omega)-\kappa(\omega)\sum_{k'=1}^{N}\overset{\circ}{\bar{C}}_{k'}e^{2\pi i\omega hk'}-\kappa(\omega)\sum_{k=1}^{N}\mathring{C}_k e^{-2\pi i\omega hk}+\sum_{k=1}^{N}\sum_{k'=1}^{N}\mathring{C}_k\overset{\circ}{\bar{C}}_{k'}\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta h(k-k')}\\&=\kappa(\omega)-\sum_{k'=1}^{N}\overset{\circ}{\bar{C}}_{k'}\left[\kappa(\omega)e^{2\pi i\omega hk'}-\sum_{k=1}^{N}\mathring{C}_k\sum_{\beta=-\infty}^{\infty}\kappa(\beta)e^{2\pi i\beta h(k-k')}\right]-\kappa(\omega)\sum_{k=1}^{N}\mathring{C}_k e^{-2\pi i\omega hk}.\end{aligned}$$

Hence, taking into account (16) to work out $\left\|\mathring{\ell}\right\|^2$ for the optimal quadrature formula (3), we have the following

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(1,0)*}}=\kappa(\omega)-\kappa(\omega)\sum_{k=1}^{N}\mathring{C}_k e^{-2\pi i\omega hk}.$$

Then, using the analytic form of the optimal coefficients, that is, by formula (6), we obtain

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(1,0)*}}=\kappa(\omega)-2\kappa^2(\omega)\sum_{k=1}^{N}\frac{e^{2h}-2e^h\cos(2\pi\omega h)+1}{e^{2h}-1}\cdot e^{2\pi i\omega hk}e^{-2\pi i\omega hk}.$$

Taking account that $e^{2\pi i\omega hk}e^{-2\pi i\omega hk}=1$ and $\sum\limits_{k=1}^{N}1=1/h$, we get expression (25) which is presented in Theorem 2, that is, Theorem 2 is proved.

Remark 1. It should be noted that from expression (25) we have the following

$$\left\|\overset{o}{\ell}\right\|^2=\frac{1}{12}h^2-\left(\frac{4\pi^2\omega^2+3}{360}\right)\cdot h^4+O(h^6),$$

i.e., the order of convergence of the optimal quadrature formula of the form (3) is $O(h)$.

From Theorem 1, it can be shown that in the work [6] for $\omega=0$, we obtain coefficients of the constructed optimal quadrature formula in the space $\widetilde{W_2}^{(1,0)}(0,1]$, that is:

Corollary 1. *Coefficients of the optimal quadrature formula of the form*

$$\int\limits_0^1\varphi(x)dx\cong\sum_{k=1}^{N}C_k\varphi(hk)$$

in the space $\widetilde{W_2}^{(1,0)}(0,1]$ *have the following form*

$$\mathring{C}_k=\frac{2(e^h-1)}{e^h+1},\ for\ k=1,2,\dots,N, \tag{26}$$

and for the norm of the error functional ℓ *of the optimal quadrature formula* (3) *the following holds*

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(1,0)}}=1-\frac{2(e^h-1)}{h(e^h+1)}.$$

AN APPROXIMATION FORMULA FOR THE FOURIER INTEGRALS ON THE INTERVAL $[a,b]$

In this section, we obtain an approximation formula for the numerical calculation of the integral (1) with real ω for functions of the space $W_2^{(1,0)}[a,b]$.

One of the expansions of this optimal quadrature formula for the case real ω is an approximation formula obtained by assuming the coefficients (6) and (26) as continuous functions with respect to $\omega\in\mathbb{R}$.

We consider construction of the optimal quadrature formula of the form

$$\int\limits_a^b e^{2\pi i\omega x}\varphi(x)dx\cong\sum_{k=1}^{N}C_{k,\omega}[a,b]\varphi(hk+a), \tag{27}$$

where $\omega\in\mathbb{R}$, $i^2=-1$, $C_{k,\omega}[a,b]$ are coefficients and $h=(b-a)/N$ with $N\in\mathbb{N}$.

Now, by the linear transformation $y=\frac{x-a}{b-a}$, we obtain the following

$$\int\limits_a^b e^{2\pi i\omega x}\varphi(x)dx=(b-a)e^{2\pi i\omega a}\int\limits_0^1 e^{2\pi i\omega(b-a)y}\varphi((b-a)y+a)dy. \tag{28}$$

Finally, applying Theorem 1 and Corollary 1 to the integral on the right-hand side of the last equality, we get the following main result of the present paper.

Theorem 3. *Coefficients of the approximation formula of the form* (27) *in the space* $W_2^{(1,0)}[a,b]$ *for* $\omega \in \mathbb{R}\setminus\{0\}$ *have the form*

$$\begin{aligned}
C_{0,\omega}[a,b] &= \left(1+e^{\frac{2h}{b-a}}+2\pi i\omega(b-a)(e^{\frac{2h}{b-a}}-1)-2e^{\frac{h}{b-a}}e^{2\pi i\omega h}\right)U_\omega e^{2\pi i\omega a},\\
C_{k,\omega}[a,b] &= 2\left(1+e^{\frac{2h}{b-a}}-2e^{\frac{h}{b-a}}\cos(2\pi\omega h)\right)U_\omega e^{2\pi i\omega(hk+a)},\ \ k=1,2,\ldots,N-1,\\
C_{N,\omega}[a,b] &= \left(1+e^{\frac{2h}{b-a}}-2\pi i\omega(b-a)(e^{\frac{2h}{b-a}}-1)-2e^{\frac{h}{b-a}}e^{2\pi i\omega h}\right)U_\omega e^{2\pi i\omega b},
\end{aligned} \tag{29}$$

where $h=\frac{b-a}{N}$ *and*

$$U_\omega=(b-a)\cdot\left[\left(4\pi^2\omega^2(b-a)^2+1\right)\left(e^{\frac{2h}{b-a}}-1\right)\right]^{-1}.$$

Remark 2. *It should be noted that for* $\omega=0$ *from approximation formula* (27) *with coefficients* (29) *we obtain the following*

$$\begin{aligned}
C_{0,0}[a,b] &= (b-a)\cdot\frac{e^{\frac{h}{b-a}}-1}{e^{\frac{h}{b-a}}+1},\\
C_{k,0}[a,b] &= 2(b-a)\cdot\frac{e^{\frac{h}{b-a}}-1}{e^{\frac{h}{b-a}}+1},\ \ k=1,2,\ldots,N-1,\\
C_{N,0}[a,b] &= (b-a)\cdot\frac{e^{\frac{h}{b-a}}-1}{e^{\frac{h}{b-a}}+1}.
\end{aligned}$$

CONCLUSION

In this paper, for the approximate calculation of the integral (1), in the space $\widetilde{W_2}^{(1,0)}(0,1]$, we studied the problem of construction the optimal quadrature formula in the sense of Sard. To do this, firstly, using the Fourier transform, the Dirac delta function and their properties, we found the extremal function of the error functional (5), and using it, we calculated the norm of the error functional ℓ for the quadrature formula (3). Next, we obtained the system of linear equations, its solution that gives a minimum of this norm. We found the optimal coefficients which are presented in Theorem 1 by solving the system, that is, we constructed the optimal quadrature formula (3), and calculated the norm of the optimal quadrature formula (3). Finally, using the obtained optimal quadrature formula, we obtained the approximation formula for numerical calculation of Fourier integrals in the space $\widetilde{W_2}^{(1,0)}(a,b]$.

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