AIP Conference Proceedings

RESEARCH ARTICLE | MARCH 11 2024

An optimal quadrature formula in the space $\widetilde{W}_2^{(2,1)}$ of periodic complex-valued functions

Umedjon Khayriev ✉; Mohinur Usmonova; Gavhar Turdieva

AIP Conf. Proc. 3004, 060037 (2024)
https://doi.org/10.1063/5.0199582

An Optimal Quadrature Formula in the Space $\widetilde{W_2}^{(2,1)}$ of Periodic Complex-Valued Functions

Umedjon Khayriev,[1, 2, a)] Mohinur Usmonova,[3, b)] and Gavhar Turdieva[2]

[1)] *V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, University str., Tashkent 100174, Uzbekistan,*
[2)] *Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan,*
[3)] *Bukhara State Pedagogical Institute, 2, Piridastgir str., Bukhara 200114, Uzbekistan.*

[a)] *Corresponding author:khayrievu@gmail.com*
[b)] *mohinurusmonova872@gmail.com*

Abstract. In the present paper, in the Hilbert space $\widetilde{W}_2^{(2,1)}(0,1]$ of periodic complex-valued functions, an optimal quadrature formula with exponential weight function is constructed. The optimality of the formula of rectangles in the space $\widetilde{W}_2^{(2,1)}(0,1]$ of periodic functions for $\omega = 0$ is shown. In addition, the numerical results show that the order of convergence of the optimal quadrature formula constructed in the space $\widetilde{W}_2^{(2,1)}(0,1]$ is equal to 2.

INTRODUCTION

Fourier transforms are widely used in science and technology, in particular, in problems of computed tomography (CT). It is known that when complete continuous X-ray data are available, CT images can be reconstructed exactly using various analytical formulas, such as the filtered back projection formula [1, 2, 3]. In this case, the Radon transform, the Fourier transform and the back projection formula are used. Since in practice we have finite discrete values of the Radon transform, we have to approximately calculate the Fourier transform in the CT filtered back projection method. This means that we need to approximately calculate the integrals of the form

$$I[\varphi,\omega] = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx. \tag{1}$$

with $\omega \in \mathbb{R}$. For large values of the parameter ω, such integrals are called *strongly oscillating*. To calculate such integrals, it is necessary to develop special efficient numerical methods. For the first time, such a method for calculating the integral was proposed by Filon [4]. Further, for various types of integrals with strongly oscillating integrands, many special efficient methods have been developed [5, 6, 7, 8, 9].

Note that in [10, 11] optimal quadrature formulas in the sense of Sard were constructed for approximate calculation of the integral (1) for integers ω in the spaces $L_2^{(m)}$ and $W_2^{(2,1)}$. Recently, in works [12, 13, 14], the optimal quadrature formulas for numerical calculation of the integral (1) with real ω were constructed in the Sobolev space $L_2^{(m)}[a,b]$. The resulting optimal quadrature formulas were applied to reconstruct CT images.

It should be noted that the optimal quadrature formulas for approximate integration of the Fourier coefficients (1) with an integer ω in the space $\widetilde{L_2^{(m)}}$ of periodic functions were obtained by Kh.M. Shadimetov [15]. Later, in the work [16] the optimal quadrature formulas for any m with $\omega = 0$ were constructed in the space $\widetilde{W_2}^{(m,m-1)}$. Recently, in [12, 13, 17, 18], the approximation formulas were obtained for numerical calculation of the integral (1) with real ω using the optimal quadrature formulas with integers ω constructed in the space $\widetilde{L_2^{(m)}}(0,1]$ of complex-valued periodic functions in the works [19], then these approximation formulas were applied for numerical reconstruction of CT images.

On the other hand, the optimal quadrature formulas for approximate integration of the Fourier coefficients (1) with an integer ω in the space $\widetilde{W}_2^{(1,0)}(0,1]$ of periodic functions were constructed in [20, 21].

The purpose of this work is to continue and develop the results of [22] for $m = 2$. Here we consider the problem of constructing an optimal quadrature formula in the sense of Sard for numerical integration of the integral (1) with integer ω. The resulting optimal quadrature formula is used for approximate calculation of the Fourier transform in specific functions.

International Scientific and Practical Conference on "Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)"
AIP Conf. Proc. 3004, 060037-1–060037-8; https://doi.org/10.1063/5.0199582
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

The rest of the paper is organized as follows. In the second section we discuss the problem of construction of optimal quadrature formula. The third section is devoted to obtaining the upper bound for the error of quadrature formula. In the fourth section we get an optimal quadrature formula for approximate calculation of integrals of the form (1) in the Hilbert space $\widetilde{W_2}^{(2,1)}(0,1]$ of complex-valued functions. In the last section we calculate the norm for the error functional of the optimal quadrature formula.

STATEMENT OF THE PROBLEM

We consider a linear space $W_2^{(2,1)}[0,1]$ of complex-valued functions φ that 1^{st} order derivative is absolute continuous and 2^{nd} order generalized derivative is square-integrable on $[0,1]$. The space $W_2^{(2,1)}$ is the Hilbert space [30] with the inner product

$$\langle \varphi, \psi \rangle_{W_2{}^{(2,1)}} = \int_0^1 (\varphi''(x) + \varphi'(x))(\overline{\psi}''(x) + \overline{\psi}'(x))dx,$$

and the corresponding norm

$$\|\varphi\|_{W_2^{(2,1)}} = \left\{ \langle \varphi, \varphi \rangle_{W_2^{(2,1)}} \right\}^{1/2}.$$

We mainly consider the corresponding Hilbert space $\widetilde{W}_2^{(2,1)}(0,1]$ of periodic complex-valued functions, a subspace of the space $W_2^{(2,1)}[0,1]$. We note that every element in this space is 1-periodic, i.e.

$$\varphi(x+\beta) = \varphi(x), \quad \text{for } x \in \mathbb{R}, \ \beta \in \mathbb{Z},$$

and is a class of functions differing from each other by a constant term.

We consider a quadrature formula of the form

$$\int_0^1 e^{2\pi i \omega x} \varphi(x) dx \cong \sum_{k=1}^{N} C_k \varphi(hk), \tag{2}$$

where $\omega \in \mathbb{Z} \setminus \{0\}$ is a parameter, φ belongs to the space $\widetilde{W}_2^{(2,1)}(0,1]$, and C_k $(k = 1,2,...,N)$ are coefficients of the quadrature formula, $i^2 = -1$, $h = 1/N$, $N \in \mathbb{N}$.

The difference between the integral and the quadrature sum is called *the error* of the quadrature formula (2). This difference defines a linear functional

$$\ell(x) = e^{2\pi i \omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta), \tag{3}$$

where $\delta(x)$ is the Dirac delta-function. $\ell(x)$ is called *the error functional* of the quadrature formula (2).

The value of the functional ℓ on a function φ gives the error of formula (2), i.e.

$$(\ell, \varphi) = \int_0^1 e^{2\pi i \omega x} \varphi(x) dx - \sum_{k=1}^{N} C_k \varphi(hk) = \int_0^1 \ell(x) \varphi(x) dx. \tag{4}$$

Since the error functional ℓ is defined on the space $\widetilde{W}_2^{(2,1)}(0,1]$, the following equality is valid as in the work [23]

$$(\ell, 1) = 0. \tag{5}$$

We obtain the following upper bound for the absolute value of the error (4)

$$|(\ell, \varphi)| \le \|\ell\|_{\widetilde{W}_2^{(2,1)*}} \|\varphi\|_{\widetilde{W}_2^{(2,1)}},$$

13 March 2024 08:31:10

where

$$\|\ell\|_{\widetilde{W}_2^{(2,1)*}} = \sup_{\varphi, \|\varphi\| \neq 0} \frac{\|(\ell,\varphi)\|)}{\|\varphi\|_{\widetilde{W}_2^{(2,1)}}}.$$

It is clear that the coefficients C_k are variable parameters of the quadrature formula (2). The quadrature formula of the form (2) which has an error functional with minimum norm in terms of the coefficients C_k for given nodes at a fixed value N is called *the optimal quadrature formula* in the space $\widetilde{W}_2^{(2,1)}(0,1]$. The coefficients C_k which give the minimum to the norm of ℓ is said to be *the optimal coefficients.* They are denoted by $\overset{\circ}{C}_k$.

Problem 1. For construction of optimal quadrature formula of the form (2) we should solve the following problems:

(a) *Calculate the norm of the error functional* (3) *for the quadrature formula* (2)*;*

(b) *Find the optimal coefficients* $\overset{\circ}{C}_k$ *which give the minimum for the norm of the error functional* (3)*;*

(c) *Calculate the following norm of the error functional* (3) *for the optimal quadrature formula* (2)

$$\left\|\overset{\circ}{\ell}\right\|_{\widetilde{W_2}^{(2,1)*}} := \inf_{C_k} \|\ell\|_{\widetilde{W_2}^{(2,1)*}}. \tag{6}$$

Further, in the present work we solve Problem 1 (a),(b) and (c).

THE NORM OF THE ERROR FUNCTIONAL FOR THE QUADRATURE FORMULA

In this section, we solve Problem 1 (a). To do this, first, we find an extremal function ψ_ℓ for the error functional (3) that satisfies the following equality (see [23, 24])

$$(\ell, \psi_\ell) = \|\ell\|_{\widetilde{W}_2^{(2,1)*}} \|\psi_\ell\|_{\widetilde{W}_2^{(2,1)}}.$$

Since the space $\widetilde{W}_2^{(2,1)}(0,1]$ is a Hilbert space, using Riesz's theorem on the general form of a linear continuous functional, we obtain the following

$$(\ell, \varphi) = \langle \varphi, \psi_\ell \rangle_{\widetilde{W}_2^{(2,1)}}, \tag{7}$$

then we get

$$(\ell, \psi_\ell) = \|\ell\|^2_{\widetilde{W}_2^{(2,1)*}}. \tag{8}$$

Integrating by parts the right-hand side of equation (7), keeping in mind periodicity of functions φ and ψ_ℓ, we have

$$\overline{\psi_\ell}^{(4)}(x) - \overline{\psi_\ell}^{(2)}(x) = \ell(x). \tag{9}$$

The following is true.

Lemma 1. *The solution of equation* (9) *is the extremal function* ψ_ℓ *of the error functional* (3) *and it is expressed as*

$$\psi_\ell(x) = \frac{e^{-2\pi i \omega x}}{(2\pi\omega)^4 + (2\pi\omega)^2} - \sum_{k=1}^{N} \overline{C}_k \sum_{\beta \neq 0} \frac{e^{2\pi i \beta (x - hk)}}{(2\pi\beta)^4 + (2\pi\beta)^2} + d_0, \tag{10}$$

where d_0 *is a constant term, and* $\overline{C}_k$ *are the complex conjugate for* C_k.

The proof of this lemma is similar to the proof of Lemma 1 of the work [13].

Theorem 1. *The norm for the error functional ℓ of the quadrature formula* (2) *with $\omega \in \mathbb{Z}\setminus\{0\}$ has the following form*

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4+(2\pi\omega)^2} - \frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}\cdot\left[\sum_{k'=1}^{N}\overline{C}_{k'}e^{2\pi i\omega hk'} + \sum_{k=1}^{N}C_k e^{-2\pi i\omega hk}\right]$$
$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}C_k\overline{C}_{k'}\sum_{\beta\neq 0}\frac{e^{-2\pi i\beta h(k-k')}}{(2\pi\beta)^4+(2\pi\beta)^2}. \tag{11}$$

Proof. To proof Theorem 1, we use equality (8)

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} = (\ell,\psi_\ell) = \int_0^1 \ell(x)\psi_\ell(x)dx,$$

and considering (3) and (10), we rewrite the last expression as follows

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} = \int_0^1\left(e^{2\pi i\omega x} - \sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\right)\cdot\left(\frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4+(2\pi\omega)^2} - \sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\gamma\neq 0}\frac{e^{2\pi i\gamma(x-hk')}}{(2\pi\gamma)^4+(2\pi\gamma)^2}+d_0\right)dx$$

Taking into account (5), simplifying the above expression, we have

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4+(2\pi\omega)^2} - \sum_{k'=1}^{N}\overline{C}_{k'}\sum_{\beta\neq 0}\frac{e^{-2\pi i\beta hk'}}{(2\pi\beta)^4+(2\pi\beta)^2}\int_0^1 e^{2\pi i(\omega+\beta)x}dx$$
$$-\frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(hk+\beta)e^{-2\pi i\omega(hk+\beta)}$$
$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}C_k\overline{C}_{k'}\sum_{\beta=-\infty}^{\infty}\sum_{\gamma\neq 0}\varepsilon_{(0,1]}(hk+\beta)\frac{e^{2\pi i\gamma(hk-hk'+\beta)}}{(2\pi\gamma)^4+(2\pi\gamma)^2}, \tag{12}$$

where

$$\varepsilon_{(0,1]}(x) = \begin{cases}0, & \text{if } x\notin(0,1],\\ 1, & \text{if } x\in(0,1].\end{cases}$$

Considering the following equalities

$$\int_0^1 e^{2\pi i(\omega+\beta)x}dx = \begin{cases}0, & \text{if } \omega+\beta\neq 0,\\ 1, & \text{if } \omega+\beta=0,\end{cases}$$
$$\sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(hk+\beta) = \sum_{\beta=-\infty}^{\infty}\varepsilon_{(0,1]}(hk) = \varepsilon_{(0,1]}(hk) = 1,$$

we make (11) from (12). Thus, Theorem 1 is proved.

It can be easily showed that $\|\ell\|^2$ is a real non-negative quantity [17,18,19].
Thus, Problem 1 (a) is solved.
Using (11), we find the optimal coefficients of the quadrature formula (2) in the next section.

OPTIMAL COEFFICIENTS OF THE QUADRATURE FORMULA

Now, we find the optimal coefficients of the quadrature formula (2), that is we solve Problem 1 (b). For this, we apply the method of Lagrange unknown multipliers, i.e., we consider the following function

$$L(\overline{C}_{k'},\mu) = \|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} + \mu\cdot(\ell,1) \text{ for } k=1,2,\dots,N,$$

where μ is a constant.

Equating all partial derivatives of the function $L(\overline{C}_{k'},\mu)$ by $\overline{C}_{k'}$ and μ, we have

$$\frac{\partial L}{\partial \overline{C}_{k'}} = -e^{2\pi i\omega hk'}\kappa(\omega)+\sum_{k=1}^{N} C_k \sum_{\beta\neq 0} e^{2\pi i\beta h(k-k')}\kappa(\beta)=0, \quad \text{for } k=1,2,\ldots,N, \tag{13}$$

$$\frac{\partial L}{\partial \mu} = \int_0^1 e^{2\pi i\omega x}dx-\sum_{k=1}^{N} C_k=0, \tag{14}$$

where

$$\kappa(w)=\frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}. \tag{15}$$

It can be proved that the system of equations (13)-(14) has a unique solution, as in the work [25]. A solution of this system which we denote by $\{\overset{\circ}{C}_k,\ k=1,2,\ldots,N\}$ is the minimum point for the function $L(\overline{C}_{k'},\mu)$. The aim of the present work is to find an analytic solution for the system (13)-(14).

The following holds.

Theorem 2. *Among the quadrature formulas of the form* (2) *with* $\omega\in\mathbb{Z}\setminus\{0\}$ *and* $\omega h\notin\mathbb{Z}$ *in the space* $\widetilde{W}_2^{(2,1)}(0,1]$ *of 1-periodic, complex-valued functions, there is a unique quadrature formula with coefficients having the representation*

$$\overset{\circ}{C}_k=\frac{2K_\omega}{(2\pi\omega)^4+(2\pi\omega)^2}\cdot e^{2\pi i\omega hk} \ \ for\ \ k=1,2,\ldots,N,\ and\ m\geq 2, \tag{16}$$

where

$$K_\omega=-\left[\frac{e^{2h}-1}{e^{2h}+1-2e^h\cos(2\pi\omega h)}+\frac{h}{\cos(2\pi\omega h)-1}\right]^{-1}.$$

Proof. We consider the case $\omega\in\mathbb{Z}$ and $\omega h\notin\mathbb{Z}$. We search a solution of the system (13)-(14) in the following form

$$\overset{\circ}{C}_k=C(\omega,h)\cdot e^{2\pi i\omega hk} \ \ \text{for}\ \ k=1,2,\ldots,N \text{ and } \mu=0, \tag{17}$$

where $C(\omega,h)$ is an unknown function.

Using designation (17), we rewrite the system (13) as follows

$$-e^{2\pi i\omega hk}\kappa(\omega)+C(\omega,h)\sum_{\beta\neq 0}e^{2\pi i\beta hk}\kappa(\beta)\sum_{k'=1}^{N}e^{2\pi i(\beta+\omega)hk'}=0, \quad k=1,2,\ldots,N,$$

where

$$\sum_{k'=1}^{N}e^{2\pi i(\beta+\omega)hk'}=\begin{cases}0, & \text{if } (\beta+\omega)h\neq t\in \mathrm{Z},\\ N, & \text{if } (\beta+\omega)h=t\in \mathrm{Z}.\end{cases}$$

Simplifying the left-hand side of the last equation, we make the following equation

$$-e^{2\pi i\omega hk}\kappa(\omega)+C(\omega,h)\cdot N\sum_{t=-\infty}^{\infty}e^{2\pi i\omega hk}\kappa(t\cdot N-\omega)=0,$$

from this equation, we get

$$C(\omega,h)=h\cdot\kappa(\omega)\left(\sum_{t=-\infty}^{\infty}\kappa(t\cdot N-\omega)\right)^{-1}, \tag{18}$$

where $\kappa(\omega)$ is defined by (15).

To find the unknown function $C(\omega, h)$, we must calculate the series in equation (18). For this, we rewrite this series as follows

$$\sum_{t=-\infty}^{\infty} \kappa(t \cdot N - \omega) = \left(\frac{h}{2\pi}\right)^4 \cdot \sum_{t=-\infty}^{\infty} \frac{1}{(t-\omega h)^2 \left((t-\omega h)^2 + \left(\frac{h}{2\pi}\right)^2\right)}. \tag{19}$$

Calculating the right-hand side of expression (19) as in [[25], p. 66], we have the following result

$$\sum_{t=-\infty}^{\infty} \kappa(t \cdot N - \omega) = -\frac{h}{2} \cdot \frac{e^{2h}-1}{e^{2h}+1-2e^h \cos(2\pi\omega h)} - \frac{h^2 \cdot \lambda E_0(\lambda)}{(1-\lambda)^2},$$

where $E_0(\lambda)$ is the Euler-Frobenius polynomials (see [26]).
Using well known formula $\frac{\lambda}{(1-\lambda)^2} = \frac{1}{2\cos(2\pi\omega h)-2}$ and $E_0(x) = 1$, we have

$$\sum_{t=-\infty}^{\infty} \kappa(t \cdot N - \omega) = -\frac{h}{2} \cdot \frac{e^{2h}-1}{e^{2h}+1-2e^h \cos(2\pi\omega h)} - \frac{h^2}{2\cos(2\pi\omega h)-2}. \tag{20}$$

Using equalities (17), (18) and (21), we get the result of Theorem 2. Theorem 2 is proved.

By proving Theorem 2, we have solved Problem 1 (b). In the next section, we solve Problem 1 (c).

CALCULATION THE NORM OF THE ERROR FUNCTIONAL FOR THE OPTIMAL QUADRATURE FORMULA

The following theorem is valid for the error functional norm of the optimal quadrature formula (2).

Theorem 3. *The norm of the error functional* (3) *for the optimal quadrature formula* (2) *has the form*

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} \cdot \left[1 - \frac{2N}{(2\pi\omega)^4 + (2\pi\omega)^2} \cdot K_\omega\right], \tag{21}$$

where $\omega \in \mathbb{Z} \setminus \{0\}$ *and* $\omega h \notin \mathbb{Z}$, *and*

$$K_\omega = -\left[\frac{e^{2h}-1}{e^{2h}+1-2e^h \cos(2\pi\omega h)} + \frac{h}{\cos(2\pi\omega h)-1}\right]^{-1}.$$

Proof. We use equalities (6) and (11) and then we have

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} - \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} \sum_{k'=1}^{N} \overline{\mathring{C}}_{k'} e^{2\pi i\omega hk'} - \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} \sum_{k=1}^{N} \mathring{C}_k e^{-2\pi i\omega hk}$$
$$+ \sum_{k=1}^{N} \sum_{k'=1}^{N} C_k \overline{\mathring{C}}_{k'} \sum_{\beta \neq 0} \frac{e^{-2\pi i\beta h(k-k')}}{(2\pi\beta)^4 + (2\pi\beta)^2} = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} - \sum_{k'=1}^{N} \overline{\mathring{C}}_{k'}$$
$$\times \left[\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4 + (2\pi\omega)^2} - \sum_{k=1}^{N} \mathring{C}_k \sum_{\beta \neq 0} \frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4 + (2\pi\beta)^2}\right] - \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} \sum_{k=1}^{N} \mathring{C}_k e^{-2\pi i\omega hk}.$$

Considering (13), we have

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W}_2^{(2,1)*}} = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} - \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2} \sum_{k=1}^{N} \mathring{C}_k e^{-2\pi i\omega hk}$$

Using $\mathring{C}_k$ is defined by (16), we rewrite the last expression as follows

$$\left\| \mathring{\ell} \right\|^2_{\widetilde{W_2}^{(2,1)*}} = \frac{1}{(2\pi\omega)^4+(2\pi\omega)^2}\left[1-\frac{2K_\omega}{(2\pi\omega)^4+(2\pi\omega)^2}\sum_{k=1}^{N}1\right],$$

Using $\sum_{k=1}^{N}1=N$, we have proved Theorem 3 completely.

Corollary 1. *We expand the right-hand side of equality* (21) *to the Taylor series, the norm for the error functional of the optimal quadrature formula* (2) *has the following form*

$$\left\| \mathring{\ell} \right\|^2_{\widetilde{W_2}^{(2,1)*}} = \frac{1}{720}h^4+\frac{640\pi^6\omega^6+304\pi^4\omega^4+32\pi^2\omega^2-1}{30240(4\pi^2\omega^2+1)^2}h^6+O(h^8).$$

CONCLUSION

In the present paper, we have constructed an optimal quadrature formula of the form (2) in the Hilbert space $\widetilde{W}_2^{(2,1)}(0,1]$ of periodic, complex-valued functions. We have found out that in the space $\widetilde{W_2}^{(2,1)}(0,1]$ the absolute value of the error of the optimal quadrature formula of the form (2) is smaller than the absolute value of the error of the optimal quadrature formula of the form (2), constructed in the space $\widetilde{L_2^{(2)}}(0,1]$.

13 March 2024 08:31:10

REFERENCES

1. T. M. Buzug, *Computed Tomography from Photon Statistics to Modern Cone-Beam CT* (Springer, Berlin, 2008).
2. T. G. Feeman, *The Mathematics of Medical Imaging, A Beginner's Guide, second ed.* (Springer, 2015).
3. A. C. Kak , M. Slaney, *Principles of Computerized Tomographic Imaging* (IEEE Press, 1988).
4. L. N. G. Filon, "On a quadrature formula for trigonometric integrals." Proc. Roy. Soc. Edinburgh. **49**, 38–47 (1928).
5. I. Babuška, "Optimal quadrature formulas," Dokl. Akad. Science, USSR. **149** (1963).
6. A. Iserles, S. P. Nörsett, "Efficient quadrature of highly oscillatory integrals using derivatives," Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. sci. **461**, 1383–1399 (2005).
7. G. V. Milovanović, "Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures," Comput. Math. Appl. **36**, 19–39 (1988).
8. E. Novak, M. Ullrich, H. Wozniakowski, "Complexity of oscillatory integration for univariate Sobolev space," J. Complexity **31**, 15–41 (2015).
9. S. Olver, *(PhD dissertation)* (University of Cambridge, 2008).
10. N. D. Boltaev, A. R. Hayotov, G. V. Milovanović, Kh. M. Shadimetov, "Optimal quadrature formulas for Fourier coefficients in $W_2^{(m,m-1)}$ space." Journal of applied analysis and computation. **7**, 1233–1266 (2017).
11. A. R. Hayotov, S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in $W_2^{(m,m-1)}$ space." AIP Conference Proceedings. **2365** (2021), 020021, https://doi.org/10.1063/5.0057127.
12. A. R. Hayotov, S. Jeon, C-O. Lee, Kh. M. Shadimetov, "Optimal quadrature formulas for non-periodic functions in Sobolev space and its application to CT image reconstruction." Filomat **35**, 4177–4195 (2021).
13. A. R. Hayotov, S. Jeon, Kh. M. Shadimetov, "Application of optimal quadrature formulas for reconstruction of CT images." Journal of Computational and Applied Mathematics. **388**, 113313 (2021).
14. A. R. Hayotov, S. Jeon, C-O. Lee, "On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$," Journal of Computational and Applied Mathematics **372**, 112713 (2020).
15. M. Kh. Shadimetov, *Optimal lattice quadrature and cubature formulas in Sobolev spaces. Monograph* (FAN, Tashkent, [in Russian], 2019) pp. 97–104.
16. A. R. Hayotov and U. N. Khayriev, "Optimal quadrature formulas in the space $\widetilde{W_2}^{(m,m-1)}$ of periodic functions." Vestnik KRAUNC. **3(40)**, 211–226 (2022), 10.26117/2079-6641-2022-40-3-211-226.
17. A. Hayotov and U. Khayriev and F. Azatov, "Exponentially Weighted Optimal Quadrature Formula with Derivative in the Space $L_2^{(2)}$," AIP Conference Proceedings **2781**, 020050 (2023), https://doi.org/10.1063/5.0144753.
18. Kh. M. Shadimetov and A. R. Hayotov, *Optimal approximation of error functionals of quadrature and interpolation formulas in spaces of differentiable functions.* (Muhr press, Tashkent, 2022) p. 246, [in Russian].
19. Kh. M. Shadimetov, "Weight optimal cubature formulas in Sobolev's periodic space," Siberian J. Numer. Math. Novosibirsk **2**, 185–196 (1999).
20. A. R. Hayotov and U. N. Khayriev, "Construction of an Optimal Quadrature Formula in the Hilbert Space of Periodic Functions." Lobachevskii Journal of Mathematics. **11(43)**, 3151–3160 (2022).

21. S. S. Babaev, A. R. Hayotov, and U. N. Khayriev, "On an optimal quadrature formula for approximation of Fourier integrals in the space $W_2^{(1,0)}$," Uzbek Mathematical Journal **2**, 23–36 (2020).
22. U. N. Khayriev, "Construction of the Exponentially Weighted Optimal Quadrature Formula In a Hilbert Space of Periodic Functions." Problems of Computational and Applied Mathematics. – Tashkent, **5/1(44)**, 134–142 (2022).
23. S. L. Sobolev and V. L. Vaskevich, *Theory of Cubature Formulas.* (Kluwer Academic Publishers Group., Dordrecht, 1997).
24. G. V. Demidenko and V. L. Vaskevich, *Selected Works of S.L. Sobolev* (Springer, New York, 2006).
25. Kh. M. Shadimetov, A. R. Hayotov, *Optimal approximation of error functionals of quadrature and interpolation formulas in spaces of differentiable functions* (Muhr press, Tashkent, 2022).
26. Kh. M. Shadimetov, A. R. Hayotov and B. Abdikayimov, "On an optimal quadrature formula in a Hilbert space of periodic functions," Algorithms **15(10)**, 344 (2022), doi.org/10.3390/a15100344.
27. K. M. Shadimetov, A. R. Hayotov, and B. I. Bozarov, "Optimal quadrature formulas for oscillatory integrals in the sobolev space," Journal of inequalities and applications. Springer. **103**, 1–21 (2022).