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The numerical solution of a Fredholm integral equation of the second kind using the Galerkin method based on optimal interpolation

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ABSTRACT

In this paper, we study the Galerkin method for obtaining approximate solutions to linear Fredholm integral equations of the second kind. The finite element solution is represented as a linear combination of basis functions, and the construction of suitable basis functions plays a crucial role in the accuracy of the approximation. We propose an optimal interpolation formula that exactly reproduces the functions e^x and e^{-x}, and derive basis functions from its coefficients. This interpolation formula is constructed within the Hilbert space $W_2^{(1,0)}$. To evaluate the effectiveness of the proposed approach, we solve several integral equations using the Galerkin method with two types of basis functions: the newly constructed exponential basis and classical piecewise linear basis functions. Numerical experiments are presented to compare the accuracy of these approaches. Graphs and tables illustrate the approximation errors, demonstrating that both basis functions achieve an error order of $O(h)$, with the optimal interpolation-based basis yielding superior accuracy in certain cases.

1. Introduction

In this work, we discuss the Galerkin method for solving the integral equation

$$u(x) - \int_a^b K(x, y)u(y)dy = f(x), \quad x \in [a, b]. \tag{1}$$

In these equations, u is an unknown function, the kernel of the integral equation K, and the function f on the right-hand side are given.

The term integral equation was first used by du Bois-Reymond [1] in 1888. Eq. (1) carry the name of Fredholm because of his contributions to the field and are called *Fredholm integral equations* of the *first* and *second kinds*, respectively. Here we consider Eq. (1) the case where the solution exists and is unique, and the kernel is sufficiently smooth (see, [2]).

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We express the integral Eq. (1) in the operator form:

$$(I-K)u=f$$

where the operator K is assumed to be compact on a Banach space $\mathbb{V}$, and I is the identity operator. The most common choices of Banach spaces are $C(a,b)$ and $L^2(a,b)$. For Galerkin's method and its generalizations, the Sobolev space $H^r(a,b)$ is also frequently used, where $H^0(a,b)\equiv L^2(a,b)$.

In practice, we select a sequence of finite-dimensional subspaces $\mathbb{V}_n\subset\mathbb{V}$ for $n\geq 1$, where each $\mathbb{V}_n$ has dimension d_n. Let $\mathbb{V}_n$ has a basis $\phi_1,\dots,\phi_{d_n}$, where we set $N\equiv d_n$ for notational simplicity. We then seek a function $u_N\in\mathbb{V}_n$, which can be expressed as

$$u_N(x)=\sum_{i=1}^{N}c_i\phi_i(x),\quad x\in[a,b]. \tag{2}$$

This is substituted into (1), and the coefficients $(c_1,\dots,c_N)$ are determined by ensuring the equation is nearly exact in a certain sense. And the error will be equal to the following

$$\begin{aligned} r_N(x) &= u_N(x)-\int_a^b K(x,y)u_N(y)dy-f(x)\\ &=\sum_{i=1}^{N}c_i\left(\phi_i(x)-\int_a^b K(x,y)\phi_i(y)dy\right)-f(x),\quad x\in[a,b]. \end{aligned} \tag{3}$$

This is known as the residual in the equation's approximation when using $u\approx u_N$. To obtain the coefficients $(c_1,\dots,c_N)$ required to $r_N(x)$ satisfy

$$(r_N,\phi_j)=0,\quad j=1,\dots,N. \tag{4}$$

To find $(c_1,\dots,c_N)$, apply (4) to (3). This yields the linear system of equations

$$\sum_{i=1}^{N}c_i\left\{(\phi_i,\phi_j)-(K\phi_i,\phi_j)\right\}=(f,\phi_j),\quad j=1,\dots,N. \tag{5}$$

This is Galerkin method for obtaining an approximate solution to (1). The system has a solution, and it is unique. The resulting sequence of approximate solutions, u_N converges to u in $\mathbb{V}$ (The proof is given in [2]).

First Bubnov in 1913 and then, in more details, Galerkin [3] in 1915 approached and extended this approximation method without relying on a minimization formulation. Later, Petrov [4] first considered the general form of the Galerkin method.

Rest of the paper is organized as follows. In Section 2, we introduce piecewise linear basis functions and their application in the Galerkin method. Section 3 details the construction of an optimal interpolation formula in the $W_2^{(1,0)}$ space. These interpolation formula coefficients are obtained as basis functions. Section 4 presents numerical results, comparing the performance of piecewise linear and exponential basis functions through error analysis.

2. Piecewise linear basis functions

For simplicity, we solve the problem for the interval $[0,1]$ instead of the interval $[a,b]$. Piecewise linear splines are good basis functions due to their simplicity and ease of use. We divide the interval $[0,1]$ into N subintervals with nodes:

$$0=x_0<x_1<\dots<x_N=1.$$

Let $h=1/N$ denote the mesh size. First, we choose piecewise linear splines as the basis functions:

$$\phi_i(x)=\begin{cases}\frac{x-x_{i-1}}{h}, & x_{i-1}\leq x<x_i,\\ \frac{x_{i+1}-x}{h}, & x_i\leq x<x_{i+1},\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

and from the last expression, we have

$$\phi_0(x)=\frac{h-x}{h}.$$

Each $\phi_i(x)$ is continuous, piecewise linear, and satisfies $\phi_i(x_j)=\delta_{ij}$ (Kronecker delta). Here, $i=\overline{1,N}$ and $j=\overline{1,N}$.

The Galerkin method requires solving the linear system $\mathbf{Ac}=\mathbf{b}$, where:

$$A_{ij}=(\phi_i,\phi_j)-(K\phi_i,\phi_j),$$

$$b_i=(f,\phi_j),\quad i=\overline{1,N},j=\overline{1,N}.$$

It is clear that ϕ_i and ϕ_j overlap only if $|i-j|\leq 1$. Thus, A_{ij} is sparse (mostly tridiagonal for 1D problems). Here, (ϕ_i,ϕ_j) can be computed analytically:

$$(\phi_i,\phi_j)=\begin{cases}\frac{2h}{3}, & \text{if } i=j,\\ \frac{h}{6}, & \text{if } |i-j|=1,\\ 0, & \text{otherwise.}\end{cases}$$

Similarly, the calculation of the term involving the kernel,

$$(K\phi_i, \phi_j) = \int_0^1 \left[\int_0^1 K(x,y)\phi_i(x)dx\right]\phi_j(y)dy$$

is performed in the same way as above. Usually, quadrature formulas are used to calculate this double integral.

Usually, different quadrature formulas are chosen depending on the given kernel $K(x,y)$. For example, integrals can be approximated using optimal quadrature formulas [5–7] with $p(x)$ weight, [8–10] with a weak singularity integral.

The choice of basis functions is also important for the Galerkin method. The construction of basis functions in different spaces and their application to solving boundary value problems for ordinary differential equations using the Galerkin method are considered in the work [11].

We emphasize that the coefficients of optimal interpolation formulas (see, [12–16]) constructed in various Hilbert and Sobolev spaces can also be used as basis functions.

3. An optimal interpolation formula

3.1. Problem statement

Here, we consider construction of an optimal interpolation by a variational method. In the variational approach, splines are elements of Hilbert or Banach spaces minimizing certain functionals.

Assume we are given a table of values $\varphi(x_\beta)$, $\beta = 0, 1, \dots, N$ of a function φ at points $x_\beta \in [0,1]$. It is required approximate the function φ by another more simple function P_φ, i.e.,

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x)\cdot\varphi(x_\beta), \tag{7}$$

which satisfies the following interpolation conditions

$$\varphi(x_\beta) = P_\varphi(x_\beta),\ \ \beta = 0, 1, \dots, N.$$

Here $C_\beta(x)$ and x_β ($\in [0,1]$) are *the coefficients* and *the nodes* of the interpolation formula (7), respectively.

We suppose that functions φ belong to the Hilbert space

$$W_2^{(1,0)}(0,1) = \{\varphi : [0,1] \to \mathbb{R} \mid \varphi \text{ is abs. cont. and } \varphi' \in L_2(0,1)\},$$

equipped with the norm

$$\left\|\varphi|W_2^{(1,0)}(0,1)\right\| = \left\{\int_0^1 \left(\varphi'(x) + \varphi(x)\right)^2 dx\right\}^{1/2}$$

and $\int_0^1 \left(\varphi'(x)+\varphi(x)\right)^2 dx < \infty$. The last equality is the semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = ke^{-x}$, here k is any real constant.

The difference $\varphi - P_\varphi$ is called *the error* of the interpolation formula (7). The value of the error at a point $z \in [0,1]$ is a linear functional on the space $W_2^{(1,0)}(0,1)$, i.e.,

$$\begin{aligned}(\ell,\varphi) &= \varphi(z) - P_\varphi(z) = \varphi(z) - \sum_{\beta=0}^{N} C_\beta(z)\varphi(x_\beta)\\ &= \int_{-\infty}^{\infty}\left(\delta(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x-x_\beta)\right)\varphi(x)dx,\end{aligned} \tag{8}$$

where $\delta(x)$ is the Dirak delta-function and

$$\ell(x,z) = \delta(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x-x_\beta) \tag{9}$$

is *the error functional* of the interpolation formula (7) and it belongs to the space $W_2^{(1,0)*}(0,1)$. The space $W_2^{(1,0)*}(0,1)$ is the conjugate to the space$W_2^{(1,0)}(0,1)$. In addition, for convenience, we denote $\ell(x,z)$ by $\ell(x)$.

By the Cauchy–Schwarz inequality the absolute value of the error (8) is estimated as follows

$$|(\ell,\varphi)| \le \|\varphi|W_2^{(1,0)}\| \cdot \|\ell|W_2^{(1,0)*}\|,$$

where,

$$\left\|\ell|W_2^{(1,0)*}\right\| = \sup_{\varphi,\ \|\varphi\|\neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|}.$$

Therefore, in order to estimate the error of the interpolation formula (7) on functions of the space $W_2^{(1,0)}(0,1)$ it is required to find the norm of the error functional ℓ in the conjugate space $W_2^{(1,0)*}(0,1)$.

From here we get:

Problem 1. Find the norm of the error functional ℓ for the interpolation formula (7) in the space $W_2^{(1,0)*}(0,1)$.

It is clear that the norm of the error functional ℓ depends on the coefficients $C_\beta(z)$ and the nodes x_β. The problem of minimization of the quantity $\|\ell\|$ by coefficients $C_\beta(z)$ is a linear problem and by nodes x_β is, in general, a complicated and non-linear problem. We consider the problem of minimization of the quantity $\|\ell\|$ by coefficients $C_\beta(z)$ when the nodes x_β are fixed.

If there are coefficients $\mathring{C}_\beta(z)$ that minimize the norm of the error functional, that is,

$$\left\|\mathring{\ell}|W_2^{(1,0)*}\right\| = \inf_{C_\beta(z)} \left\|\ell|W_2^{(1,0)*}\right\| \tag{10}$$

then they are called *the optimal coefficients* and the corresponding interpolation formula

$$\mathring{P}_\varphi(z) = \sum_{\beta=0}^{N} \mathring{C}_\beta(z)\varphi(x_\beta)$$

is called *the optimal interpolation formula* in the space $W_2^{(1,0)}(0,1)$.

Thus, in order to construct the optimal interpolation formula in the space $W_2^{(1,0)}(0,1)$ we need to solve the next problem.

Problem 2. Find the coefficients $\mathring{C}_\beta(z)$ that give the quantity (10) when the nodes x_β are fixed.

3.2. The norm of the error functional

The main aim of the present paper is to construct the optimal interpolation formulas in the space $W_2^{(1,0)}(0,1)$ and to find explicit formulas for the optimal coefficients. The first such problem was stated and studied by Sobolev in [17], where the extremal function of the interpolation formula was found in the Sobolev space $W_2^{(m)}$.

To find the explicit form of the norm of the error functional ℓ in the space $W_2^{(1,0)*}(0,1)$, we use concept of an extremal function introduced by Sobolev [17,18]. The function ψ_ℓ from $W_2^{(1,0)}(0,1)$ space is called *the extremal function* for the error functional ℓ if the following equality is fulfilled

$$(\ell,\psi_\ell) = \left\|\ell\left|W_2^{(1,0)*}\right.\right\| \cdot \left\|\psi_\ell\left|W_2^{(1,0)}\right.\right\|.$$

The space $W_2^{(1,0)}(0,1)$ is a Hilbert space and the inner product in this space is defined by the following formula

$$\langle\varphi,\psi\rangle = \int_0^1 \left(\varphi'(x)+\varphi(x)\right)\left(\psi'(x)+\psi(x)\right)dx. \tag{11}$$

According to the Riesz representation theorem, any continuous linear functional ℓ in a Hilbert space can be represented in the form of an inner product. So, in our case, for any function φ from $W_2^{(1,0)}(0,1)$ space, we have

$$(\ell,\varphi) = \langle\psi_\ell,\varphi\rangle. \tag{12}$$

Here ψ_ℓ is the function from $W_2^{(1,0)}(0,1)$ is defined uniquely by the functional ℓ and is the extremal function.

The extremal function has the following form (see, [14])

$$\psi_\ell(x) = -\ell(x) * G_1(x) + de^{-x}, \tag{13}$$

where

$$G_1(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2}\right), \tag{14}$$

d is a real constant, $*$ is the operation of convolution which for the functions f and g is defined as follows

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

It is easy to see from (12) that the error functional ℓ, defined on the space $W_2^{(1,0)}(0,1)$, satisfies the following equality

$$(\ell, e^{-x}) = 0. \tag{15}$$

The last equality means that our interpolation formula is exact for the function e^{-x}.

Now we obtain the norm of the error functional ℓ. Since the space $W_2^{(1,0)}(0,1)$ is the Hilbert space then by the Riesz theorem, we have

$$\left(\ell,\psi_\ell\right) = \|\ell\|\cdot\|\psi_\ell\| = \|\ell\|^2.$$

Hence, using (9) and (13), taking into account (15), we get

$$\begin{aligned}\|\ell\|^2 &= (\ell,\psi_\ell) = \int_{-\infty}^{\infty} \ell(x,z)\psi_\ell(x)dx \\ &= -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta(z)C_\gamma(z)G_1(x_\beta - x_\gamma) + 2\sum_{\beta=0}^{N} C_\beta(z)G_1(z-x_\beta).\end{aligned} \tag{16}$$

3.3. The coefficients of the optimal quadrature formula

Let us assume that the nodes x_β in the interpolation formula (7) are fixed. The error functional (9) meets the requirement in (15). The norm of this error functional ℓ is a multidimensional function that depends on the coefficients $C_\beta(z)$ (where β ranges from 0 to N). To find the point where the expression (16) reaches its conditional minimum, while satisfying the condition in (15), we use the Lagrange method:

$$\Psi(C_0(z), C_1(z), \dots, C_N(z), d(z)) = \|\ell\|^2 + 2d(z)\,(\ell, e^{-x})\,.$$

Equating to 0 the partial derivatives of the function Ψ by $C_\beta(z)$ ($\beta = \overline{0, N}$), and $d(z)$, we get the following system of linear equations

$$\sum_{\gamma=0}^{N} C_\gamma(z) G_1(x_\beta - x_\gamma) + d(z) e^{-x_\beta} = G_1(z - x_\beta), \tag{17}$$

$\beta = 0, 1, \dots, N,$

$$\sum_{\gamma=0}^{N} C_\gamma(z)\; e^{-x_\gamma} = e^{-z}, \tag{18}$$

where $G_1(x)$ is defined by equality (14).

Therefore, in fixed values of the nodes x_β the square of the norm of the error functional ℓ, being quadratic function of the coefficients $C_\beta(z)$, has a unique minimum in some certain values $C_\beta(z) = \mathring{C}_\beta(z)$.

In this work, we do not focus on the algorithm for solving the system of Eqs. (17) and (18). For more details, refer to [14]. Instead, we simply present the solution.

Theorem ([14]). *Coefficients of the optimal interpolation formula (7) with equally spaced nodes in the space $W_2^{(1,0)}(0,1)$ have the following form*

$$\begin{aligned}\mathring{C}_\beta(z) \;=\; &\frac{1}{2(1-e^{2h})} \Bigg[\operatorname{sgn}(z - h\beta - h) \cdot (e^{h\beta + 2h - z} - e^{z - h\beta}) \\ &+ \operatorname{sgn}(z - h\beta + h) \cdot (e^{h\beta - z} - e^{z - h\beta + 2h}) \\ &+ (1 + e^{2h}) \cdot \operatorname{sgn}(z - h\beta) \cdot (e^{z - h\beta} - e^{h\beta - z}) \Bigg], \quad \beta = 0, 1, \dots, N.\end{aligned} \tag{19}$$

Hence we obtain

$$\|\ell\|^2_{W_2^{1,0}(0,1)} = \varepsilon_{(h\beta, h(\beta+1)]}(z) \cdot \frac{e^{2z - 2h\beta} + e^{2h\beta - 2z} - 1 - e^{2h}}{2(1 - e^{2h})}, \quad \beta = 0, 1, \dots, N-1,$$

where $\varepsilon_{(h\beta,h(\beta+1)]}(z)$ is the indicator of the interval $(h\beta, h(\beta+1)]$.

3.4. Optimal coefficients as basis function

Now, we discuss the use of these coefficient in place of the basis function in (6).

We define the basis functions ϕ_1, …, ϕ_{N-1} based on (19) as follows:

$$\phi_i(x) = \begin{cases} \frac{e^{-x+hi} - e^{2h+x-hi}}{1-e^{2h}}, & h(i-1) \le x < hi, \\ \frac{e^{x-hi} - e^{2h-x+hi}}{1-e^{2h}}, & hi \le x < h(i+1), \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

We write $\phi_0(x)$ and $\phi_N(x)$ separately according to (19)

$$\phi_0(x) = \frac{e^{x-x_0} - e^{2h-x+x_0}}{1-e^{2h}},$$

$$\phi_N(x) = \frac{e^{1-x} - e^{2h+x-1}}{1-e^{2h}}.$$

These functions are boundary basis functions included to satisfy the boundary conditions of the finite element space constructed for the Galerkin method.

3.5. Theoretical analysis and convergence

Let u be the exact solution to the Fredholm integral equation of the second kind and u_N be the Galerkin approximation constructed in the finite-dimensional subspace $\mathbb{V}_n \subset W_2^{(1,0)}$. Under the assumption that the kernel $K(x,t)$ is continuous and the operator is compact, the Galerkin method satisfies the Céa's Lemma:

$$\|u - u_N\| \le C \inf_{v \in \mathbb{V}_n} \|u - v\|,$$

where C is a constant that depends on the operator norm and problem geometry.

Since the basis functions derived from the optimal interpolation formula provide an approximation order of $\mathcal{O}(h)$, the overall error satisfies:

$$\|u - u_N\| = \mathcal{O}(h).$$

This confirms that the Galerkin method with these basis functions is convergent with the same order as the interpolation accuracy in the underlying space $W_2^{(1,0)}$.

4. Numerical results

In this section, we present an example of an algorithm for solving an integral equation using the Galerkin method. In this section, we mainly use the functions (6) and (20) as basis functions in the Galerkin method. And we compare the approximations to the exact solution of the integral equation in both cases.

Algorithm 1 Galerkin Method for Solving an Integral Equation

Define the problem:
Given the Fredholm integral equation of the second kind:

$$u(x) - \int_a^b K(x,t)u(t)dt = f(x)$$

where $K(x,t)$ is the kernel function, $f(x)$ is the given function.
Set parameters:
a and b ▷ Integration limits
n ▷ Number of basis functions
$h \leftarrow \frac{b-a}{n}$ ▷ Step size
Define the initial approximation:

- For choosing piecewise linear basis

$$\psi_0(x) = \begin{cases} \frac{x_0 - x}{h}, & 0 \le x < h, \\ 0, & \text{otherwise.} \end{cases}$$

- For choosing exponential basis

$$\psi_0(x) = \begin{cases} \frac{e^{x-x_0} - e^{2h-x+x_0}}{1-e^{2h}}, & 0 \le x < h, \\ 0, & \text{otherwise.} \end{cases}$$

Define basis functions:
Choose one of the following:

- Piecewise linear basis (6) for $i = 1, \dots, n$.
- Exponential basis (20) for $i = 1, \dots, n$.

Compute stiffness matrix A**:**
for $i = 1$ to n **do**
  for $j = 1$ to n **do**
    $A_{ij} \leftarrow \int_a^b \phi_i(x)\phi_j(x)dx$
  end for
end for
Compute coupling matrix B**:**
for $i = 1$ to n **do**
  for $j = 1$ to n **do**
    $B_{ij} \leftarrow \int_a^b \phi_i(x) \int_a^b \phi_j(t)K(x,t)\,dt\,dx$
  end for
end for
Assemble load vector b**:**
for $i = 1$ to n **do**
  $b_i \leftarrow \int_a^b \phi_i(x) \left(f(x) - \psi_0(x) + \int_a^b K(x,t)\psi_0(t)dt \right) dx$
end for
Solve the linear system:
$c \leftarrow (A - B)^{-1} b$
Compute the approximate solution:
$u_N(x) \leftarrow \psi_0(x) + \sum_{i=1}^n c_i\phi_i(x)$
Return $u_N(x)$ as the approximate solution.

Example 1. Consider the integral equation

$$u(x) - \int_0^1 tu(t)dt = e^x - 1, \quad 0 \le x \le 1.$$

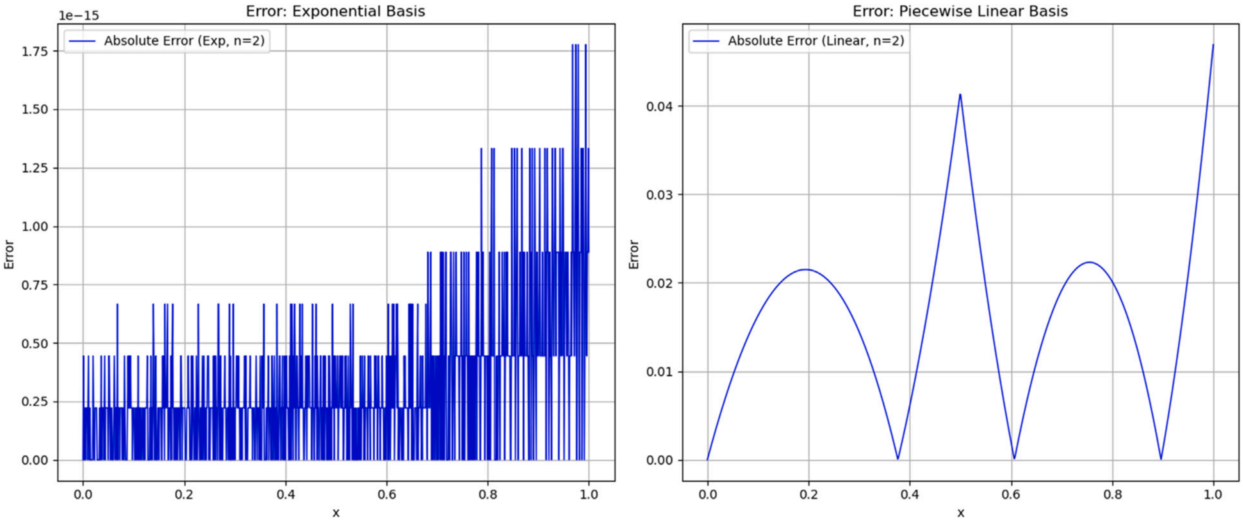

Fig. 1. Graphs of absolute errors of obtained approximate solutions using the Galerkin method with linear and exponential basis for N=2 finite elements.

Table 1
Mean absolute error for exponential basis (MAEEB) and piecewise linear basis (MAEPLB)

n	MAEEB	MAEPLB
2	0.0681	0.0347
4	0.0146	0.0073
8	0.0034	0.0017

The exact solution of this integral equation is $u(x) = e^x$. We solve this integral equation using Algorithm 1. We analyze the graphs of the absolute errors of approximations to the exact solution of the approximate solutions found using both basis sets for various values of N (see Fig. 1).

The graphs clearly demonstrate that the exponential basis functions yield significantly lower approximation errors compared to the piecewise linear basis, especially for lower values of N. This behavior is expected since the exponential basis is constructed to exactly reproduce functions like e^x and e^{-x}, which are closely aligned with the nature of the exact solutions considered in the test problems.

Example 2. Consider the integral equation

$$u(x) - \frac{1}{2}\int_0^{\pi/2} \sin(x)u(t)dt = \cos(x), \quad 0 \le x \le \frac{\pi}{2}.$$

The exact solution of this integral equation is $u(x) = \sin(x)+\cos(x)$. We approximately solve this integral equation using Algorithm 1. We analyze the graphs of the absolute errors of approximations to the exact solution of the approximate solutions found using both basis sets for various values of N (see Figs. 2–4).

Table 1 quantifies the error behavior for both basis sets. We observe that as N increases, the error decreases in both cases, consistent with the expected convergence of the Galerkin method. Notably, the exponential basis achieves approximately twice the accuracy of the piecewise linear basis across all tested values of N. For instance, at $N = 8$, the mean absolute error for the exponential basis is 0.0034, compared to 0.0017 for the linear basis. It is clear from this that the basis function should be chosen depending on the given integral equation.

These numerical findings confirm that the optimal interpolation-based exponential basis functions provide superior approximation properties when the exact solution is exponential. However, for functions with trigonometric or polynomial behavior, piecewise linear bases may offer better performance. Therefore, the choice of basis should be guided by the underlying properties of the solution.

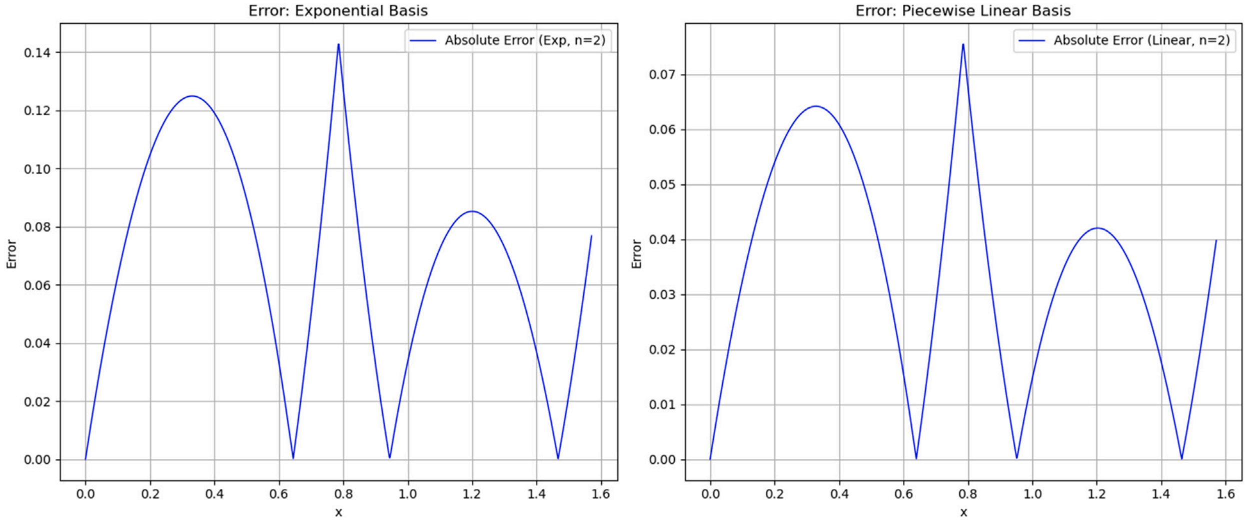

Fig. 2. Graphs of absolute errors of obtained approximate solutions using the Galerkin method with linear and exponential basis for N=2 finite elements.

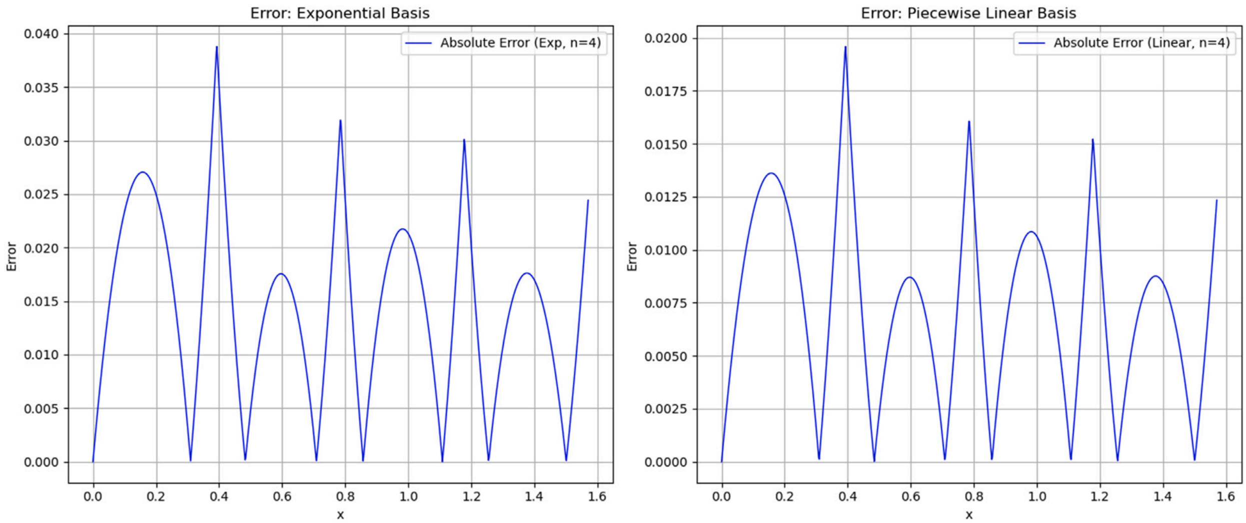

Fig. 3. Graphs of absolute errors of obtained approximate solutions using the Galerkin method with linear and exponential basis for N=4 finite elements.

Conclusion and discussion of numerical results

In this study, we explored the Galerkin method for solving Fredholm integral equations of the second kind, utilizing basis functions derived from the optimal interpolation formula in the Hilbert space $W_2^{(1,0)}$. Piecewise linear and exponential basis functions were employed to construct approximate solutions.

The numerical results reveal several important insights into the performance of the proposed basis functions. Firstly, the exponential basis functions derived from the optimal interpolation formula in $W_2^{(1,0)}$ produce errors close to machine zero for problems where the exact solution exhibits exponential behavior, as seen in Example 1. This high level of accuracy confirms the theoretical optimality of the constructed basis.

Secondly, for the problem in Example 2 involving trigonometric terms, the piecewise linear basis yields slightly better accuracy, particularly for small N. This suggests that while optimal interpolation-based bases are powerful, their effectiveness may depend on the class of functions being approximated.

Finally, the comparison of error magnitudes in Table 1 shows that both bases have convergence of order $\mathcal{O}(h)$, but their constants differ. This highlights the importance of selecting basis functions that align closely with the characteristics of the underlying solution.

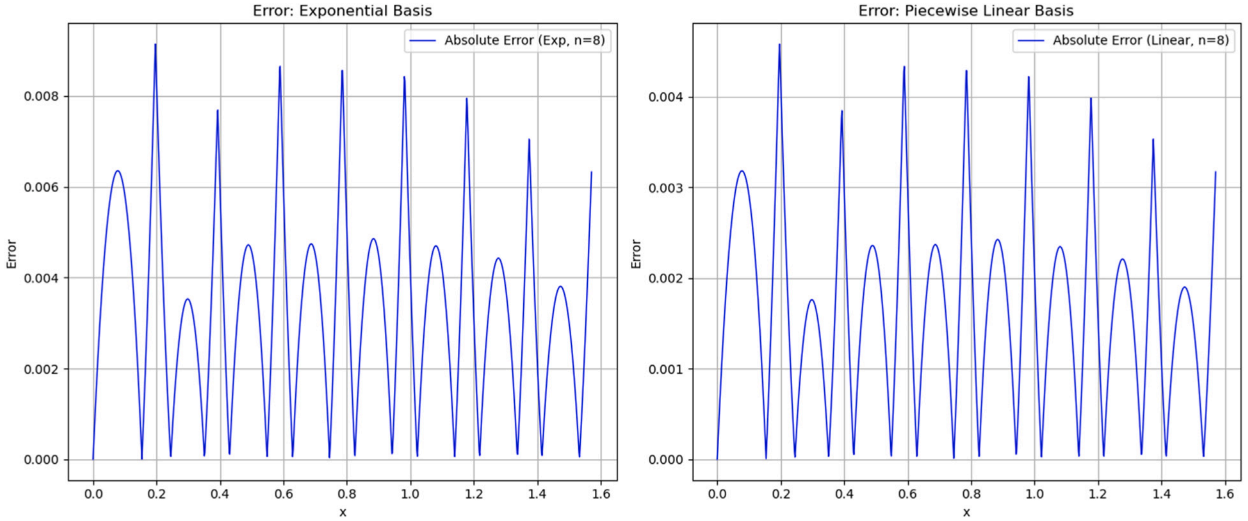

Fig. 4. Graphs of absolute errors of obtained approximate solutions using the Galerkin method with linear and exponential basis for N=8 finite elements.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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