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MATEMATIKA * MATHEMATICS *** МАТЕМАТИКА**		

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CALCULATION OF COEFFICIENTS OF OPTIMAL QUADRATURE FORMULA BASED ON THE φ-FUNCTION METHOD

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Abstract. *In this work, we study the problem of constructing an optimal quadrature formula in the sense of Sard. The φ-function method is used to construct the quadrature formula. The error of the formula is calculated using the integral of the square of the φ-function from the Hilbert space. Next, the function φ is chosen such that the integral of the square of the function φ takes the smallest value on the given interval. Finally, using the φ function obtained, the coefficients of the optimal quadrature formula are calculated. This work shows that the optimal quadrature formula obtained in Hilbert space $W_{2,\sigma}^{(1,0)}$ is exact for exponential functions with non-zero sigma parameters.*

Keywords: *Hilbert space, optimal quadrature formula, φ-function method, error of the quadrature formula.*

РАСЧЁТ КОЭФФИЦИЕНТОВ ОПТИМАЛЬНОЙ КВАДРАТУРНОЙ ФОРМУЛЫ НА ОСНОВЕ МЕТОДА φ-ФУНКЦИИ

Аннотация. *В данной работе изучается задача построения оптимальной квадратурной формулы в смысле Сарда. Для построения квадратурной формулы используется метод φ-функции. Погрешность формулы вычисляется с использованием интеграла квадрата функции φ из Гильбертова пространства. Затем функция φ выбирается таким образом, чтобы интеграл квадрата функции φ принимал наименьшее значение на заданном интервале. Наконец, с использованием полученной функции φ вычисляются коэффициенты оптимальной квадратурной формулы. В данной работе показано, что оптимальная квадратурная формула, полученная в гильбертовом пространстве $W_{2,\sigma}^{(1,0)}$, является точной для экспоненциальных функций с ненулевыми сигма-параметрами.*

Ключевые слова: *гильбертово пространство, оптимальная квадратурная формула, метод φ-функций, погрешность квадратурной формулы.*

φ- FUNKSIYA USULI ASOSIDA OPTIMAL KUADRATURA FORMULA KOEFFITSIYENTLARINI HISOBLASH

Annotatsiya. *Bu ishda biz Sard ma'nosida optimal kvadratura formula qurish masalasini o‘rganamiz. Kvadratur formula qurish uchun φ-funksiya usuli qo‘llaniladi. Formulaning xatoligi Gilbert fazosida φ funksiya kvadratining integralidan foydalanib hisoblanadi. Keyinchalik φ funksiyasi shunday tanlanadiki, bu funksiya kvadratining integrali berilgan oraliqda eng kichik qiymatni olsin. Nihoyat, olingan funksiyadan foydalanib, optimal kvadratur formulaning koeffitsientlari hisoblanadi. Bu ish shuni ko‘rsatadiki, $W_{2,\sigma}^{(1,0)}$*

Gilbert fazosida olingan optimal kvadratur formula nolga teng bo'lmagan sigma parametrlari bo'lgan eksponensial funksiyalar uchun aniqdir.

Kalit so'zlar: *Gilbert fazosi, optimal kvadratur formula, φ-funksiya usuli, kvadratur formula xatoligi.*

Introduction. Many problems in science and technology lead to integral and differential equations. Often these integrals cannot be calculated exactly. It is necessary to calculate the approximate value of such integrals. The optimal quadrature formulas are a method aimed at approximately calculating integrals and ensuring high accuracy. Many researchers have developed various quadrature formulas based on certain data. The following scientific constructed quadrature formulas in their works: Gregory, Newton-Cotes, Simpson, Euler, Gauss, Chebyshev, and others.

Gregory's quadrature formula allows you to calculate the integral of a function by selecting points. In this method, increasing the number of steps helps reduce the error in the integral calculation. Although Gregory's formulas are considered early developments in the approximation of integrals, they are less commonly used today.

The Newton-Cotes quadrature formulas were developed by Isaac Newton and Roger Cotes. They provide a way to calculate the integral of a function using interpolation methods. These formulas include many types of quadratures, including: trapezoidal formula is first-order interpolation. Simpson's rule is second-order interpolation and Boole's rule is fourth-order interpolation.

Simpson's quadrature formula is a type of Newton-Cotes method, developed by Thomas Simpson. The main advantage of this method that is a second-order interpolation of the Newton-Cotes methods, providing relatively high accuracy among quadrature formulas.

Euler developed a number of approaches to the development of quadrature formulas. He developed some methods of integral calculus and created many formulas. Euler's work was focused on optimality and uncertainty reduction.

Gauss's quadrature formula (or Gauss-Legendre quadrature) is known as one of the most efficient methods for integral calculus. It produces a large number of linear and exact solutions. Gauss's formula uses special points and weights to calculate definite integrals, which allows the method to minimize errors.

Chebyshev played an important role in developing optimal quadrature formulas. Polynomials developed by Chebyshev are widely used to reduce errors in integration and obtain optimal solutions. Chebyshev's quadrature formula is used to distribute points and weights in the most efficient way.

The construction of quadrature formulas and the study of their error based on functional analysis methods were first presented in the scientific works of A. Sard [1] and S. M. Nikolsky [2], and the emergence of the theory of cubature formulas is associated with the scientific research of S. L. Sobolev [3]. There are spline methods, φ-function methods, and Sobolev methods for constructing optimal formulas obtained by minimizing the functional error norm with respect to the coefficients at given nodes. A. Sard [4], L. F. Meyers [5], G. Coman [6,7], S. D. Silliman [8] based on the spline method, A. Ghizzetti and A. Ossicini [9], F. Lanzara [10], T. Catinaş and G. Coman [11] constructed optimal quadrature formulas in $L_2^{(m)}$ space using the φ-function method. Recent results on optimal formulas obtained by Sobolev's method can be found, for example, in the works [12,13]. The results of this work are related to the results of works [14-17] devoted to the construction of optimal quadrature formulas using the Sobolev method. It should be noted that our result generalizes the results of recent works [18]. Since the research in this article is related to the latter approaches, we will consider results in this direction.

The methods developed by mathematicians working on optimal quadrature formulas serve to minimize errors in integral calculations and increase efficiency. The scientific works listed above have made a significant contribution to the development of this field.

Statement of the problem. In this work, we study the construction of an optimal quadrature formula using the method of φ-functions. In this regard, consider quadrature formulas of the form

$$\int_a^b f(x)dx = \sum_{k=0}^{n} C_k f(x_k) + R_n(f), \tag{1}$$

where C_k are the coefficients of the quadrature formula, $x_k \in [a,b]$, $k = 0,1,\ldots,N$ are the nodes, $R_n(f)$ is the error of quadrature formula (1), $f(x)$ are elements of the Hilbert space $W_{2,\sigma}^{(1,0)}(a,b)$ is defined by

$$W_{2,\sigma}^{(1,0)}=\left\{ f:[a,b]\to R \;\middle|\; f(x) \text{ is absolutely continuous}, \int_a^b (f'(x))^2dx<\infty \right\},$$

The inner product of two arbitrary functions $f(x)$ and $g(x)$ in the Hilbert space $W_{2,\sigma}^{(1,0)}$ is defined as follows

$$\langle f(x),g(x)\rangle_{W_{2,\sigma}^{(1,0)}}=\int_a^b \left(f'(x)+\sigma f\right)\left(g'(x)+\sigma g(x)\right)dx, \qquad (2)$$

where $\sigma\in\mathbb{R}$ and $\sigma\neq 0$.

The norm of a function in this space is

$$\|f(x)\|_{W_{2,\sigma}^{(1,0)}}=\left\{\int_a^b (f'(x)+\sigma f(x))^2dx\right\}^{1/2}. \qquad (3)$$

In this paper, we consider the optimality problem of a quadrature formula in the Sard sense. One of the important problems in the theory of quadrature formulas is the optimality problem of a quadrature formula with respect to the error of this formula.

Definition 1. The quadrature formula (1) is called *optimal in the sense of Nikolsky* in the space $W_{2,\sigma}^{(1,0)}$, if the value reaches its smallest value relative to C and X

$$F_n(W_{2,\sigma}^{(1,0)},C,X)=\sup_{f\in W_{2,\sigma}^{(1,0)}}|R_n(f)| \qquad (4)$$

where $C=(C_0,C_1,...,C_n)$ are the coefficients and $X=(x_0,x_1,...,x_n)$ are the nodes.

Definition 2. The quadrature formula (1) is called *optimal in the sense of Sard* in the space $W_{2,\sigma}^{(1,0)}$ if the quantity

$$F_n(W_{2,\sigma}^{(1,0)},C)=\sup_{f\in W_{2,\sigma}^{(1,0)}}|R_n(f)| \qquad (5)$$

reaches its smallest value relative to C for fixed X, where $C=(C_0,C_1,...,C_n)$ are the coefficients and $X=(x_0,x_1,...,x_n)$ are the nodes.

Method of φ-function in the Hilbert space $W_{2,\sigma}^{(1,0)}(a,b)$

Let's assume that the function $f(x)$ belongs to the space $W_{2,\sigma}^{(1,0)}$. Let $n\in N$ $a=x_0<x_1<...<x_n=b$ be the node points for a given $n\in N$. For each interval $[x_{k-1},x_k], k=1,...,n$, we consider a function $\varphi_k, k=1,...,n$ with the following property:

$$\varphi'_k(x)-\sigma\varphi_k(x)=1, k=1,...,n, \qquad (6)$$

where the function φ is defined as:

$$\varphi\big|_{[x_{k-1},x_k]}=\varphi_{k,} k=1,...,n. \qquad (7)$$

Theorem 1. *In the Hilbert space $W_{2,\sigma}^{(1,0)}(a,b)$, among the quadrature formulas of the form (1), there is a unique optimal quadrature formula in the Sard sense whose coefficients are defined as follows*

$$\overset{\circ}{C}_0=\frac{e^{\sigma x_1}-e^{\sigma x_0}}{\sigma(e^{\sigma x_1}+e^{\sigma x_0})},$$

$$\overset{\circ}{C}_k=\frac{2e^{\sigma x_k}(e^{\sigma x_{k+1}}-e^{\sigma x_{k-1}})}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})(e^{\sigma x_k}-e^{\sigma x_{k-1}})},\quad k=1,2,...,n-1, \qquad (8)$$

$$\overset{\circ}{C}_n=\frac{e^{\sigma x_n}-e^{\sigma x_{n-1}}}{\sigma(e^{\sigma x_n}+e^{\sigma x_{n-1}})}.$$

$$a=x_0<x_1<...<x_n=b,$$

where C_0,C_k,C_n are the coefficients of the optimal quadrature formula.

Now we prove that these coefficients can be derived.

We enter the following definition:

$$J(f)=\int_a^b f(x)dx, \tag{9}$$

$$K_n(f)=\sum_k^n C_k f(x_k)\,. \tag{10}$$

Now, using the additivity property of the definite integral and the (6) expressions, we get the following:

$$J(f)=\int_a^b f(x)dx=\sum_{k=1}^n\int_{x_{k-1}}^{x_k}1\cdot f(x)dx=\sum_{k=1}^n\int_{x_{k-1}}^{x_k}(\varphi'_k(x)-\sigma\varphi_k(x))\cdot f(x)dx=$$

$$=\sum_{k=1}^n(\int_{x_{k-1}}^{x_k}\varphi'_k(x)\cdot f(x)dx-\sigma\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot f(x)dx)=$$

$$=\sum_{k=1}^n(\varphi_k(x)\cdot f(x)\Big|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot f'(x)dx)-\sigma\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot f(x)dx)=$$

$$=\sum_{k=1}^n(\varphi_k(x)\cdot f(x)\Big|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx)$$

$$=\sum_{k=1}^n(\varphi_k(x_k)\cdot f(x_k)-\varphi_k(x_{k-1})\cdot f(x_{k-1}))-\sum_{k=1}^n(\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx)$$

$$=\sum_{k=1}^n\varphi_k(x_k)\cdot f(x_k)-\sum_{k=1}^n\varphi_k(x_{k-1})\cdot f(x_{k-1})-\sum_{k=1}^n(\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx)$$

$$=\sum_{k=1}^n\varphi_k(x_k)\cdot f(x_k)-\sum_{k=0}^{n-1}\varphi_{k+1}(x_k)\cdot f(x_k)-\sum_{k=1}^n(\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx)$$

$$=\varphi_n(x_n)\cdot f(x_n)+\sum_{k=1}^{n-1}\varphi_k(x_k)\cdot f(x_k)-\sum_{k=1}^{n-1}\varphi_{k+1}(x_k)\cdot f(x_k)$$

$$-\varphi_0(x_0)\cdot f(x_0)-\sum_{k=1}^n(\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx)$$

$$=-\varphi_0(x_0)\cdot f(x_0)+\sum_{k=1}^{n-1}(\varphi_k(x_k)-\varphi_{k+1}(x_k))\cdot f(x_k)+\varphi_n(x_n)\cdot f(x_n)$$

$$-\sum_{k=1}^n(\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx).$$

We enter designation as follows:

$$\begin{aligned}&C_0=-\varphi_0(x_0),\\&C_k=\varphi_k(x_k)-\varphi_{k+1}(x_k),\quad k=1,2,...,n-1,\\&C_n=\varphi_n(x_n),\end{aligned} \tag{11}$$

$$J(f)=C_0 f(x_0)+\sum_{k=1}^{n-1}C_k\cdot f(x_k)+C_n f(x_n)+R_n[f] \tag{12}$$

and the error $R_n[f]$ as follows:

$$R_n[f]=-\sum_{k=1}^n\int_{x_{k-1}}^{x_k}\varphi_k(x)\cdot(f'(x)+\sigma f(x))dx=-\int_a^b\varphi(x)\cdot(f'(x)+\sigma f(x))dx \tag{13}$$

Remark 1. Knowing the function φ, we can find the coefficients C_k, $k=0,1,...,n$. $k=0,1,...,n$ according to formula (11). This method of constructing the quadrature formula is called the method of φ-function.

Remark 2. It is clear from expression (13) that the quadrature formula in form (1) is true for all solutions of the equation $f'(x)+\sigma f(x)=0$.

Optimality problem for formula (1) and definition of function φ_k.

We look at the space $W_{2,\sigma}^{(1,0)}$. The function space $f(x)$ is absolutely continuous on the

interval $[a,b]$ and first-order derivative belongs to the space $L_2(a,b)$. (13) formulas and from the Cauchy-Schwarz inequality for the absolute value of the error we get the following:

$$|R_n(f)| \le \|f'(x)+\sigma f(x)\|_{L_2(a,b)} \cdot \left(\int_a^b \varphi^2(x)dx\right)^{1/2} = \|f(x)\|_{W_{2,\sigma}^{(1,0)}} \cdot \|\varphi(x)\|_{L_2(a,b)}. \qquad (14)$$

Thus, the optimal quadrature formula in the form (1) is

$$F(C)=\int_a^b \varphi^2(x)dx$$

is to find the C that gives the smallest value to the magnitude.

Now we find the φ_k that are solutions of the following equation in each interval $[x_{k-1},x_k], k=1,...,n$

$$y'-\sigma y=1 \qquad (15)$$

we look for the solution of this function in the form of a product of the functions $I(x)$ and $y_1(x)$

$$y=I(x)\cdot y_1(x),$$

where $y_1(x)$ is the solution to the following homogeneous equation

$$y_1'-\sigma y_1=0, \qquad (16)$$

one of the solutions of the equation in the form (16) has the following form

$$y_1(x)=e^{\sigma x}\cdot e^B,$$

$$B=0,\ y_1(x)=e^{\sigma x}. \qquad (17)$$

Now we solve equation (15). By hypothesis, the solution to equation (15) is as follows

$$y(x)=Iy_1(x), \qquad (18)$$

where $y_1(x)$ is defined by equation (17). Then we just need to find $I(x)$. To find it, we first calculate the first order derivative of the unknown function $y(x)$ in formula (18). It is as follows

$$y'(x)=I'y_1(x)+Iy_1'(x). \qquad (19)$$

Substituting $y(x)$ found in formula (18) and $y_1(x)$ in formula (17) into expression (15), we get the following

$$I'(x)e^{\sigma x}+\sigma I(x)e^{\sigma x}-\sigma I(x)e^{\sigma x}=1, \qquad (20)$$

where

$$I'(x)=e^{-\sigma x},\ I(x)=-\frac{1}{\sigma}e^{-\sigma x}+B \qquad (21)$$

the solution to equation (18) is as follows

$$y(x)=I(x)y_1(x)=\left(-\frac{1}{\sigma}e^{-\sigma x}+B\right)\cdot e^{\sigma x}=-\frac{1}{\sigma}+Be^{\sigma x} \qquad (22)$$

Then, for each interval $[x_{k-1},x_k], k=1,...,n$ we take the function of the form (22) as φ_k

$$\varphi_k(x)=-\frac{1}{\sigma}+B_1^{(k)}e^{\sigma x},\ [x_{k-1},x_k], k=1,...,n. \qquad (23)$$

To find in each interval $\left[x_{k-1}, x_k\right], k=1,...,n$ the functions φ_k, we need to find the coefficients $B_1^{(k)}, k=1,...,n$ that give the smallest value to the solution of the following equation

$$F(C)=\int_a^b \varphi^2(x)dx.$$

We find the $B_1^{(k)}, k=1,...,n$ for which the integral of the square of the function φ_k, determined by the equation in the form (23), has the smallest value

$$F_k(B_1^{(k)})=\int_{x_{k-1}}^{x_k}\left(\varphi_k\left(x\right)\right)^2 dx\ ,\ k=1,...,n-1. \qquad (24)$$

The value $B_1^{(k)}$ is the smallest value of the function $F_k(B_1^{(k)})$ in the interval $\left[x_{k-1}, x_k\right]$

$$B_1^{(k)}=\frac{2(e^{\sigma x_{k+1}}-e^{\sigma x_k})}{\sigma(e^{2\sigma x_{k+1}}-e^{2\sigma x_k})}=\frac{2}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})}. \qquad (25)$$

Then, by substituting the value in (25) into expression (23), we obtain the following

$$\varphi_k(x)=-\frac{1}{\sigma}+\frac{2e^{\sigma x}}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})}\ ,\ x\in\left[x_{k-1}, x_k\right],\ k=1,...,n. \qquad (26)$$

The coefficients $C_k, k=1,...,n$ of the optimal quadrature formula in form (1) are found using expressions in form (11) and (26).

We find the first coefficient C_0 using $\varphi_k(x), k=1,...,n-1$, $\varphi_k(x), k=0$.

$$C_0=-\left(\frac{2e^{\sigma x_0}-e^{\sigma x_1}-e^{\sigma x_0})}{\sigma(e^{\sigma x_1}+e^{\sigma x_0})}\right)=\frac{e^{\sigma x_1}-e^{\sigma x_0}}{\sigma(e^{\sigma x_1}+e^{\sigma x_0})} \qquad (27)$$

We find the coefficients $C_k, k=1,...,n-1$ using $\varphi_k(x), k=n$ $\varphi_k(x), k=1,...,n-1$

$$C_k=\varphi_k(x_k)+\varphi_{k+1}(x_k)=\frac{2e^{\sigma x_k}(e^{\sigma x_{k+1}}-e^{\sigma x_{k-1}})}{\sigma(e^{\sigma x_{k+1}}+e^{\sigma x_k})(e^{\sigma x_k}-e^{\sigma x_{k-1}})}. \qquad (28)$$

Finally, we find the coefficient C_n using $\varphi_k(x), k=n$.

$$C_n=\varphi_n(x_n)=\frac{1}{\sigma}\left(\frac{2e^{\sigma x_n}}{e^{\sigma x_n}+e^{\sigma x_{n-1}})}-1\right)=\frac{e^{\sigma x_n}-e^{\sigma x_{n-1}}}{\sigma(e^{\sigma x_n}+e^{\sigma x_{n-1}})} \qquad (29)$$

Thus, by generalizing the results in (27), (28), and (29), we have proved the main theorem of this work.

Conclusion. In this paper, we have considered determining the coefficients of the optimal quadrature formula in the space $W_{2,\sigma}^{(1,0)}$. Here $W_{2,\sigma}^{(1,0)}$ is the Hilbert space of absolutely continuous functions whose first-order derivatives are integrable by their squares on the interval $\left[a,b\right]$. The function $f(x)$ in the quadrature formula is a linear combination of the values of $f(x_k)$ at the nodes $x_k\in\left[a,b\right]$, where $a=x_0<...<x_n=b$. The error $R_n(f)$ of the quadrature formula (1) is estimated from above using the product of the norm of the function $f(x)$ in the space $W_{2,\sigma}^{(1,0)}(a,b)$ and the norm of the special φ-function in the $L_2(a,b)$ space. To estimate $R_n(f)$, we need to find the coefficients that provide the smallest value for the norm of the φ-function. Here, we found the optimal coefficients C_k. In the future, we will calculate the error of the optimal quadrature formula using the obtained optimal coefficients.

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