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UZBEK MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named by "Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk". Since 1991 it is known as "Uzbek Mathematical Journal". It has 4 issues annually.

Volume 65 **Issue 3** **2021**

Uzbek Mathematical Journal is abstracting and indexing by

MathSciNet **Zentralblatt** **Math VINITI**

Starting from 2018 all papers published in Uzbek Mathematical Journal you can find in **EBSCO** and **CrossRef**.

TASHKENT - 2021

Uzbek Mathematical Journal
2021, Volume 65, Issue 3, pp.93-100
DOI: 10.29229/uzmj.2021-3-9

Optimal quadrature formulas in the space $\widetilde{W_2}^{(1,0)}$ of periodic functions

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Abstract. This paper is devoted to the process of constructing an optimal quadrature formula in the Hilbert space of real-valued, periodic functions. Here, in order to obtain the upper bound for the absolute error of the considered quadrature formula, the norm of the error functional is calculated. In the calculation of the norm of the error functional the extremal function of the quadrature formula is used. Further, the system of linear equations for coefficients of the quadrature formula that give the minimum value to the norm of the error functional is obtained.

Keywords: Optimal quadrature formula, Hilbert space, the error functional, Fourier transform.

MSC (2010): 65D30, 65D32.

1 Introduction. Statement of the problem

In this paper, we are concerned with obtaining optimal quadrature formulas. It is assumed here that the integrand belongs to the Hilbert space $W_2^{(m,m-1)}$. Recall the definition of this Hilbert space, based, for example, on the work [9].

$W_2^{(m,m-1)}(0,1)$ is the Hilbert space of real-valued functions and it is defined as follows

$$\begin{aligned} W_2{}^{(m,m-1)} \quad = \quad & \{\varphi : [0,1] \to \mathbb{R} \mid \varphi^{(m-1)} \text{ are absolute continuous} \\ & \text{and } \varphi^{(m)} \in L_2(0,1)\}. \end{aligned}$$

This space is equipped by the norm

$$\|\varphi|W_2^{(m,m-1)}(0,1)\| = \left(\int_0^1 \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x)\right)^2 dx\right)^{1/2}.$$

This equality is a semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = P_{m-2}(x) + de^{-x}$, where $P_{m-2}(x)$ is a polynomial of degree $m-2$ and d is a constant. Every element of the space $W_2^{(m,m-1)}$ is a set of functions which are differ from each other on linear combination of any polynomial of degree $m-2$ and e^{-x}. So, the space $W_2^{(m,m-1)}(0,1)$ is a factor space.

We denote by $\widetilde{W}_2^{(m,m-1)}(0,1)$ the subspace of $W_2^{(m,m-1)}(0,1)$ consisting of real-valued, 1-periodic functions.

In the present paper we deal with the case $m = 1$ and we consider the Hilbert space $\widetilde{W}_2^{(1,0)}$ of 1-periodic, real-valued functions. Notice that every element of the space $\widetilde{W_2}^{(1,0)}$ satisfies the following condition of 1-periodicity

$$\varphi(x+\beta) = \varphi(x) \text{ for } x \in \mathbb{R} \text{ and } \beta \in \mathbb{Z}.$$

In this space, the inner product is defined as

$$\langle \varphi, \psi \rangle_{\widetilde{W_2}^{(1,0)}} = \int_0^1 (\varphi'(x) + \varphi(x))(\psi'(x) + \psi(x))dx, \tag{1.1}$$

and the corresponding norm of the function φ is denoted by

$$\|\varphi\|_{\widetilde{W_2}^{(1,0)}} = \langle \varphi, \varphi \rangle^{1/2}$$

and $\int_0^1 (\varphi'(x) + \varphi(x))^2 dx < \infty$.

We consider the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{k=1}^{N} C_k \varphi(hk) \tag{1.2}$$

with the error

$$(\ell, \varphi) = \int_0^1 \varphi(x)dx - \sum_{k=1}^{N} C_k \varphi(hk), \tag{1.3}$$

where

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx,$$

and the corresponding error functional is

$$\ell(x) = \left(\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k \delta(x - hk) \right) * \phi_0(x). \tag{1.4}$$

Here C_k are the coefficients of formula (1.2), $h = \frac{1}{N}$, $N \in \mathbb{N}$, $\varepsilon_{(0,1]}(x)$ is the indicator of the interval $(0,1]$, δ is the Dirac's delta-function, $\varphi \in \widetilde{W_2}^{(1,0)}$, $\phi_0(x) = \sum\limits_{\beta=-\infty}^{\infty} \delta(x-\beta)$.

The error (1.3) of the quadrature formula (1.2) is a linear functional in $\widetilde{W_2}^{(1,0)*}$, where $\widetilde{W_2}^{(1,0)*}$ is the conjugate space for the space $\widetilde{W_2}^{(1,0)}$. The absolute value of the error (1.3) is estimated by Cauchy-Schwarz inequality as follows

$$|(\ell, \varphi)| \le \|\varphi\|_{\widetilde{W_2}^{(1,0)}} \cdot \|\ell\|_{\widetilde{W_2}^{(1,0)*}},$$

where

$$\|\ell\|_{\widetilde{W_2}^{(1,0)*}} = \sup_{\|\varphi\|_{\widetilde{W_2}^{(1,0)}}=1} |(\ell,\varphi)| \tag{1.5}$$

is the norm of the error functional (1.4).

Hence, in order to get the minimum of the upper bound of the error for the quadrature formula (1.2) we should solve the following.

Problem 1. Find the coefficients $\overset{\circ}{C}_k$ that give minimum value to $\|\ell\|_{\widetilde{W_2}^{(1,0)*}}$, and calculate

$$\left\|\overset{\circ}{\ell}\right\|_{\widetilde{W_2}^{(1,0)*}} = \inf_{C_k} \|\ell\|_{\widetilde{W_2}^{(1,0)*}}.$$

We note that the coefficients $\overset{\circ}{C}_k$ which are the solution of Problem 1 are called *the optimal coefficients* and the quadrature formula (1.2) with these coefficients is *the optimal quadrature formula in the sense of Sard* [8].

The solution to Problem 1 was first proposed by Sobolev [14], later the problem was solved in the space $W_2^{(m,m-1)}$ of non-periodic functions in the works [9, 1, 2, 3, 4, 13, 6] and in the space $\widetilde{L_2}^{(m)}$ of periodic functions in the works [10, 12, 5, 11, 7].

Further, in the next sections in order to solve Problem 1 we do the following.

- First, we find the extremal function of the quadrature formula (1.2).
- Using the extremal function we calculate the norm of the error functional (1.4).
- We get the system for optimal coefficients which give the minimum to the norm of the error functional.

2 The norm of the error functional (1.4)

To calculate the norm (1.5), we use *the extremal function* ψ_ℓ for the error functional ℓ (see [14]) that satisfies the following equality:

$$(\ell,\psi_\ell) = \|\ell\|_{\widetilde{W_2}^{(1,0)*}} \cdot \|\psi_\ell\|_{\widetilde{W_2}^{(1,0)}}. \tag{2.1}$$

Since $\widetilde{W_2}^{(1,0)}$ is the Hilbert space by the Riesz theorem for the error functional ℓ in any φ from $\widetilde{W_2}^{(1,0)}$ there is ψ_ℓ which satisfies the equality

$$(\ell,\varphi) = \langle \psi_\ell, \varphi\rangle_{\widetilde{W_2}^{(1,0)}}, \tag{2.2}$$

where $\langle \psi_\ell, \varphi\rangle_{\widetilde{W_2}^{(1,0)}}$ is the inner product of the functions ψ_ℓ and φ defined by (1.1). In addition, the equality $\|\ell\|_{\widetilde{W_2}^{(1,0)*}} = \|\psi_\ell\|_{\widetilde{W_2}^{(1,0)}}$ is fulfilled. So, taking into account (2.1), we derive

$$(\ell,\psi_\ell) = \|\ell\|^2_{\widetilde{W_2}^{(1,0)*}}.$$

Integrating by parts in the right-hand side of (2.2), keeping in mind periodicity of functions, for ψ_ℓ we have

$$\psi_\ell''(x) - \psi_\ell(x) = -\ell(x). \tag{2.3}$$

For the solution of the last equation the following holds.

Theorem 1. *The solution of the boundary value problem (2.3) is the extremal function ψ_ℓ of the error functional ℓ, expressed as*

$$\psi_\ell(x) = -\left(\int_0^1 G_1(x-y)dy - \sum_{k=1}^{N} C_k G_1(x-hk)\right), \tag{2.4}$$

where

$$G_1(x) = \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta x}}{(2\pi i\beta)^2 - 1}. \tag{2.5}$$

Proof. Using the expression (2.2), we get the following

$$(\ell,\varphi) = \langle\psi_\ell,\varphi\rangle_{\widetilde{W_2}^{(1,0)}} = \int_0^1 \left(\psi_\ell'(x) + \psi_\ell(x)\right)\left(\varphi'(x) + \varphi(x)\right) dx \tag{2.6}$$

Suppose that $\varphi(x)$ belongs to the space $C^1(0,1]$, where $C^1(0,1]$ is the space of the first order differentiable, and finite, functions defined in the interval $(0,1]$. Then from (2.6), integrating by parts, we obtain

$$\langle\psi_\ell,\varphi\rangle_{\widetilde{W_2}^{(1,0)}} = -\int_0^1 \left(\psi_\ell''(x) - \psi_\ell(x)\right)\varphi(x)dx. \tag{2.7}$$

Keeping in the mind (2.7), from (2.2) we get

$$\psi_\ell''(x) - \psi_\ell(x) = -\ell(x). \tag{2.8}$$

Then, we find the periodic solution of equation (2.8) using Fourier Transform. For this, we use the following properties of Fourier Transform (see, for instance [3],[4])

$$\begin{aligned}
F[\varphi] &= \int_{-\infty}^{\infty} \varphi(x)e^{2\pi ipx}dx,\\
F^{-1}[\varphi] &= \int_{-\infty}^{\infty} \varphi(p)e^{-2\pi ipx}dp,\\
F[\varphi^{(\alpha)}] &= (-2\pi ip)^{\alpha}F[\varphi],\ (\alpha\in\mathbb{N}),\\
F[\varphi * g] &= F[\varphi]\cdot F[g],\\
F[\varphi\cdot g] &= F[\varphi] * F[g],\\
F[\phi_0(x)] &= \phi_0(p),\\
F^{-1}[F[\varphi(x)]] &= \varphi(x).
\end{aligned}$$

Here * is the convolution operation.

We apply Fourier Transform to both sides of equation (2.8)

$$F[\psi_\ell'' - \psi_\ell] = -F[\ell].$$

Since, the Fourier Transform is linear operator, we have

$$\left((2\pi ip)^2 - 1\right) F[\psi_\ell] = -F\left[\left(\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k \delta(x - hk)\right) * \phi_0(x)\right]$$

or

$$\left((2\pi ip)^2 - 1\right) F[\psi_\ell] = -\left(F\left[\varepsilon_{(0,1]}(x)\right] - \sum_{k=1}^{N} C_k e^{2\pi iphk}\right) \cdot \phi_0(p),$$

where

$$F[\delta(x - hk)] = \int_{\infty}^{\infty} \delta(x - hk) e^{2\pi ipx} dx = e^{2\pi iphk}.$$

$$\begin{aligned}
F\left[\psi_\ell(x)\right] &= -\frac{F\left[\varepsilon_{(0,1]}(x)\right] - \sum\limits_{k=1}^{N} C_k e^{2\pi iphk}}{(2\pi ip)^2 - 1} \cdot \phi_0(p) \\
&= -\frac{F\left[\varepsilon_{(0,1]}(x)\right] \cdot \phi_0(p)}{(2\pi ip)^2 - 1} + \sum_{k=1}^{N} C_k \frac{e^{2\pi iphk} \cdot \phi_0(p)}{(2\pi ip)^2 - 1} \\
&= -\frac{F\left[\varepsilon_{(0,1]}(x)\right] \cdot \sum\limits_{\beta=-\infty}^{\infty} \delta(p - \beta)}{(2\pi ip)^2 - 1} + \sum_{k=1}^{N} C_k \frac{e^{2\pi iphk} \cdot \sum\limits_{\beta=-\infty}^{\infty} \delta(p - \beta)}{(2\pi ip)^2 - 1}.
\end{aligned}$$

Using the property $f(x)\delta(x - a) = f(a)\delta(x - a)$ of delta-function, we have the following

$$F\left[\psi_\ell(x)\right] = -F\left[\varepsilon_{(0,1]}(x)\right] \cdot \sum_{\beta=-\infty}^{\infty} \frac{\delta(p - \beta)}{(2\pi i\beta)^2 - 1} + \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta hk} \delta(p - \beta)}{(2\pi i\beta)^2 - 1}.$$

Then, we apply the Inverse Fourier Transform to the above equality and we obtain the following

$$\begin{aligned}
F^{-1}\left[F\left[\psi_\ell\right]\right] &= -\varepsilon_{(0,1]}(x) * \sum_{\beta=-\infty}^{\infty} \frac{F^{-1}\left[\delta(p - \beta)\right]}{(2\pi i\beta)^2 - 1} \\
&+ \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta hk}}{(2\pi i\beta)^2 - 1} F^{-1}\left[\delta(p - \beta)\right].
\end{aligned}$$

Hence consequently we get

$$\begin{aligned}\psi_\ell(x) &= -\left(\varepsilon_{(0,1]}(x) * \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta x}}{(2\pi i\beta)^2-1} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta hk}\cdot e^{-2\pi i\beta x}}{(2\pi i\beta)^2-1}\right)\\ &= -\left(\varepsilon_{(0,1]}(x) * G_1(x) - \sum_{k=1}^{N} C_k G_1(x-hk)\right)\\ &= -\left(\int_{-\infty}^{\infty} \varepsilon_{(0,1]}(y) G_1(x-y)dy - \sum_{k=1}^{N} C_k G_1(x-hk)\right)\\ &= -\left(\int_{0}^{1} G_1(x-y)dy - \sum_{k=1}^{N} C_k G_1(x-hk)\right)\end{aligned}$$

where, $G_1(x)$ is defined by expression (2.5).
And so, theorem 1 is proved from the last equality.

Using (2.4) for the norm of the error functional ℓ with (2.6), get

$$\begin{aligned}\|\ell\|^2_{\widetilde{W_2}^{(1,0)*}} &= (\ell, \psi_\ell)\\ &= \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)dx = -\int_{-\infty}^{\infty} \ell(x)\cdot(\ell(x) * G_1(x))dx.\end{aligned}$$

From the above equality, we obtain the following

$$\begin{aligned}\|\ell\|^2_{\widetilde{W_2}^{(1,0)*}} &= -\Bigg[\sum_{k=1}^{N}\sum_{\gamma=1}^{N} C_k C_\gamma G_1(hk-h\gamma) - 2\sum_{k=1}^{N} C_k \int_0^1 G_1(x-hk)dx\\ &+ \int_0^1\int_0^1 G_1(x-y)dxdy\Bigg]. \qquad (2.9)\end{aligned}$$

Thus, we get expression (2.9) for the norm of the error functional (1.4).
In the next section, we solve Problem 1.

3 Minimization of the expression (2.6) by coefficients C_k

Problem 1 is equivalent to the minimization of (2.9) in C_k.
Now we consider the function

$$\Psi(C_1, C_2, ..., C_N) = \|\ell\|^2_{\widetilde{W_2}^{(1,0)*}}.$$

Taking the partial derivatives of Ψ with respect to C_k $(k=\overline{1,N})$ and equating them to zero, we get the following system of linear equations:

$$\sum_{\gamma=1}^{N} C_\gamma G_1(hk-h\gamma)=\int_0^1 G_1(x-hk)dx,\ \ k=1,...,N. \tag{3.1}$$

It can be proved that the system of equations (3.1) has a unique solution, as in the work [9]. The solution of the system (3.1) gives the minimum to the norm of the error functional determined by equality (2.9) in certain value of $C_k=\overset{\circ}{C}_k$. The aim of our next work is to find analytic solution of the system (3.1).

Conclusion

Here for approximation of integrals in the Hilbert space of real-valued, periodic functions the optimal quadrature formula in the sense of Sard is constructed and the analytic form of the norm of the error functional is found.

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