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Initial-boundary problem for $2D$ system of viscoelasticity

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2 oʻlchamli yopishqoq elastiklik sistemasi uchun boshlangʻich-chegaraviy masala.
Izotropik muhitda ikki oʻlchamli yopishqoq elastiklik integro-differensial tenglamalar sistemasi uchun kuchlanish vektori va tezligini aniqlashning boshlangʻich-chegaraviy masalasi oʻrganilgan. Masala yechimining fazoviy oʻzgaruvchilardan biri boʻyicha Furye almashtirishiga nisbatan ikkinchi tur Volterra tipidagi integral tenglamalar sistemasini yechishga keltirilgan. Uzluksiz funksiyalar sinfida bu sistemaga ketma-ket yaqinlashishlar usuli qoʻllanilgan. Qoʻyilgan masalaning yechimlari mavjudligi va yagonaligini ifodalovchi teorema isbotlangan.
Kalit soʻzlar: giperbolik sistema; toʻgʻri masala; integral tenglamalar; qoʻzgʻalmas nuqta haqidagi teorema.

Начально-краевая задача для 2D-системы вязкоупругости.
Для двумерной системы интегро-дифференциальных уравнений вязкоупругости в изотропной среде изучается прямая задача определения вектора напряжения и скорости частиц. Задача сводится к эквивалентной системе интегральных уравнений второго рода типа Вольтерра относительно преобразования Фурье по одной из пространственных переменных решения прямой задачи. Далее к этой системе применяется метод последовательных приближений в классе непрерывных функций. Таким образом, доказывается существование и единственность решения поставленной задачи.
Ключевые слова: гиперболическая система; прямая задача; система уравнений вязкоупругости, интегральное уравнение, принцип сжатых отображений.

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Introduction. Canonical form of viscoelasticity equations

In this work, the deformation of flat bodies (for example, fiberglass) is considered taking into account the viscoelastic properties. Writing for this case the system of equations of the theory of elasticity in stresses and velocities of particles as a system of equations of the first order, for it the direct problem is studied. The direct problem is the initial-boundary problem for this system.

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Let $\overline{x} = (x_1, x_2) \in \mathbb{R}^2$. We denote by σ_{ij} the projection onto the axis x_i of the stress acting on the area with the normal parallel to the axis x_j, and $\overline{u}_i$ is the projection onto the axis x_i of the particle displacement vector. According to Hooke's law for viscoelastic media, stresses and deformations are related by the formulas [1, p.242]:

$$\sigma_{ij}(\overline{x},t) = \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \mathrm{div}\overline{u} +$$

$$+\int_0^t K_{ij}(t-\tau)\left[\mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \mathrm{div}\overline{u}\right](\overline{x},\tau)\,d\tau, \tag{1}$$

where $\lambda = \lambda(x_2),\ \mu = \mu(x_2)$ are Lame coefficients, $\lambda > 0,\ \mu > 0,\ \delta_{ij}$ is the Kronecker symbol, $K_{ij}(t)$ are the functions corresponding to the viscosity the medium and $K_{ij} = K_{ji},\ i,j = 1,2$.

The equations of motion of particles of a flat body in the absence of external forces have the form

$$\rho\frac{\partial^2 \overline{u}_i}{\partial t^2} = \sum_{j=1}^{2}\frac{\partial \sigma_{ij}}{\partial x_j}, i = 1,2, \tag{2}$$

where $\rho = \rho(x_2) > 0$ is the medium density, $\overline{u}(\overline{x},t) = \big(\overline{u}_1(\overline{x},t), \overline{u}_2(\overline{x},t)\big)$ is the displacement vector.

Note that (1) can be considered as the integral equations of Volterra of the second kind with respect to the expression $\mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \mathrm{div}\overline{u}$. For each fixed pair (i,j) solving these equations, we obtain

$$\sigma_{ij}(\overline{x},t) = \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \mathrm{div}\overline{u} + \int_0^t r_{ij}(t-\tau)\sigma_{ij}\left(\overline{x},\tau\right)d\tau, \tag{3}$$

here r_{ij} are the resolvents of the kernels K_{ij} and they are related by integral relations [2]:

$$r_{ij}(t) = -K_{ij}(t) - \int_0^t K_{ij}(t-\tau)r_{ij}(\tau)d\tau, \quad i,j = 1,2. \tag{4}$$

From the condition $K_{ij} = K_{ji}$ implies $r_{ij} = r_{ji}$.

Differentiating (3) with respect to t and introducing the notation $u_i = \frac{\partial}{\partial t}\overline{u}$, we obtain

$$\frac{\partial}{\partial t}\sigma_{ij}(\overline{x},t) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda \mathrm{div}u + r_{ij}(0)\sigma_{ij}(\overline{x},t) + \int_0^t r'_{ij}(t-\tau)\sigma_{ij}\left(\overline{x},\tau\right)d\tau. \tag{5}$$

Taking this into account, the system of equations (1) and (2) for the velocity u_i and stress σ_{ij} can be written in the form of a system of five first-order integro-differential equations. For convenience, denoting $x_1 = x,\ x_2 = y$, we have

$$A\frac{\partial U}{\partial t} + B\frac{\partial U}{\partial x} + C\frac{\partial U}{\partial y} + DU = \int_0^t R(t-\tau)U(\overline{x},\tau)d\tau, \tag{6}$$

where $U = (u_1, u_2, \sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21})^*$, $*$ is the sign transposition,

$$A = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -(\lambda+2\mu) & 0 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & -(\lambda+2\mu) & 0 & 0 & 0 \\ -\mu & 0 & 0 & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_{11}(0) & 0 & 0 \\ 0 & 0 & 0 & -r_{22}(0) & 0 \\ 0 & 0 & 0 & 0 & -r_{12}(0) \end{pmatrix},$$

$$R(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r'_{11}(t) & 0 & 0 \\ 0 & 0 & 0 & -r'_{22}(t) & 0 \\ 0 & 0 & 0 & 0 & -r'_{12}(t) \end{pmatrix}.$$

System (6) can be reduced to a symmetric hyperbolic system [3, pp.162-169].

System (6) is reduced to canonical form with respect to the variables t and y. For this, we multiply (6) on the left hand on A^{-1} and write the equation

$$|A^{-1}C - \nu I| = 0, \tag{7}$$

where I is the unit matrix, the dimension 5. No, we find roots of (7) respect to the ν :

$$\nu_{1,2} = \pm\nu_s = \sqrt{\frac{\mu}{\rho}},\ \nu_{3,4} = \pm\nu_p = \sqrt{\frac{\lambda+2\mu}{\rho}},\ \nu_5 = 0. \tag{8}$$

Here, ν_s and ν_p determine the velocities of the transverse and longitudinal seismic waves, respectively.

Now we choose a nondegenerate matrix $T(y,t)$, so

$$T^{-1}A^{-1}CT = \Lambda, \tag{9}$$

where Λ is a diagonal matrix, the diagonal of which contains the eigenvalues (for each fixed x, defined by (8)) of the matrix $A^{-1}C$, i.e.

$$\Lambda = diag(\nu_s, -\nu_s, \nu_p, -\nu_p, 0).$$

According to the Formula (9), implies

$$A^{-1}CT = T\Lambda,$$

which means that the i th column of the matrix T is an eigenvector of the matrix $A^{-1}T$, corresponding to the eigenvalue λ_i. Direct calculations show that a matrix T, satisfying the above conditions can be chosen as (not the only way)

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{\rho(\lambda+2\mu)}} & \frac{1}{\sqrt{\rho(\lambda+2\mu)}} & 0 \\ 0 & 0 & \frac{\lambda}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & 1 \\ 0 & 0 & 1 & 1 & 0 \\ -\sqrt{\mu\rho} & \sqrt{\mu\rho} & 0 & 0 & 0 \end{pmatrix}.$$

We introduce the vector function U by the equality

$$U = T\vartheta.$$

Carring out this change in equation (6) and then multiplying it on the left by T^{-1}, we obtain

$$I\frac{\partial\vartheta}{\partial t} + \Lambda\frac{\partial\vartheta}{\partial y} + B_1\frac{\partial\vartheta}{\partial x} + D_1\vartheta = \int_0^t R_1(y,t-\tau)\vartheta(\overline{x},\tau)d\tau, \tag{10}$$

where

$$B_1 = T^{-1}A^{-1}BT = \begin{pmatrix} 0 & 0 & a_1 & a_2 & -\frac{1}{2\rho} \\ 0 & 0 & a_2 & a_1 & -\frac{1}{2\rho} \\ a_3 & a_4 & 0 & 0 & 0 \\ a_4 & a_3 & 0 & 0 & 0 \\ a_5 & a_5 & 0 & 0 & 0 \end{pmatrix},$$

$a_i = a_i(y),\ a_1 = -\frac{\sqrt{\mu}}{2\rho\sqrt{\lambda+2\mu}} - \frac{\lambda}{2\rho(\lambda+2\mu)},\ a_2 = \frac{\sqrt{\mu}}{2\rho\sqrt{\lambda+2\mu}} - \frac{\lambda}{2\rho(\lambda+2\mu)},\ a_3 = -\frac{\lambda}{2} - \frac{\sqrt{\mu(\lambda+2\mu)}}{2},\ a_4 = -\frac{\lambda}{2} + \frac{\sqrt{\mu(\lambda+2\mu)}}{2},$
$a_5 = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu},$

$$D_1(y,t) = T^{-1}A^{-1}C\frac{\partial T}{\partial y} + T^{-1}DT = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{pmatrix},$$

$c_{ij} = c_{ij}(y),\ i,j = \overline{1,5}\ \ c_{11} = -c_{12} = \frac{1}{2\rho}\frac{\partial}{\partial y}\left(\sqrt{\mu\rho}\right) - \frac{r_{33}(0)}{2},\quad c_{21} = -c_{22} = \frac{1}{2\rho}\frac{\partial}{\partial y}\left(\sqrt{\mu\rho}\right) + \frac{r_{33}(0)}{2},\quad c_{33} = c_{43} = \frac{\lambda+2\mu}{2}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{\rho(\lambda+2\mu)}}\right) - \frac{r_{22}(0)}{2},\quad c_{34} = c_{44} = \frac{\lambda+2\mu}{2}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{\rho(\lambda+2\mu)}}\right) - \frac{r_{22}(0)}{2},\quad c_{53} = c_{54} = \frac{\lambda}{\lambda+2\mu}\left(r_{22}(0) - r_{11}(0)\right),$
$c_{55} = -r_{11}(0),\ \ c_{lp} = c_{pl} = 0,\ l = 1,2,\ p = 3,4,5,\ \ c_{35} = c_{45} = 0,$

$$R_1(y,t) = T^{-1}A^{-1}RT = \begin{pmatrix} \widetilde{r}_{11} & \widetilde{r}_{12} & \widetilde{r}_{13} & \widetilde{r}_{14} & \widetilde{r}_{15} \\ \widetilde{r}_{21} & \widetilde{r}_{22} & \widetilde{r}_{23} & \widetilde{r}_{24} & \widetilde{r}_{25} \\ \widetilde{r}_{31} & \widetilde{r}_{32} & \widetilde{r}_{33} & \widetilde{r}_{34} & \widetilde{r}_{35} \\ \widetilde{r}_{41} & \widetilde{r}_{42} & \widetilde{r}_{43} & \widetilde{r}_{44} & \widetilde{r}_{45} \\ \widetilde{r}_{51} & \widetilde{r}_{52} & \widetilde{r}_{53} & \widetilde{r}_{54} & \widetilde{r}_{55} \end{pmatrix},$$

$\widetilde{r}_{ij} = \widetilde{r}_{ij}(y,t),\ i,j = \overline{1,5}\ \ \widetilde{r}_{lp} = \widetilde{r}_{pl} = 0,\ l = 1,2,\ p = 3,4,5,\quad \widetilde{r}_{35} = \widetilde{r}_{45} = 0,\quad \widetilde{r}_{11} = -\widetilde{r}_{12} = -\widetilde{r}_{21} = \widetilde{r}_{22} = -\frac{r'_{12}(t)}{2},\quad \widetilde{r}_{ij} = -\frac{r'_{22}(t)}{2},\quad i,j = 3,4,\ \widetilde{r}_{53} = \widetilde{r}_{54} = \frac{\lambda}{\lambda+2\mu}\left(r'_{22}(t) - r'_{11}(t)\right),\quad \widetilde{r}_{55} = -r'_{11}(t).$

System (10) is convenient in that it disintegrated with respect to the derivatives t and y, and turns out to be split only through $\frac{\partial\vartheta}{\partial x}$ and ϑ. The components $\vartheta_i,\ i = \overline{1,4}$ of the vector of the function are called the Riemann invariants of system (6). They remain constant along the characteristic of the system (10) in the case when $B_1 = 0,\ D_1 = 0,\ R_1 = 0.$

The statement of a problem

Let us consider the system of equations (10) in the domain

$$D = \{(x,y,t) : x \in \mathbb{R}, 0 < y < H, t > 0\},\quad H = const$$

with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$:

$$\Gamma_0 = \{(x,y,t) : x \in \mathbb{R}, 0 \le y \le H, t = 0\},$$

$$\Gamma_1 = \{(x,y,t) : x \in \mathbb{R}, y = 0, t > 0\},$$

$$\Gamma_2 = \{(x,y,t) : x \in \mathbb{R}, y = H, t > 0\}.$$

For this system, we pose the direct problem as follows: determine the solution of the system of equations (10) in $\overline{D} = D \cup \Gamma$ from the data on Γ

$$\vartheta_i\Big|_{\Gamma_0} = \varphi_i(x,y),\ \ i = \overline{1,5}, \tag{11}$$

$$\vartheta_i\Big|_{\Gamma_1} = \psi_i(x,t),\ \ i = 1,3,\quad \vartheta_i\Big|_{\Gamma_2} = \psi_i(x,t),\ \ i = 2,4. \tag{12}$$

It is known that [4,p.164–5, sec.4] problem (10), (11), (12) is well posed. Suppose that the functions $\varphi_i(x,y),\ \psi_i(x,y)$ are finite in x for each fixed y,t and have smoothness of some order. Note that, the class of functions satisfying these conditions are not empty (see, for example, [6]).

The existence of a finite domain of dependence for system (10) and the finiteness in x of the data (11) and (12) implies the finiteness in x of solutions v_i to problem (10), (11) and (12). Then to functions ϑ_i can be applied Fourier transform in x. Denote $V_i(y,t) := \widetilde{\vartheta}_i(\xi,y,t)\Big|_{\xi=0}$, where $\widetilde{\vartheta}_j(\xi,y,t) = \int\limits_{\mathbb{R}} e^{i\xi x}\vartheta_j(x,y,t)dx,\ j = \overline{1,4},\ \xi \in \mathbb{R}$ parameter. Direct calculation show that, $V(y,t) = V_i, (i = \overline{1,5})$ satisfies the equation

$$I\frac{\partial V}{\partial t} + \Lambda\frac{\partial V}{\partial y} + D_1 V = \int\limits_0^t R_1(y,t)V(y,t-\tau)d\tau. \tag{13}$$

Relations (11), (12) correspond to the conditions

$$V_i\Big|_{\widetilde{\Gamma}_0} = \widetilde{\varphi}_i(y),\ \ i = \overline{1,5}, \tag{14}$$

$$V_i\Big|_{\widetilde{\Gamma}_1} = \widetilde{\psi}_i(t),\ i=1,3,\ V_i\Big|_{\widetilde{\Gamma}_2} = \widetilde{\psi}_i(t),\ i=2,4, \tag{15}$$

where $\widetilde{\Gamma}_i$ is the projection of $\Gamma_j,\ j=0,1,2$ onto the plane $y,t,\ \widetilde{\varphi}_i(y),\ i=\overline{1,5},\ \widetilde{\psi}_i(t),\ i=\overline{1,4}$ are the Fourier images of the corresponding functions from (11), (12) for $\xi=0$. We also denote by D_H the projection of D onto the plane y,t. In what follows, we will consider the system of equations (13) in the domain $D_H\cup\widetilde{\Gamma}$ under the conditions (14) and (15).

Main result and its proof

Let us pass from equalities (13)-(15) to integral relations for the components of the vector V. For this, in the plane of the variables η,τ, consider an arbitrary point $(y,t)\in D_H\cup\widetilde{\Gamma}$ and passing through its characteristic corresponding to the ith diagonal element of the matrix Λ. Recall that the characteristics corresponding to ν_s and ν_p have a positive slope, and the characteristics corresponding to $-\nu_s$ and $-\nu_p$ have a negative slope. We denote

$$\mu_1(y)=\int_0^y \frac{d\beta}{\nu_s(\beta)},\quad \mu_2(y)=\int_0^y \frac{d\beta}{\nu_p(\beta)}.$$

The functions inverse to $t=\mu_1(y), t=\mu_2(y)$ will be denoted by $y=\mu_1^{-1}(t), y=\mu_2^{-1}(t)$. Using the introduced functions, the equation of characteristics passing through the points (y,t) on the plane of the variables η,τ can be written in the form

$$\tau=t+(-1)^{i+1}(\mu_j(\eta)-\mu_j(y)),\quad j=1,2,\quad i=\overline{1,4} \tag{16}$$

for the equations of system (13).

Consider an arbitrary point $(y,t)\in D_H$ on the plane of variables η,τ and draw through it the characteristic of the ith equation of system (13) up to the intersection in the region $\tau\le t$ with the boundary $\widetilde{\Gamma}$. Point intersections are denoted by (y_0^i,t_0^i). For the first and third equations, this point lies either on $\widetilde{\Gamma}_0$ or $\widetilde{\Gamma}_1$, and for the second and fourth equations, either on $\widetilde{\Gamma}_0$ or $\widetilde{\Gamma}_2$. By integrating the first four equations of system (13) along the corresponding characteristics from the point (y_0^i,t_0^i) to the point (y,t) we find

$$V_i(y,t)=V_0^i(y_0^i,t_0^i)+$$

$$+\int_{t_0^i}^t\left[\sum_{j=1}^4 c_{ij}(\eta)V_j(\eta,\tau)+\int_0^\tau\sum_{j=1}^4\widetilde{r}_{ij}(\eta,\alpha)V_j(\eta,\tau-\alpha)d\alpha\right]\Bigg|_{\eta=\mu_{m(i)}^{-1}\left[(-1)^{i+1}(\tau-t)+\mu_{m(i)}(y)\right]}d\tau,\ i=\overline{1,4}, \tag{17}$$

where $m(i)=\begin{cases}1, & i=1,2\\ 2, & i=3,4.\end{cases}$

The fifth equation of (13) is equivalent to the following integral equation:

$$V_5(y,t)=\int_0^t\left[\sum_{j=1}^5 c_{ij}(y)V_j(y,\tau)+\int_0^\tau\sum_{j=1}^5\widetilde{r}_{ij}(y)V_j(y,\tau)d\alpha\right]d\tau. \tag{18}$$

Define t_0^i in (17), (18). It depends on the coordinates of the point (y,t). It is easy to see that $t_0^i(y,t)$ has the form

$$t_0^i(y,t)=\begin{cases} t+(-1)^i\mu_{m(i)}(y)+\gamma_i\mu_{m(i)}(H), & t\ge\mu_{m(i)}(y),\\ 0, & 0<t<\mu_{m(i)}(y),\quad i=\overline{1,4},\end{cases}$$

where $\gamma_i=\begin{cases}1, & i=1,3,\\ 0, & i=2,4.\end{cases}$

Then, from the condition that the pair (y_0^i, t_0^i) satisfies the equation (17), it follows

$$y_0^i(y,t) = \begin{cases} \gamma_i H, & t \geq \mu_{m(i)}(y), \\ \mu_1^{-1}\left(\mu_{m(i)}(y) + (-1)^i t\right), & 0 < t < \mu_{m(i)}(y), \quad i = \overline{1,4}. \end{cases}$$

The free terms of integral equations (17) are defined in terms of the initial boundary conditions (14) and (15) as follows:

$$V_0^i(y_0^i, t_0^i) = \begin{cases} \widetilde{\psi}_i\left(t + (-1)^i \mu_{m(i)}(y) + \gamma_i \mu_{m(i)}(H)\right), & t \geq \mu_{m(i)}(y), \\ \widetilde{\varphi}_i\left(\mu_{m(i)}^{-1}(\mu_{m(i)}(y) + (-1)^i t)\right), & 0 < t < \mu_{m(i)}(y), \quad i = \overline{1,4}. \end{cases}$$

We require the functions $\widetilde{V}_i(z_0^i, t_0^i)$ to be continuous in the domain D. Note that, to satisfy these conditions, the given functions $\widetilde{\varphi}_i$ and $\widetilde{\psi}_i$ should be satisfy the matching conditions at the corner points of the domain D_H:

$$\widetilde{\varphi}_i(0) = \widetilde{\psi}_i(0), \quad i = 1,3; \qquad \widetilde{\varphi}_i(0) = \widetilde{\psi}_i(H), \quad i = 2,4. \tag{19}$$

Here and below, the values of the functions $\widetilde{\psi}_i$ at $t = 0$ and functions $\widetilde{\varphi}_i$ at $y = 0$ and $y = H$ are understood as the limit at these points as the argument tends from the right side of the point where these functions are defined

Suppose that all given functions in (17), (18) are continuous functions of their arguments in D_H. Then this system of equations is a closed system of Voltaire-type integral equations of the second kind with continuous kernels and free terms. As usual, such a system has a unique solution in the bounded subdomain $D_{HT} = \{(y,t) : 0 < y < H,\ 0 < t < T\}$, where $T > 0$ is some fixed number, of D_H.

Let us introduce the vector function $\omega(y,t) = \frac{\partial V}{\partial t}(y,t)$. To obtain a problem for a function $\omega(y,t)$ similar to (13) - (15), we differentiate equations (13) and boundary conditions (17) with respect to the variable t, and the condition at $t = 0$ is found using equations (13) and initial conditions (14). Thus we get

$$\frac{\partial \omega_i}{\partial t} + \lambda_i \frac{\partial \omega_i}{\partial y} + \sum_{j=1}^{4} c_{ij}(y)\omega_j(y,t) = \int_0^t \sum_{j=1}^{4} \widetilde{r}_{ij}(\eta,\tau)\omega_j(\eta, t-\tau)d\tau + \sum_{j=1}^{4} \widetilde{r}_{ij}(y,t)\widetilde{\psi}_j(y), \quad i = \overline{1,4}, \tag{20}$$

$$\frac{\partial \omega_5}{\partial t} + \sum_{j=1}^{5} c_{5j}(y)\omega_j(y,t) = \int_0^t \sum_{j=1}^{5} \widetilde{r}_{5j}(y,\tau)\omega_j(y, t-\tau)d\tau + \sum_{j=1}^{5} \widetilde{r}_{ij}(y,t)\widetilde{\psi}_j(y), \tag{21}$$

$$\omega_i(y,t)\Big|_{t=0} = \Phi_i(y), i = \overline{1,5}, \tag{22}$$

$$\omega_i(y,t)\Big|_{y=0} = \frac{d\widetilde{\psi}_i(t)}{dt}, i = 1,3, \ \omega_i\Big|_{y=H} = \frac{d\widetilde{\psi}_i(t)}{dt}, i = 2,4, \tag{23}$$

where

$$\Phi_i(y) = -\lambda_i \frac{d\widetilde{\varphi}_i(y)}{dy} - \sum_{j=1}^{5} c_{ij}(y)\widetilde{\varphi}_i(y), \quad i = \overline{1,5}. \tag{24}$$

The main result of this paper is following assertion:

Theorem. *Let be $\rho(y) \in C^1[0,\infty)$, $\mu(y) \in C^1[0,\infty)$, $\lambda(y) \in C^1[0,\infty)$, $\widetilde{\varphi}(y) \in C^1[0,\infty)$, $\widetilde{\phi}(t) \in C^1[0,\infty]$, $\rho(y) > 0$, $\lambda(y) > 0$, $\mu(y) > 0$, $K_{ij}(t) \in C^1[0,\infty)$, $i,j = 1,2$, and conditions (19), (28) and (29) are met. Then, there is a unique classical solution to the problem (20)-(23) in the domain D_{HT}.*

Proof. Integration along the corresponding characteristics again leads to the problem (20)-(23) to the integral equations

$$\omega_i(y,t) = \omega_i(y_0^i, t_0^i) + \int_{t_0^i}^{t} \Big[\sum_{j=1}^{4} (-c_{ij}(\eta)\omega_j(\eta,\tau) + \widetilde{r}_{ij}(\eta,\tau)\widetilde{\varphi}_j(\eta)) +$$

$$+\int_0^\tau \sum_{j=1}^4 \widetilde{r}_{ij}(\eta,\alpha)\omega_j(\eta,\tau-\alpha)d\alpha\Big]\Big|_{\eta=\mu^{-1}_{m(i)}\left[(-1)^{i+1}(\tau-t)+\mu_{m(i)}(y)\right]} d\tau, \quad i=\overline{1,4}, \tag{25}$$

$$\omega_5(y,t)=\Phi_5(y)+\int_0^t \left[\sum_{j=1}^5 \left(-c_{5j}(y)\omega_j(y,\tau)+\widetilde{r}_{5j}(y,\tau)\widetilde{\varphi}_j(y)\right)+\int_0^\tau \sum_{j=1}^5 \widetilde{r}_{5j}(y,\alpha)\omega_j(y,\tau-\alpha)d\alpha\right]d\tau. \tag{26}$$

For functions ω_i, additional conditions (15) look like

$$\omega_i\Big|_{\Gamma_2,\xi=0}=\frac{d\widetilde{h}_i(t)}{dt},\ i=1,3,\quad \omega_i\Big|_{\Gamma_1,\xi=0}=\frac{d\widetilde{h}_i(t)}{dt},\ i=2,4,5. \tag{27}$$

In equations (25) $\omega_i(z_0^i,t_0^i)$ are defined as follows:

$$\omega_0^i(y_0^i,t_0^i)=\begin{cases} \frac{d\widetilde{\psi}_i}{dt}\left(t+(-1)^i\mu_{m(i)}(y)+\gamma_i\mu_{m(i)}(H)\right), & t\geq \mu_{m(i)}(y),\\ \Phi_i\left(\mu_{m(i)}(y)+(-1)^i t\right), & 0<t<\mu_{m(i)}(y), \quad i=\overline{1,4}.\end{cases}$$

Let the following conditions are fulfilled:

$$-\lambda_i\left[\frac{d\widetilde{\varphi}_i(y)}{dy}\right]_{y=0}-\sum_{j=1}^5 c_{ij}(0)\widetilde{\varphi}_i(0)=\left[\frac{d\widetilde{\psi}_i(t)}{dt}\right]_{t=0}, i=1,3, \tag{28}$$

$$-\lambda_i\left[\frac{d\widetilde{\varphi}_i(y)}{dy}\right]_{y=H}-\sum_{j=1}^5 c_{ij}(H)\widetilde{\varphi}_i(H)=\left[\frac{d\widetilde{\psi}_i(t)}{dt}\right]_{t=0}, i=2,4. \tag{29}$$

It is not difficult to notice that the conditions for the agreement of the initial (22) and boundary (23) data at the corner points of the domain D coincide with relations (28) and (29). Hence, it is clear that under the same equalities (28) and (29) the equations (25), (26) will have unique continuous solutions $\omega_i(y,t)$, or the same $\left(\frac{d}{dt}\right)V_i(y,t)$. The theorem is proved. □

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