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Exponentially Weighted Optimal Quadrature Formula with Derivative in the Space $L_2^{(2)}$

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Abstract. The present paper is devoted to the construction of an optimal quadrature formula with derivative for approximation of Fourier integrals in the Sobolev space $L_2^{(2)}$ of complex-valued functions. Using the constructed optimal quadrature formula, the Fourier transform of some functions is approximated.

INTRODUCTION

The problem of constructing optimal quadrature formulas for the numerical calculation of the Fourier coefficients based on the Sobolev method in [1]

$$I(\varphi,\omega)=\int_0^1 e^{2\pi i\omega x}\varphi(x)dx \tag{1}$$

for at $\omega\in\mathbb{Z}$ was studied in the Hilbert spaces $L_2^{(m)}(0,1)$ and $W_2^{(m,m-1)}(0,1)$. In these studies, analytic formulas were obtained for the optimal coefficients for $m\geq 1$. In particular, the order of approximation of the optimal quadrature formulas for the case $m=1$ was studied.
In [2] and [3] the case problem of construction of optimal quadrature formulas in the sense of Sard for approximate calculation of Fourier integrals of the form (1) with $\omega\in\mathbb{R}$ were studied in the space $L_2^{(1)}(0,1)$ and $W_2^{(1,0)}(0,1)$.
There are several methods to construct optimal quadrature formulas in the sense of Sard, such as the spline method, the ϕ-function method (see [4], [5]) and the Sobolev method. The Sard problem has been studied by many authors in various spaces based on these methods, for example, [4, 5, 6, 7, 8, 9, 10].
Among these formulas, the Euler-Maclaurin type quadrature formulas are very important for the numerical integration of differentiable functions and are widely used in applications. The optimality of quadrature and cubature formulas of the Euler-Maclaurin type in the different spaces was studied, for example, in [5, 11, 12, 13].
In the present paper we study the problem of construction of the Euler-Maclaurin type optimal quadrature formula in the sense of Sard for approximate calculation of Fourier integrals of the form (1) with $\omega\in\mathbb{R}\setminus\{0\}$ in the space $L_2^{(2)}(0,1)$.

CONSTRUCTION OF THE OPTIMAL QUADRATURE FORMULA

The Euler-Maclaurin quadrature formulas can be viewed as well as an extension of the trapezoidal rule at $\omega=0$ by the inclusion of correction terms. It should be noted that in applications and in solution of practical problems numerical integration formulas are interesting for functions with small smoothness.

We consider a quadrature formula of the form

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)dx\cong\sum_{\beta=0}^{N}\left(C_0[\beta]\varphi[\beta]+C_1[\beta]\varphi'[\beta]\right) \tag{2}$$

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where $C_0[\beta]$ are the coefficients given in [2]

$$\begin{aligned} C_0[0] &= h \cdot \frac{1+2\pi i\omega h - e^{2\pi i\omega h}}{(2\pi\omega h)^2}, \\ C_0[\beta] &= h \cdot \frac{2(1-\cos(2\pi\omega h))}{(2\pi\omega h)^2} \cdot e^{2\pi i\omega h\beta}, \qquad \beta = 1,2,...,N-1, \\ C_0[N] &= h \cdot \frac{1-2\pi i\omega h - e^{-2\pi i\omega h}}{(2\pi\omega h)^2} \cdot e^{2\pi i\omega}, \end{aligned} \tag{3}$$

and $i^2=-1$, $\omega \in \mathbb{R}\setminus\{0\}$, $h=\frac{1}{N}$, $N \in \mathbb{N}$, $[\beta]=h\beta$ and $C_1[\beta]$ are unknown coefficients of the formula (2), and they should be found.

We suppose that a function φ belongs to $L_2^{(2)}(0,1)$ and

$$L_2^{(2)} = \{\varphi : [0,1] \to \mathbb{C} \mid \varphi' \text{ is absolute continuous and } \varphi'' \in L_2(0,1)\}.$$

The inner product for the functions φ and ψ in this space is defined as $\langle \varphi,\psi\rangle_{L_2^{(2)}} = \int_0^1 \varphi''(x)\cdot\bar{\psi}''(x)dx$, where $\bar{\psi}$ is the complex conjugate function to the function ψ and the corresponding semi-norm of the function φ is defined by the formula $\|\varphi\|_{L_2^{(2)}} = \langle\varphi,\varphi\rangle^{1/2}$ and $\int\limits_0^1 \varphi''(x)\cdot\bar{\varphi}''(x)dx < \infty$ (see [10]).

The difference

$$(\ell,\varphi) = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx - \sum_{\beta=0}^{N}\left(C_0[\beta]\varphi[\beta] + C_1[\beta]\varphi'[\beta]\right) = \int\limits_{-\infty}^{\infty} \ell(x)\varphi(x)dx, \tag{4}$$

is called *the error* and the corresponding error functional has the form

$$\ell(x) = e^{2\pi i\omega x}\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N}\left(C_0[\beta]\delta(x-h\beta) - C_1[\beta]\delta'(x-h\beta)\right), \tag{5}$$

here $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$, δ is the Dirac delta-function and $\varphi \in L_2^{(2)}$.
For the error functional (5) to be defined on the space $L_2^{(2)}(0,1)$ it should be imposed the following conditions

$$(\ell,1) := \int_0^1 e^{2\pi i\omega x}dx - \sum_{\beta=0}^{N} C_0[\beta] = 0, \tag{6}$$

$$(\ell,x) := \int_0^1 e^{2\pi i\omega x}x dx - \sum_{\beta=0}^{N}\left(C_0[\beta]\cdot h\beta + C_1[\beta]\right) = 0. \tag{7}$$

The last two equalities mean that the quadrature formula (2) is exact for any linear function $ax+b$. The coefficients $C_0[\beta], \beta=0,1,...,N$ determined by equality (3) we have chosen such that they satisfy the condition (6). Therefore, we have only condition (7) for coefficients $C_1[\beta], \beta=0,1,...,N$.
The error functional ℓ of the formula (2) is a linear continuous functional in $L_2^{(2)*}(0,1)$, where $L_2^{(2)*}(0,1)$ is the conjugate space to the space $L_2^{(2)}(0,1)$.
According to the Cauchy-Schwarz inequality, we have

$$|(\ell,\varphi)| \le \|\ell\|_{L_2^{(2)*}(0,1)} \cdot \|\varphi\|_{L_2^{(2)}(0,1)}.$$

Therefore, the error (4) of the formula (2) is estimated by the norm $\|\ell\|_{L_2^{(2)*}(0,1)} = \sup_{\|\varphi\|_{L_2^{(2)}(0,1)}=1} |(\ell,\varphi)|$.

The main aim of this work is to find the minimum of the norm for the error functional ℓ by coefficients $C_1[\beta]$ for given $C_0[\beta]$ in the space $L_2^{(2)}$. That is the problem is to find the coefficients $C_1[\beta]$ that satisfy the following equality

$$\|\overset{\circ}{\ell}\|_{L_2^{(2)*}} = \inf_{C_1[\beta]} \|\ell\|_{L_2^{(2)*}}. \tag{8}$$

The coefficients $C_1[\beta]$ which satisfy the last equality are called *optimal coefficients* and are denoted by $\overset{\circ}{C}_1[\beta]$.
Thus, to obtain the optimal quadrature formula of the form (2) in the sense of Sard in the space $L_2^{(2)}(0,1)$, we need to solve the following problems.
Problem 1. Find the norm of the error functional (5) of the quadrature formula (2) in the space $L_2^{(2)*}$
Problem 2. Find the coefficients $\overset{\circ}{C}_1[\beta]$ that satisfy equality (8).
Further, we solve Problem 1 for the case $\omega \in \mathbb{R}$ with $\omega \neq 0$ by finding the norm (8), and we solve Problem 2 minimizing the norm $\|\ell\|_{L_2^{(2)*}}$ by the coefficients $C_1[\beta]$.

THE NORM OF THE ERROR FUNCTIONAL (5)

In this section, we solve Problem 1. For finding the norm (8) we use the *extremal function* for the error functional ℓ (see, [9, 10]) which satisfies the following equality

$$(\ell, \psi_\ell) = \|\ell\|_{L_2^{(2)*}(0,1)} \cdot \|\psi_\ell\|_{L_2^{(2)}(0,1)}$$

In [10] for a linear functional ℓ defined on the Sobolev space $L_2^{(m)}$ the extremal function was found and it was shown that the extremal function ψ_ℓ is the solution to the boundary value problem

$$\psi_\ell^{(2m)}(x) = (-1)^m \bar{\ell}(x), \tag{9}$$

$$\psi_\ell^{(m+s)}(x)|_{x=0}^{x=1} = 0, \quad s = 0,1,...,m-1, \tag{10}$$

where $\bar{\ell}$ is the complex conjugate to ℓ.

In the case $m = 2$ from the result about extremal function of [14] the following holds
Theorem 1. *The solution of the boundary value problem* (9)–(10) *for $m = 2$ is the extremal function ψ_ℓ of the error functional ℓ and it has the following form*

$$\psi_\ell(x) = \bar{\ell}(x) * G_2(x) + d_0 \cdot x + d_1, \tag{11}$$

where d_0 and d_1 are complex constants, $*$ is the operation of convolution and

$$G_2(x) = \frac{|x|^3}{12}. \tag{12}$$

Theorem 1 for any $m \in \mathbb{Z}$ was proved by S.L.Sobolev (see, for instance [9, 10]). In particular, we get $\|\ell\|_{L_2^{(2)*}} = \|\psi_\ell\|_{L_2^{(2)}}$ and

$$(\ell, \psi_\ell) = \|\ell\|^2_{L_2^{(2)*}}. \tag{13}$$

Then keeping in mind (13) and using (6), (7) and (11), we obtain

$$\begin{aligned}
\|\ell\|^2_{L_2^{(2)*}} &= (\ell, \psi_\ell) = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)dx = \int_{-\infty}^{\infty} \ell(x) \cdot \left(\bar{\ell}(x) * G_2(x)\right) dx \\
&= \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} \left(C_0[\beta]\bar{C}_0[\gamma]G_2(h\beta - h\gamma) + \left(\bar{C}_1[\beta]C_0[\gamma] + C_1[\beta]\bar{C}_0[\gamma]\right) G_2'(h\beta - h\gamma) - C_1[\beta]\bar{C}_1[\gamma]G_2''(h\beta - h\gamma)\right) \\
&- \sum_{\beta=0}^{N} \int_0^1 \left(\bar{C}_0[\beta]e^{2\pi i\omega x} + C_0[\beta]e^{-2\pi i\omega x}\right) G_2(x - h\beta)dx \\
&+ \sum_{\beta=0}^{N} \int_0^1 \left(\bar{C}_1[\beta]e^{2\pi i\omega x} + C_1[\beta]e^{-2\pi i\omega x}\right) G_2'(x - h\beta)dx \\
&+ \int_0^1 \int_0^1 e^{2\pi i\omega(x-y)} G_2(x-y)dxdy,
\end{aligned} \tag{14}$$

where

$$G_2'(x) = \frac{\text{sgn}x \cdot |x|^2}{4} \text{ and } G_2''(x) = \frac{|x|}{2} = G_1(x). \tag{15}$$

It is easy to show that the right-hand side of the equality (14) is real. Indeed, let

$$C_0[\beta] = C_0^R[\beta] + iC_0^I[\beta], \text{ and } C_1[\beta] = C_1^R[\beta] + iC_1^I[\beta], \tag{16}$$

where $C_k^R[\beta]$ and $C_k^I[\beta]$ are real, $k = 0,1$. Using Euler's formula $e^{2\pi i\omega x} = \cos(2\pi\omega x) + i\sin(2\pi\omega x)$.
From (14) for square of the norm of ℓ, we obtain

$$\begin{aligned}
\|\ell\|^2_{L_2^{(2)*}} &= \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\left(C_0^R[\beta]C_0^R[\gamma] + C_0^I[\beta]C_0^I[\gamma]\right)G_2(h\beta - h\gamma) + 2\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\left(C_1^R[\beta]C_0^R[\gamma] + C_1^I[\beta]C_0^I[\gamma]\right)G_2'(h\beta - h\gamma) \\
&- \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\left(C_1^R[\beta]C_1^R[\gamma] + C_1^I[\beta]C_1^I[\gamma]\right)G_2''(h\beta - h\gamma) \\
&- 2\sum_{\beta=0}^{N}\int_0^1\left(C_0^R[\beta]\cos(2\pi\omega x) + C_0^I[\beta]\sin(2\pi\omega x)\right)G_2(x - h\beta)dx \\
&+ 2\sum_{\beta=0}^{N}\int_0^1\left(C_1^R[\beta]\cos 2\pi\omega x + C_1^I[\beta]\sin(2\pi\omega x)\right)G_2'(x - h\beta)dx \\
&+ \int_0^1\int_0^1\cos(2\pi\omega(x-y))G_2(x-y)dxdy.
\end{aligned} \tag{17}$$

And from (7), we obtain the following equalities

$$\sum_{\beta=0}^{N}\left(C_0^R[\beta]\cdot h\beta + C_1^R[\beta]\right) = \int_0^1 x\cdot\cos(2\pi\omega x)dx, \tag{18}$$

$$\sum_{\beta=0}^{N}\left(C_0^I[\beta]\cdot h\beta + C_1^I[\beta]\right) = \int_0^1 x\cdot\sin(2\pi\omega x)dx. \tag{19}$$

Thus, Problem 1 is solved.
In the next section we solve Problem 2.

FINDING THE MINIMUM OF THE EXPRESSION (17) BY COEFFICIENTS $C_1[\beta]$

We use the Lagrange method to find the minimum of the expression (17) under the conditions (18) - (19).
We consider the function.

$$\begin{aligned}
\Psi &\ \left(C_1^R[0], C_1^R[1], \ldots, C_1^R[N], C_1^I[0], C_1^I[1], \ldots, C_1^I[N], d^R, d^I\right) \\
&= \|\ell\|^2_{L_2^{(2)*}} + 2d^R\left(\int_0^1 x\cdot\cos(2\pi\omega x)dx - \sum_{\beta=0}^{N}\left(C_0^R[\beta]\cdot h\beta + C_1^R[\beta]\right)\right) \\
&+ 2d^I\left(\int_0^1 x\cdot\sin(2\pi\omega x)dx - \sum_{\beta=0}^{N}\left(C_0^I[\beta]\cdot h\beta + C_1^I[\beta]\right)\right),
\end{aligned}$$

where d^R and d^I are real constants.

Equating to zero the partial derivatives of Ψ by $C_1^R[\beta], C_1^I[\beta], (\beta=\overline{0,N})$ and d^R, d^I, we obtain the following system of linear equations

$$\sum_{\gamma=0}^{N} C_1^R[\gamma]G_2''(h\beta-h\gamma)+d^R=\int_0^1\cos(2\pi\omega x)G_2'(x-h\beta)dx+\sum_{\gamma=0}^{N} C_0^R[\gamma]G_2'(h\beta-h\gamma),\quad \beta=0,1,...,N, \quad (20)$$

$$\sum_{\gamma=0}^{N} C_1^I[\gamma]G_2''(h\beta-h\gamma)+d^I=\int_0^1\sin(2\pi\omega x)G_2'(x-h\beta)dx+\sum_{\gamma=0}^{N} C_0^I[\gamma]G_2'(h\beta-h\gamma),\quad \beta=0,1,...,N, \quad (21)$$

$$\sum_{\gamma=0}^{N} C_1^R[\gamma]=\int_0^1 x\cdot\cos(2\pi\omega x)dx-\sum_{\gamma=0}^{N} C_0^R[\gamma]\cdot h\gamma, \quad (22)$$

$$\sum_{\gamma=0}^{N} C_1^I[\gamma]=\int_0^1 x\cdot\sin(2\pi\omega x)dx-\sum_{\gamma=0}^{N} C_0^I[\gamma]\cdot h\gamma. \quad (23)$$

Now multiplying both sides of (21) and (23) by i, then adding to both sides of (20) and (22) accordingly, using expression (16) and $d=d^R+id^I$, we get the following system of $N+2$ linear equatons with $N+2$ unknowns $C_1[\gamma]$, $\gamma=0,1,...,N$ and d

$$\sum_{\gamma=0}^{N} C_1[\gamma]G_2''(h\beta-h\gamma)+d=f(h\beta),\quad \beta=0,1,...,N, \quad (24)$$

$$\sum_{\gamma=0}^{N} C_1[\gamma]=g_0, \quad (25)$$

where

$$f(h\beta) = \int_0^1 e^{2\pi i\omega x}G_2'(x-h\beta)dx+\sum_{\gamma=0}^{N} C_0[\gamma]G_2'(h\beta-h\gamma)=(h\beta)^2\cdot q_1+h\beta\cdot q_2+q_3([\beta]), \quad (26)$$

$$\begin{aligned} g_0 &= \int_0^1 x\cdot e^{2\pi i\omega x}dx-\sum_{\gamma=0}^{N} C_0[\gamma]\cdot h\gamma=-\frac{(2\pi i\omega-1)e^{2\pi i\omega}+1}{(2\pi\omega)^2}-\frac{1-2\pi i\omega h-e^{-2\pi i\omega h}}{(2\pi\omega)^2 h}\cdot e^{2\pi i\omega} \\ &- \frac{2(1-\cos(2\pi\omega h))}{(2\pi\omega)^2 h}\cdot\frac{\left((1-h)e^{2\pi i\omega h}-1\right)e^{2\pi i\omega}+he^{2\pi i\omega h}}{(e^{2\pi i\omega h}-1)^2}, \end{aligned} \quad (27)$$

and

$$\begin{aligned} q_1 &= \frac{-e^{2\pi i\omega h}+2\cos(2\pi\omega h)-1+\left(e^{-2\pi i\omega h}-2\cos(2\pi\omega h)+1\right)e^{2\pi i\omega}}{16(\pi\omega)^2 h} \\ &- \frac{(1-\cos(2\pi\omega h))\left(e^{2\pi i\omega h(N+1)}+1\right)}{8(\pi\omega)^2(e^{2\pi i\omega h}-1)h}, \end{aligned} \quad (28)$$

$$\begin{aligned} q_2 &= -\frac{\left(e^{-2\pi i\omega h}-2\cos(2\pi\omega h)+1+h\right)e^{2\pi i\omega}+h}{8(\pi\omega)^2 h} \\ &- \frac{(1-\cos(2\pi\omega h))\left(he^{2\pi i\omega h}+(1+h)e^{2\pi i\omega h(N+1)}-e^{2\pi i\omega h(N+2)}\right)}{4(\pi\omega)^2(e^{2\pi i\omega h}-1)^2 h}, \end{aligned} \quad (29)$$

$$\begin{aligned} q_3(h\beta) &= \frac{\left(e^{-2\pi i\omega h}-2\cos(2\pi\omega h)+1+2h\right)e^{2\pi i\omega}}{16(\pi\omega)^2 h}+\frac{e^{2\pi i\omega}-2e^{2\pi i\omega h\beta}+1}{16(\pi i\omega)^3}+\frac{1-cos(2\pi\omega h)}{8(\pi\omega)^2 h} \\ &\times\left(\frac{e^{2\pi i\omega h}\left(e^{2\pi i\omega h}+1\right)\left(2e^{2\pi i\omega h\beta}-e^{2\pi i\omega}-1\right)}{(e^{2\pi i\omega h}-1)^3}\cdot h^2+\frac{2e^{2\pi i\omega h(N+1)}}{(e^{2\pi i\omega h}-1)^2}\cdot h-\frac{e^{2\pi i\omega h(N+1)}}{e^{2\pi i\omega h}-1}\right). \end{aligned} \quad (30)$$

In this system $C_0[\gamma]$ are defined by (3). The system (24)-(25) is called *the discrete system of Wiener-Hopf type* for the optimal coefficients. The system has a unique solution for any fixed N, $(N\in\mathbb{N})$ and it gives the minimum to $\|\ell\|^2_{L_2^{(2)*}}$. Here we bypassed the proof of the existence and is uniqueness for the solution of this system. These assertions can be proved as the proof of the existence and uniqueness of the solution of the discrete Wiener-Hopf type system of the optimal coefficients in the space $L_2^{(m)}(0,1)$ for quadrature formulas with the form $\int\limits_0^1 \phi(x) \cong \sum\limits_{\beta=0}^{N} C[\beta]\phi[\beta]$ (see [9, 10]).

We are interested in finding explicit formulas for the optimal coefficients $\mathring{C}_1[\beta], \beta = 0,1,...,N$ and unknown d satisfying the system (24)-(25). The system (24)-(25) is solved similarly as the system (34)-(35) of [14] by Sobolev's method, using the discrete analogue $D_1(h\beta)$ of the differential operator $\frac{d^2}{dx^2}$. Therefore, we present the following theorem without proof.

Theorem 2. *Coefficients of the optimal quadrature formula of the form* (2) *in the sense of Sard for* $\omega \in \mathbb{R}/\{0\}$ *in the space* $L_2^{(2)}(0,1)$ *have the following form*

$$\begin{aligned}
\mathring{C}_1[0] &= hq_1 + q_2 + \frac{g_0}{2} + \frac{q_3(h) - q_3(0)}{h}, \\
\mathring{C}_1[\beta] &= 2hq_1 + \frac{q_3(h\beta + h) - 2q_3(h\beta) + q_3(h\beta - h)}{h}, \quad \beta = 1,2,...,N-1, \\
\mathring{C}_1[N] &= (h-2)q_1 - q_2 + \frac{g_0}{2} + \frac{q_3(1-h) - q_3(1)}{h},
\end{aligned}$$

where $q_1, q_2, q_3(h\beta)$ *and* g_0 *are defined by equalities* (28)*,* (29)*,* (30) *and* (27)*, respectively.*
And so, Problem 2 is also solved.

CONCLUSION

Here for approximation of Fourier integrals in the space $L_2^{(2)}(0,1)$ the optimal quadrature formula with derivative in the sense of Sard is constructed. The obtained optimal quadrature formula with derivative has been used to calculate the Fourier integrals of some functions. It can be showed that the optimal quadrature formula (2) gives more accurate results than the optimal quadrature formula in the space $L_2^{(2)}(0,1)$.

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