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INVERSE PROBLEM FOR MOORE-GIBSON-THOMPSON EQUATION WITH INTEGRAL OVERDETERMINATION CONDITION

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Abstract. This article is dedicated to the solution of the initial-boundary value problem for the Moore-Gibson-Thompson equation and introduces the inverse problem of identifying the kernel using an additional integral condition. First, we prove the existence and uniqueness of the solution to the direct problem and provide a priori estimates for it. Then we consider a new problem that is equivalent to the direct problem, and we use it to investigate the inverse problem. By applying a fixed point theorem in a suitable Sobolev space, we obtain global existence and uniqueness results for the inverse problem.

Mathematics Subject Classification (2020): 35D30, 35N30, 45D05.
Key words: Moore-Gibson-Thompson equation, initial-boundary value problem, inverse problem, global existence, uniqueness.

1. Introduction. The theory of thermoelasticity has garnered significant interest from researchers and scientists due to its numerous applications across various fields. The fields of architecture, structural engineering, plasma physics, geophysics, aeronautics, missile technology, and steam turbine generators are among the most significant domains where this theory is applied. In a novel development, an uncoupled principle of thermoelasticity was proposed, in which elastic strain is considered

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independent of heat transfer, and vice versa. However, this hypothesis proved untenable when tested against the practical outcomes of numerous concrete problems. Subsequently, researchers proposed alternative theories of coupled thermoelasticity, which gained widespread recognition as the general thermoelastic theory. The concept has gained prominence in recent years for its aim to resolve the contradiction of the unlimited heat propagation rate. This description now applies, to some extent, when discussing practical applications such as high-speed energy transport and low-temperature, high-heat transfer engineering.

The classical coupled thermoelasticity theory (CTE) [4], proposed by Biot, predicts an implausible, infinite rate of heat propagation. To resolve the apparent contradiction between the extraordinary phenomenon of unlimited speed and the CTE theory, a new class of non-classical thermoelasticity models has been introduced. These new models, known as generalized thermoelasticity, aim to address the limitations of classical thermoelasticity by incorporating additional physical effects. The Lord–Shulman [26], Green–Lindsay [16], and Green–Naghdi [17]-[19] theories represent significant advancements in generalized thermoelasticity and are currently the primary focus of research in this field. Lord and Shulman [26] integrated the principle of heat flow rate into Fourier's law with thermal relaxation time, thereby formulating a theory of extended thermoelasticity that incorporates a thermal flux rate. Green and Lindsay [16] modified the energy equation, along with the relationship between Duhamel and Neumann, to yield two relaxation periods.

Green and Roychoudhuri proposed theories of thermoelasticity with and without energy dissipation [10] and introduced the three-phase-lag (TPL) thermoelastic model.

This model introduces three distinct phase delays: one in the heat flux vector, another in the temperature gradient, and the third in the gradient of the thermal displacement function. These delays aim to provide a more accurate representation of heat propagation in materials under thermoelastic conditions. The prevalent problems of thermoelasticity based on these new models have become increasingly significant [1, 2, 32].

The Moore-Gibson-Thompson equation (MGT) has garnered significant attention in recent years. This equation is derived from the linearization of a model for wave propagation in viscous, thermally relaxing fluids. The behaviour of acoustic waves depends on the properties of the medium, particularly regarding dispersion, dissipation, and nonlinear effects. The model equation accounts for the effects of viscosity and heat conductivity, as well as the influence of heat radiation on sound propagation.

In this study, we examine Stokes' work [31], which asserts that the MGT equation was introduced for the first time. The mathematical and physical analysis of the model equation is highly intriguing, and these authors have suggested several potential applications in nonlinear acoustics and thermal relaxation in viscous gases and fluids (see, e.g., [27], [29]). Additionally, we refer to Lasiecka et al. [8, 11, 20] for a list of references regarding the physical and mathematical motivations behind the MGT model.

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set, with a smooth boundary Γ, and let $T > 0$. Then, we consider the following direct problem

$$\begin{cases} u_{ttt} + u_{tt} - \Delta u_t - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau, x)d\tau = 0, & (0,T)\times\Omega, \\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1,\ u_{tt}(0,\cdot) = u_2, & \Omega, \\ u = 0, & [0,T]\times\Gamma. \end{cases} \tag{1}$$

The problem presented in equation (1) has garnered significant attention. First, Lasiecka and Wang established the case where the memory kernel exhibits a more general decay rate in the non-critical regime [21, 22, 23], and further results followed. See, for instance, [5]-[6], which studied the existence and uniqueness of solutions for a new class of MGT equation concerning the non-local mixed boundary value problem.

Now we set up the *inverse problem* of recovering the unknown the relaxation memory kernel coming from materials under viscoelastic effects, from knowledge of next the data for the solution u,

$$\int_\Omega \eta(x)u(t,x)dx = h(t),\ t \in [0,T]. \tag{2}$$

Condition (2) represents a space average measurement of the temperature. This inverse formulation is important for modelling various practical applications related to unknown potential and temperature. For instance, this formulation applies to various fields of human activity, including medicine, seismology, desalination of seawater, and the movement of liquids in porous media.

The theory of inverse problems in mathematical physics is widely applied to solve practical issues across nearly all fields of science, particularly in physics, economics, and ecology. Inverse problems for various types of partial differential equations or systems have been studied in many papers [13], [30]. In particular, the solvability of inverse problems in various formulations with different overdetermination conditions for partial differential equations has been extensively studied in papers [12], [14]. In [3, 24, 25], the authors studied the inverse problem of recovering a space-dependent coefficient of the MGT equation using knowledge of the trace of the solution on an open subset of the boundary. They proved the uniqueness of the coefficient and demonstrated Lipschitz stability using Carleman estimates.

Now, we provide some notations that will be used repeatedly in the subsequent sections. First, we define an operator A in $L^2(\Omega)$ by

$$Au := -\Delta u, \quad D(A) = H^2(\Omega)\cap H_0^1(\Omega).$$

Since A is a symmetric uniformly elliptic operator, the spectrum of A is entirely composed of eigenvalues and counting according to the multiplicities, we can set $0 < \lambda_1 < \lambda_2 \leq \dots$. By $e_k(x) \in H^2(\Omega)\cap H_0^1(\Omega)$ we denote the orthonormal eigenfunction corresponding to $\lambda_k : Ae_k = \lambda_k e_k$.

Let γ be an arbitrary real number. We introduce the power of operator A acting in $L^2(\Omega)$ according to the rule (see, e.g., [28])

$$A^\gamma g = \sum_{n=1}^{\infty} \lambda_n^\gamma g_n e_n,$$

where g_n is the Fourier coefficients of a function $g \in L^2(\Omega) : g_n = (g, e_n)$. The domain of this operator has the form

$$D(A^\gamma) = \left\{ p \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\gamma} (p, e_k)^2 < \infty \right\}, \quad A^\gamma p = \sum_{k=1}^{\infty} \lambda_k^{\gamma} (p, e_k) e_k$$

with the inner product $(p, v)_{D(A^\gamma)} = (A^\gamma p, A^\gamma v)_{L^2(\Omega)}$ and, respectively, the norm

$$\|p\|^2_{D(A^\gamma)} = \sum_{k=1}^{\infty} \lambda_k^{2\gamma} (p, e_k)^2.$$

We have $D(A^\gamma) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$ and $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$. Since $D(A^\gamma) \subset L^2(\Omega)$, identifying the dual $(L^2(\Omega))'$ with itself, we have $D(A^\gamma) \subset L^2(\Omega) \subset (D(A^\gamma))' = D(A^{-\gamma})$. For $f \in D(A^{-\gamma})$ and $p \in D(A^\gamma)$, by ${}_{-\gamma}\langle f, p \rangle_\gamma$ we denote the value which is obtained by operating f to p. Moreover, we denote $D(A^{-\gamma})$ is a Hilbert space with the norm:

$$\|p\|_{D(A^{-\gamma})} = \left(\sum_{k=1}^{\infty} \lambda_k^{-2\gamma} |{}_{-\gamma}\langle p, e_k \rangle_\gamma|^2 \right)^{\frac{1}{2}}.$$

We further note that

$${}_{-\gamma}\langle f, p \rangle_\gamma = (f, p) \quad \text{if } f \in L^2(\Omega) \text{ and } p \in D(A^\gamma)$$

(see e.g., Chapter V in [9]).

Here we give some notations:

- $F(t, \cdot) := g(t) * \Delta u(t, \cdot) \equiv \int\limits_0^t g(t-s) \Delta u(s, \cdot) ds$;
- $(f, l) = (f, l)_{L^2(\Omega)} = \int_\Omega f(x) l(x) dx$;
- For a given Banach space V on Ω, we use the notation $C([0,T]; V)$ to denote the following spaces:

$$C([0,T]; V) := \{u : \|u(\cdot, t)\|_V \text{ is continuous in } t \text{ on } [0,T]\};$$

- Norm of space $C([0,T]; V)$: $\|u\|_{C([0,T];V)} = \max\limits_{0 \le t \le T} \|u(t, \cdot)\|_V$;
- $X_0^T := C\big([0,T]; D(A^{3/2})\big)$;
- $Y_0^T := X_0^T \times C[0,T]$;
- $\|(u, g)\|_{Y_0^T} := \|u\|_{X_0^T} + \|g\|_{C[0,T]}$.

Definition. We call u a weak solution to (1), if Equation (1) holds in $L^2(\Omega)$ and $u(\cdot, t) \in H_0^1(\Omega)$ for almost all $t \in (0, T)$ and $u, u_t, u_{tt} \in C([0,T]; D(A^{-\gamma}))$,

$$\lim_{t \to 0} \|u(t, \cdot) - u_0\|_{D(A^{-\gamma})} = \lim_{t \to 0} \|\partial_t u(t, \cdot) - u_1\|_{D(A^{-\gamma})} = \lim_{t \to 0} \|\partial_{tt} u(t, \cdot) - u_2\|_{D(A^{-\gamma})} = 0, \quad (3)$$

with some $\gamma > 0$.

We make the following assumptions:
(K1) $u_0(x) \in D(A^{3/2})$, $u_1(x) \in D(A)$, $u_2(x) \in D(A^{1/2})$;
(K2) $h(t) \in C^4[0,T]$, $\eta(x) \in H^2(\Omega) \cap H_0^1(\Omega)$.
We divide this paper into the following: In Section 2, we will show the existence and uniqueness of a weak solution to a direct problem (1). Then we apply the Banach fixed point method to prove the local existence and uniqueness in time of the inverse problem (1)–(2), and in the fourth part we prove the global existence in the time of the inverse problem (1)–(2).

2. Direct problem. In this section, we will study the direct problem (1). We shall prove the existence and uniqueness of a weak solution to the problem. The Fourier method and Lebesgue's theorem are employed to examine the direct problem (1). It is worth noting that these methods are highly efficient and have been widely applied by numerous researchers in their scientific studies (e.g., [7, 14, 15, 34].

Assume that problem (1) has a solution $u(t,x)$ which is given by

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) e_k(x), \tag{4}$$

where

$$u_k(t) = \big(u(t,\cdot), e_k\big)_{L^2(\Omega)}.$$

Applying the method of separation of variables to determine the desired coefficients $u_k(t)$, $k = 1, 2, ...$ of the function $u(t,x)$ from (1), we obtain:

$$u_k'''(t) + u_k''(t) + \lambda_k u_k'(t) + \lambda_k u_k(t) - F_k(t) = 0, \tag{5}$$

$$u_k(0) = u_{0k},\ u_k'(0) = u_{1k},\ u_k''(0) = u_{2k}, \tag{6}$$

where

$$F_k(t) = \big(F(t,\cdot), e_k\big),\ u_{ik} = \big(u_i, e_k\big),\ i = 0,1,2.$$

Solving problem (5), (6), it is easy to conclude that it is equivalent to the following integral equations:

$$u_k(t) = \frac{1}{\lambda_k + 1}\left[\lambda_k e^{-t} + \sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right) + \cos\left(\sqrt{\lambda_k}t\right)\right] u_{0k}$$

$$+\frac{1}{\sqrt{\lambda_k}}\sin\left(\sqrt{\lambda_k}t\right) u_{1k} + \Psi_k(t) u_{2k} - \int_0^t F_k(\tau)\Psi_k(t-\tau)d\tau, \tag{7}$$

where

$$\Psi_k(t) = \frac{1}{\lambda_k + 1}\left[e^{-t} + \frac{1}{\sqrt{\lambda_k}}\sin\left(\sqrt{\lambda_k}t\right) - \cos\left(\sqrt{\lambda_k}t\right)\right].$$

After substituting the expression of (7) into (4), we obtained of the solution to problem (1) to be

$$u(t,x)=\sum_{k=1}^{\infty}\frac{1}{\lambda_k+1}\left[\lambda_k e^{-t}+\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right)+\cos\left(\sqrt{\lambda_k}t\right)\right]u_{0k}e_k(x)$$

$$+\sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{\lambda_k}}\sin\left(\sqrt{\lambda_k}t\right)u_{1k}+\Psi_k(t)u_{2k}-\int_0^t F_k(\tau)\Psi(t-\tau)d\tau\right]e_k(x). \quad (8)$$

Let us prove the uniqueness of the solution of (1).

Theorem 1. *The problem (1) cannot have more than one weak solution in* $u(t,x)\in X_0^T$.

Proof. Suppose that there exist two different weak solutions $u^{(1)}(t,x)\in X_0^T$ and $u^{(2)}(t,x)\in X_0^T$ for the problem (1). Then the difference $U=u^{(1)}-u^{(2)}$ solves

$$\begin{cases} U_{ttt}+U_{tt}+AU_t+AU-g*AU=0, & (0,T)\times\Omega,\\ U(0,\cdot)=0,\ U_t(0,\cdot)=0,\ U_{tt}(0,\cdot)=0, & \Omega,\\ U=0, & [0,T]\times\Gamma. \end{cases} \quad (9)$$

Using (8), we can write the solution to (9) as

$$U(t,x)=-\int_0^t\int_0^\tau g(\tau-s)\sum_{k=1}^{\infty}\lambda_k U_k(s)\Psi_k(t-\tau)e_k(x)ds d\tau, \quad (10)$$

where $U_k(t)=u_k^{(1)}(t)-u_k^{(2)}(t)$ and by (7), we have

$$U_k(t)=\lambda_k\int_0^t\int_0^\tau g(\tau-s)U_k(s)\Psi_k(t-\tau)ds\, d\tau, \quad k=1,2,\dots .$$

This is a homogeneous second kind Volterra integral equation with respect to $U_k(t)$, and for each $k\in\mathbb{N}$ and $t\in[0,T]$, we have $U_k(t)=0$ for $g\in C[0,T]$. From (10), we can conclude that $U(t,x)=0$, for all $t\in[0,T]$. The proof of the theorem is complete. □

In the next step, we shall prove the existence of a weak solution to a direct problem.

Theorem 2. *Let* $F=0$ *and* $u_0\in D(A)$, $u_1\in D(A)$, $u_2\in D(A^{1/2})$. *Then there exist a unique solution* $u\in C([0,T];D(A))$ *to* (1). *Moreover there exists a constant* $C>0$ *such that*

$$\|Au(t,\cdot)\|^2_{L^2(\Omega)}+\|Au_t(t,\cdot)\|^2_{L^2(\Omega)}+\|u_{tt}(t,\cdot)\|^2_{L^2(\Omega)} \le C\left(\|u_0\|^2_{D(A)}+\|u_1\|^2_{D(A)}+\|u_2\|^2_{D(A^{1/2})}\right). \quad (11)$$

Proof. We will show that (4) certainly gives the weak solution to (1). To do this, we will check the conditions of the definition one by one. Therefore, we consider the estimate

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} \le 3\sum_{k=1}^{\infty}\frac{1}{(\lambda_k+1)^2}\left[\lambda_k e^{-t}+\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right)+\cos\left(\sqrt{\lambda_k}t\right)\right]^2 u_{0k}^2$$

$$+3\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\sin^2\left(\sqrt{\lambda_k}t\right)u_{1k}^2+3\sum_{k=1}^{\infty}\Psi_k^2(t)u_{2k}^2 \le C_1\sum_{k=1}^{\infty}|u_{0k}|^2+C_1\sum_{k=1}^{\infty}\frac{1}{k^{2/n}}|u_{1k}|^2$$

$$+C_1\sum_{k=1}^{\infty}\frac{1}{k^{4/n}}|u_{2k}|^2 \le C_1\left[\|u_0\|^2_{D(A)}+\|u_1\|^2_{D(A)}+\|u_2\|^2_{D(A^{1/2})}\right]. \qquad (12)$$

In relation to other variables involved in the equation, the following estimates are appropriate

$$\|Au(t,\cdot)\|^2_{L^2(\Omega)} \le C_2\left(\sum_{k=1}^{\infty}\lambda_k^2u_{0k}^2+\sum_{k=1}^{\infty}\lambda_k u_{1k}^2+\sum_{k=1}^{\infty}u_{2k}^2\right)$$

$$\le C_2\left(\|u_0\|^2_{D(A)}+\|u_1\|^2_{D(A)}+\|u_2\|^2_{D(A^{1/2})}\right), \qquad (13)$$

$$\|Au_t(t,\cdot)\|^2_{L^2(\Omega)} \le C_3\left(\sum_{k=1}^{\infty}\lambda_k^2u_{0k}^2+\sum_{k=1}^{\infty}\lambda_k^2 u_{1k}^2+\sum_{k=1}^{\infty}\lambda_k u_{2k}^2\right)$$

$$\le C_3\left[\|u_0\|^2_{D(A)}+\|u_1\|^2_{D(A)}+\|u_2\|^2_{D(A^{1/2})}\right]. \qquad (14)$$

Moreover, we have

$$\|u_{tt}(t,\cdot)\|^2_{L^2(\Omega)} \le C_4\left[\|u_0\|^2_{D(A)}+\|u_1\|^2_{D(A)}+\|u_2\|^2_{D(A^{1/2})}\right]. \qquad (15)$$

Hereinafter, C_i, $i=1,2,3,4$ and $C=C_2+C_3+C_4$ are various positive constant values depending on u_i, $i=0,1,2$. From (12)-(15), we get (11). By (1) we see that $u_{ttt}\in C\left([0,T];L^2(\Omega)\right)$.

Now we check that the solution satisfies the initial conditions, that is

$$\lim_{t\to 0}\left\|\partial_t^i u(t,\cdot)-u_i\right\|_{L^2(\Omega)}=0, \quad i=0,1,2,\ n=1,2,3. \qquad (16)$$

In fact,

$$\|u(t,\cdot)-u_0\|^2_{L^2(\Omega)}=\sum_{k=1}^{\infty}\left[\,(u(t,x)-u_0(x))\,e_k\right]^2$$

$$=\sum_{k=1}^{\infty}\left[\frac{\lambda_k e^{-t}+\sqrt{\lambda_k}\sin(\sqrt{\lambda_k}t)+\cos(\sqrt{\lambda_k}t)-\lambda_k-1}{\lambda_k+1}u_{0k}\right.$$

$$\left.+\frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}}u_{1k}+\Psi_k(t)u_{2k}\right]^2$$

$$\leq 12\sum_{k=1}^{\infty}u_{0k}^2+3\sum_{k=1}^{\infty}\frac{1}{\lambda_k}u_{1k}^2+9\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}u_{2k}^2<\infty, \tag{17}$$

$$\|u_t(t,\cdot)-u_1\|_{L^2(\Omega)}^2=\sum_{k=1}^{\infty}\left[(u_t(t,x)-u_1(x),e_k(x))\right]^2$$

$$=\sum_{k=1}^{\infty}\Bigg[\frac{-\lambda_k e^{-t}+\lambda_k\cos(\sqrt{\lambda_k}t)-\sqrt{\lambda_k}\sin(\sqrt{\lambda_k}t)}{\lambda_k+1}u_{0k}$$

$$+\left(\cos(\sqrt{\lambda_k}t)-1\right)u_{1k}+\Psi_k'(t)u_{2k}\Bigg]^2$$

$$\leq 3\sum_{k=1}^{\infty}u_{0k}^2+3\sum_{k=1}^{\infty}u_{1k}^2+3\sum_{k=1}^{\infty}\frac{1}{\lambda_k}u_{2k}^2<\infty. \tag{18}$$

In the same way

$$\|u_{tt}(t,\cdot)-u_2\|_{L^2(\Omega)}^2\leq 6\sum_{k=1}^{\infty}\lambda_k u_{0k}^2+3\sum_{k=1}^{\infty}\lambda_k u_{1k}^2+6\sum_{k=1}^{\infty}\frac{1}{\lambda_k}u_{2k}^2<\infty, \tag{19}$$

for $t\in[0,T]$. According to theorem Lebesgue and relations (17)-(19), we will get (16). The theorem is proved. □

Theorem 3. *Let $u_0=u_1=u_2=0$ and $F\in C([0,T];D(A^{1/2}))$. Then there exists a unique weak solution $u\in C([0,T];D(A))$ to (1). In particular, for any γ satisfying $\gamma>\frac{n}{4}$, we have $u\in C([0,T];D(A^{-\gamma}))$,*

$$\lim_{t\to 0}\left\|\partial_t^i u(t,\cdot)\right\|_{D(A^{-\gamma})}=0,\quad i=0,1,2. \tag{20}$$

and if $n=1,2,3$, then

$$\lim_{t\to 0}\left\|\partial_t^i u(t,\cdot)\right\|_{L^2(\Omega)}=0,\quad i=0,1,2.$$

Moreover there exist a constant $C^*>0$ such that

$$\|Au(t,\cdot)\|_{L^2(\Omega)}+\|Au_t(t,\cdot)\|_{L^2(\Omega)}+\|u_{tt}(t,\cdot)\|_{L^2(\Omega)}\leq C^*\|F\|_{C([0,T];D(A^{1/2}))}. \tag{21}$$

Proof. Let us start with the estimate (21). Indeed, by virtue of the generalized Minkowski inequality,

$$\|Au(t,\cdot)\|_{L^2(\Omega)}^2=\|-\Delta u(t,\cdot)\|_{L^2(\Omega)}^2$$

$$\leq\left(\int_0^t\Big(\sum_{k=1}^{\infty}(\lambda_k F_k(\tau)\Psi_k(t-\tau))^2\Big)^{1/2}d\tau\right)^2\leq\left(\int_0^t\Big(\sum_{k=1}^{\infty}(F_k(\tau))^2\Big)^{1/2}d\tau\right)^2$$

$$\leq\|F\|_{C([0,T];L_2(\Omega))}^2\,t^2, \tag{22}$$

$$\|Au_t(t,\cdot)\|^2_{L^2(\Omega)} = \|-\Delta u_t(t,\cdot)\|^2_{L^2(\Omega)}$$

$$= \sum_{k=1}^{\infty} \lambda_k^2 \Big(\int_0^t F_k(\tau)\Psi_k'(t-\tau)\,d\tau \Big)^2 \le \Big(\int_0^t \Big(\sum_{k=1}^{\infty} (\lambda_k^{1/2} F_k(\tau))^2 \Big)^{1/2} d\tau \Big)^2$$
$$\le \|F\|^2_{C([0,T];D(A^{1/2}))}\, t^2, \quad (23)$$

$$\|u_{tt}(t,\cdot)\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \Big(\int_0^t F_k(\tau)\Psi_k''(t-\tau)\,d\tau \Big)^2 \le \Big(\int_0^t \Big(\sum_{k=1}^{\infty} (F_k(\tau)\Psi_k''(t-\tau))^2 \Big)^{1/2} d\tau \Big)^2$$
$$\le \Big(\int_0^t \Big(\sum_{k=1}^{\infty} (F_k(\tau))^2 \Big)^{1/2} d\tau \Big)^2 \le \|F\|^2_{C([0,T];L_2(\Omega))}\, t^2. \quad (24)$$

Finally, we have to show (20). By (8), we have

$$\|u_{tt}(t,\cdot)\|^2_{D(A^{-\gamma})} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2\gamma}} \left(\int_0^t F_k(\tau)\Psi_k''(t-\tau)\,d\tau \right)^2$$

$$\le 2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2\gamma}} \sup_{0\le\tau\le t} |F_k(\tau)|^2\, t^2 \le 2\,\|F\|^2_{C([0,T];L_2(\Omega))}\, t^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2\gamma}}. \quad (25)$$

Since $\lambda_k \ge C_5 k^{\frac{2}{n}}$ (see [15], p. 356), we have

$$\frac{1}{\lambda_k^{2\gamma}} \le \frac{C_5'}{k^{\frac{4\gamma}{n}}}.$$

By $\gamma > \frac{n}{4}$, we get $\frac{4\gamma}{n} > 1$, and $\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2\gamma}} < \infty$. The Lebesgue theorem implies $\lim_{t\to 0} \|u_{tt}(t,\cdot)\|_{D(A^{-\gamma})} = 0$. We can show $\lim_{t\to 0} \|u(t,\cdot)\|_{D(A^{-\gamma})} = 0$ and $\lim_{t\to 0} \|u_t(t,\cdot)\|_{D(A^{-\gamma})} = 0$ similarly to the above equality. The proof of Theorem 3 is complete. □

3. Solvability of the inverse problem. In this section, we consider the problem of simultaneously determining the unknown functions $u(x,t) \in X_0^T$, $g(t) \in C[0,T]$ from the integro-differential Equation (1) with initial-boundary condition, and additional condition.

3.1. Equivalence of the inverse problem. Now, to study the problem (1), (2). The following lemma is valid.

Lemma 1. *Let (K1), (K2) be held and that the following compatibility conditions are fulfilled*

$$(K3): \qquad \int_\Omega \eta(x)u_i(x)dx - h^{(i)}(0) = 0, \quad h^{(i)}(t) = \frac{d^i}{dt^i}h(t),\ i = 0,1,2.$$

Then the inverse problem (1)-(2) is equivalent to following problem:

$$\begin{cases} u_{ttt} + u_{tt} + Au_t + Au - g * Au = 0, & \Omega \times (0,T), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ u_{tt}(0,x) = u_2(x), & \Omega, \\ u = 0, & [0,T] \times \Gamma, \end{cases} \tag{26}$$

and

$$h'''(t) + h''(t) + (Au_t(t,\cdot),\eta) + (Au(t,\cdot),\eta) + (F(t,\cdot),\eta) = 0. \tag{27}$$

Proof. Let $\{u(x,t), g(t)\}$ be a solution of inverse problem (1), (2). The solution $\{u(x,t), g(t)\}$ is also a solution to the problem (26), (27). Because the problem (26) is the same as (1). Therefore, we should show only (27). Multiplying both sides of Equation (1) by a function $\eta(x)$ and integrating over Ω with respect to x gives

$$\frac{d^3}{dt^3}\int_\Omega \eta(x)u(t,x)dx + \frac{d^2}{dt^2}\int_\Omega \eta(x)u(t,x)dx + \frac{d}{dt}\int_\Omega \eta(x)Au(t,x)dx$$

$$+ \int_\Omega \eta(x)Au(t,x)dx + \int_\Omega \eta(x)F(t,x)dx = 0, \tag{28}$$

for all $t \in [0,T]$. Taking into account the condition $h(t) \in C^4[0,T]$ and additional condition (2), we have

$$\frac{d^3}{dt^3}\int_\Omega \eta(x)u(t,x)dx = h'''(t),\ \frac{d^2}{dt^2}\int_\Omega \eta(x)u(t,x)dx = h''(t),\ t \in [0,T]. \tag{29}$$

Hence, from (28), taking into account (2) and (29), we arrive at (27).

Now, suppose that $\{u(t,x), g(t)\}$ is a solution to the problem (26), (27). In order to prove that $\{u(t,x), g(t)\}$ is also a solution of the problem (1)-(2), it suffices to prove that $\{u(t,x), g(t)\}$ satisfies (2). From (27) and (28), we obtained following Cauchy problem

$$\begin{cases} y'''(t) + y''(t) = 0, & (0,T), \\ y^{(i)}(0) = 0, & i = 0,1,2, \end{cases} \tag{30}$$

where $y(t) = \int_\Omega \eta(x)u(t,x)dx - h(t)$.

The problem (30) has only a trivial solution. Then, $y(t) \equiv 0$, $0 \le t \le T$ i.e., the condition (2) is satisfied. The proof is complete. □

3.2. Investigation of the inverse problem. This section is devoted to the local solvability of problems (19) and (20), which is equivalent to the inverse problem (1) and (2).

Let

(K4) $M=\sum\limits_{k=1}^{\infty}\lambda_k\eta_k u_{0k}\neq 0$, and $m=\frac{1}{|M|}>0$, for all $t\in[0,T]$.

Now, we differentiate equality (27) with respect to t and using equality (8), after simple converting, we obtain the following integral equation for determining $g(t)$

$$g(t)=\Phi(t)+\int_0^t g(t-\tau)G_1(\tau)d\tau+\int_0^t\int_0^\tau g(\tau-s)G_2([u],\cdot)dsd\tau$$
$$+\int_0^t\int_0^\tau\int_0^s g(t-\tau)g(s-\sigma)G_3([u],\cdot)d\sigma dsd\tau. \tag{31}$$

where

$$\Phi(t)=\frac{1}{M}\left[\frac{d^4}{dt^4}h(t)+\frac{d^3}{dt^3}h(t)-\sum_{k=1}^{\infty}\lambda_k^{3/2}\eta_k u_{0k}\sin\left(\sqrt{\lambda_k}t\right)\right]$$

$$-\frac{1}{M}\left[\sum_{k=1}^{\infty}\lambda_k\eta_k u_{1k}\left[\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right)-\cos\left(\sqrt{\lambda_k}t\right)\right]+\sum_{k=1}^{\infty}\lambda_k\eta_k u_{2k}\cos\left(\sqrt{\lambda_k}t\right)\right],$$

$$G_1(t)=\frac{1}{M}\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k+1}\eta_k u_{0k}\left[\lambda_k e^{-t}-\lambda_k\cos\left(\sqrt{\lambda_k}t\right)+\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right)\right]$$

$$-\frac{1}{M}\sum_{k=1}^{\infty}\lambda_k\eta_k u_{1k}\cos\left(\sqrt{\lambda_k}t\right)$$
$$+\frac{1}{M}\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k+1}\eta_k u_{2k}\left[-e^{-t}+\cos\left(\sqrt{\lambda_k}t\right)+\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right)\right],$$

$$G_2([u],\cdot)=\frac{1}{M}\sum_{k=1}^{\infty}\lambda_k^2\eta_k u_k\cos\left(\sqrt{\lambda_k}t\right),$$

$$G_3([u],\cdot)=\frac{1}{M}\sum_{k=1}^{\infty}\frac{\lambda_k^2}{\lambda_k+1}\eta_k u_k\left[-e^{-t}+\cos\left(\sqrt{\lambda_k}t\right)+\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}t\right)\right].$$

We are now in a position to prove the local existence of a solution to our inverse problem, which proceeds by a fixed point argument. We define the function set.

$$B_{r,T}=\Big\{(\overline{u},\overline{g})\in Y_0^T:\overline{u}(0,x)=u_0(x),\ \overline{u}_t(0,x)=u_1(x),\ \overline{u}_{tt}(0,x)=u_2(x),\ x\in\Omega,$$

$$\overline{u}\mid_\Gamma=0,\ t\in[0,T],\ \|\overline{u}\|_{X_0^T}+\|\overline{g}\|_{C[0,T]}\leq r\Big\},$$

where r is a large constant depending on the initial and measurement data u_1, u_2, h. Hereinafter, we use C to denote a constant which depends on the initial data $u_i,\ i=0,1,2$, the known function η and measurement data h, but independent of r and T. We consider

$$\begin{cases} u_{ttt}+u_{tt}+Au_t+Au+\overline{F}=0, & \Omega\times(0,T),\\ u(0,\cdot)=u_0,\ u_t(0,\cdot)=u_1,\ u_{tt}(0,\cdot)=u_2, & \Omega,\\ u=0, & \Gamma\times[0,T], \end{cases} \tag{32}$$

and

$$g(t)=\Phi(t)+\int_0^t\overline{g}(t-\tau)G_1(\tau)d\tau+\int_0^t\int_0^\tau\overline{g}(\tau-s)G_2([u],\cdot)dsd\tau$$
$$+\int_0^t\int_0^\tau\int_0^s\overline{g}(t-\tau)\overline{g}(s-\sigma)G_3([u],\cdot)d\sigma dsd\tau, \tag{33}$$

where $(\overline{u},\overline{g})\in B_{r,T}$, and $\overline{F}(x,t)=-\overline{g}*A\overline{u}$.

The unique solution $u(t,x)\in X_0^T$ of the problem (32), given by (8) satisfies

$$\|u\|_{X_0^T}\le C\left[\|u_0\|_{D(A^{3/2})}+\|u_1\|_{D(A)}+\|u_2\|_{D(A^{1/2})}+T\left\|\overline{F}\right\|_{C\left([0,T];D(A^{1/2})\right)}\right]. \tag{34}$$

We get the estimate for the function $\overline{F}(t,x)$ in the following form

$$\left\|\overline{F}\right\|^2_{C\left([0,T];D(A^{1/2})\right)}=\max_{t\in[0,T]}\left|\sum_{k=1}^\infty\left(\lambda_k^{1/2}\overline{F}(t,\cdot),e_k\right)^2\right|\le\lambda_1^{-2}T^2\|\overline{g}\|^2_{C[0,T]}\|\overline{u}\|^2_{X_0^T}. \tag{35}$$

Using this result, we have $\overline{F}(t,x)\in C\left([0,T];D(A^{1/2})\right)$. By Theorem 2, that there exists the unique solution $u(t,x)\in X_0^T$ of problem (32). Then (33) defines the unknown function in terms of $u(t,x)$. By (33) we obtained

$$\|g\|_{C[0,T]}\le m\|h\|_{C^4[0,T]}$$
$$+m\left[2\|\eta\|_{D(A^{1/2})}\|u_0\|_{D(A^{3/2})}+\|\eta\|_{D(A)}\|u_1\|_{D(A)}+\|\eta\|_{D(A)}\|u_2\|_{D(A^{1/2})}\right]$$
$$+m\|\overline{g}\|_{C[0,T]}T\left[\|\eta\|_{L^2(\Omega)}\|u_0\|_{D(A)}+\|\eta\|_{D(A^{1/2})}\|u_1\|_{D(A^{1/2})}+\|\eta\|_{D(A^{1/2})}\|u_2\|_{L^2(\Omega)}\right]$$
$$+m\left[\|\overline{g}\|_{C[0,T]}\|u\|_{X_0^T}\|\eta\|_{D(A)}T^2+\|\overline{g}\|^2_{C[0,T]}\|u\|_{X_0^T}\|\eta\|_{D(A^{1/2})}T^3\right]. \tag{36}$$

This implies that $g\in C[0,T]$. Thus, the mapping

$$S:B_{r,T}\to Y_0^T,\quad(\overline{u},\overline{g})\to(u,g), \tag{37}$$

given by (32) and (33), well defined.

Now we show that S maps $B_{r,T}$ into itself for sufficiently small $T>0$. We have the following result.

Lemma 2. *Let (K1)-(K4) be held. Then there exist a sufficiently small τ and a suitable large r such that S is a contraction map on $B_{r,T}$ for all $T\in(0,\tau]$, where*

τ and r are depending on the known functions u_i, $i = 0, 1, 2$, and the measurement data h.

Proof. First, we prove $S(B_{r,T}) \subset B_{r,T}$ for sufficiently small T and suitable large r. To simplify the calculations, we restrict $T \in (0, 1]$. By (34), we obtain

$$\|u\|_{X_0^T} \leq C + Cr^2T^2. \tag{38}$$

On the other hand, we have

$$\|g\|_{C[0,T]} \leq C + CrT + CrT^2 \|u\|_{X_0^T} + Cr^2T^3 \|u\|_{X_0^T}. \tag{39}$$

Then, adding up (38)–(39) leads to

$$\|(u, g)\|_{Y_0^T} \leq$$

$$\leq 2C + CrT \left[1 + rT + CT + CrT^2 + Cr^2T^3 + Cr^3T^4\right] = 2C + C\zeta(r, T), \tag{40}$$

and therefore satisfies $\lim\limits_{T\to+0} \zeta(r, T) = 0$. We first fix r such that $r = 2C$. Then we can choose sufficiently small $\tau_1 > 0$ such that

$$\|(\overline{u}, \overline{g})\|_{Y_0^T} \leq r, \tag{41}$$

for all $T \in (0, \tau_1]$, that is, S maps $B_{r,T}$ into itself for each fixed $T \in (0, \tau_1]$.

Next, we estimate the increment of the mapping S. Let $(u, g) = S(\overline{u}, \overline{g})$ and $(v, q) = S(\overline{v}, \overline{q})$. Then we obtain that $(u - v, g - q)$ satisfies that

$$\begin{cases} L_0(u - v) + L_0(u - v)_t - \overline{g} * A\overline{u} + \overline{q} * A\overline{v} = 0, & \Omega \times (0, T), \\ (u - v)(\cdot, 0) = 0,\ (u - v)_t(\cdot, 0) = 0,\ (u - v)_{tt}(\cdot, 0) = 0, & \Omega, \\ u - v = 0, & \Gamma \times (0, T), \end{cases} \tag{42}$$

and

$$(g - q)(t) = \int_0^t \int_0^\tau \Big[\overline{g}(\tau - s)G_2([u], \cdot) - \overline{q}(\tau - s)G_2([v], \cdot)\Big] ds d\tau$$

$$+ \int_0^t \int_0^\tau \int_0^s \Big[\overline{g}(t - \tau)\overline{g}(s - \sigma)G_3([u], \cdot) - \overline{q}(t - \tau)\overline{q}(s - \sigma)G_3([v], \cdot)\Big] d\sigma ds d\tau. \tag{43}$$

where L_0 is the wave operator given by

$$L_0 := \partial_t^2 + A.$$

Then together with (34) and (35), we furthermore have

$$\|u - v\|_{X_0^T} \leq CrT^2 \Bigg[\|\overline{u} - \overline{v}\|_{X_0^T} + \|\overline{g} - \overline{q}\|_{C[0,T]} \Bigg]. \tag{44}$$

Further, using (43), and (44), we get

$$\|g - q\|_{C[0,T]} \leq CrT^2(1 + r)(1 + Cr) \Bigg[\|\overline{u} - \overline{v}\|_{X_0^T} + \|\overline{g} - \overline{q}\|_{C[0,T]} \Bigg]. \tag{45}$$

Therefore, from (44) and (45), we deduce that

$$\|(u-v, g-q)\|_{Y_0^T} \leq CrT^2\big((1+r)(1+Cr)+1\big)\,\|(\overline{u}-\overline{v}, \overline{g}-\overline{q})\|_{Y_0^T}. \tag{46}$$

Because of $\lim_{T\to+0} CrT^2\big((1+r)(1+Cr)+1\big)=0$, we can obtain (41), if we choose τ_2 sufficiently small, such that

$$CrT^2\big((1+r)(1+Cr)+1\big) \leq \frac{1}{2},$$

for all $T\in(0,\tau_2]$ to obtain

$$\|S(\overline{u},\overline{g})-S(\overline{v},\overline{q})\|_{Y_0^T} \leq \frac{1}{2}\,\|\overline{u}-\overline{v}, \overline{g}-\overline{q}\|_{Y_0^T}. \tag{47}$$

Estimates (41) and (47) show that S is a contraction map on $B_{r,T}$ for all $T\in(0,\tau]$, if we choose $\tau\leq\min\{\tau_1,\tau_2\}$. The proof is complete. □

The main result of this paper is the following local existence and uniqueness result for an inverse problem.

Theorem 4. *Let the assumptions* (K1)-(K4) *hold. Then, the inverse problem has a unique solution* $(u,g)\in Y_0^\tau$ *for sufficiently small* $\tau>0$.

Proof. Lemma 2 shows that there exists a sufficiently small $\tau>0$, such that S is a contraction mapping on $B_{r,T}$. Hence, the Banach fixed point theorem guarantees the existence of a unique solution $(u,g)\in Y_0^\tau$ to the system (26)-(27), for sufficiently small τ. As a consequence, the problem constituted by (1), (2) also admits a unique solution (u,g) in $[0,\tau]$ by Lemma 1. The proof is complete. □

4. Main results.

In this section, we give proof of the global time existence of solutions to our inverse problem.

Theorem 5. *Under hypotheses (K1)-(K4), there exists a solution* $(u,g)\in Y_0^T$ *of the inverse problem (1)-(2) for any* $T>0$.

Proof. For S is contraction map on $B_{r,T}$ for all $T\in(0,\min\{\tau_1,\tau_2\}]$, the Banach fixed point theorem concludes that there exists solution $(u,q)\in X_0^T\times C[0,T]$ of the inverse problem (26), (27).

Now, we show that we could extend the solution $(u,g)\in Y_0^T$ to a larger interval $[\tau,2\tau]$. To do this, we consider

$$\begin{cases} w_{ttt}+w_{tt}+Aw_t+Aw+W=0, & \Omega\times(\tau,T),\\ w(\tau,\cdot)=u(\tau,\cdot),\ w_t(\tau,\cdot)=u_t(\tau,\cdot),\ w_{tt}(\tau,\cdot)=u_{tt}(\tau,\cdot), & \Omega,\\ w=0, & \Gamma\times(\tau,T), \end{cases} \tag{48}$$

and

$$q(t)=\Phi(t)+\int_\tau^t q(t-s)G_1(s)ds+\int_\tau^t\int_\tau^s g(s-\sigma)G_2([w],\cdot)d\sigma ds$$

$$+\int_\tau^t\int_\tau^s\int_\tau^\sigma q(t-s)q(\sigma-\theta)G_3([w],\cdot)d\theta d\sigma ds,\ t\in[\tau,T], \tag{49}$$

where, $W(t,x)=-q*Aw$. If we prove that there exists a solution $(w,q)\in Y_\tau^T$ with some $T\le 2\tau$, then the function

$$(\widetilde{u},\widetilde{g})=\begin{cases}(u,g),\ t\in[0,\tau],\\(w,q),\ t\in[\tau,2\tau],\end{cases} \tag{50}$$

is a solution of the problem (32) and (33) on the interval $[0,2\tau]$. We prove the existence of (w,q) by to fixed point argument. Define an operator

$$Z:\widetilde{B}_{\rho,T}\to Y_\tau^T,\ (\overline{w},\overline{q})\to(w,q), \tag{51}$$

with $(\overline{w},\overline{q})\in\widetilde{B}_{\rho,T}$, where

$$\widetilde{B}_{\rho,T}=\Big\{(\overline{w},\overline{q})\in Y_\tau^T:\overline{w}(\tau,x)=u(\tau,x),\ \overline{w}_t(\tau,x)=u_t(\tau,x),$$

$$\overline{w}_{tt}(\tau,x)=u_{tt}(\tau,x),\ x\in\overline{\Omega},\ \overline{w}\,|_\Gamma=0,\ t\in(\tau,T),\|\overline{w}\|_{X_\tau^T}+\|\overline{g}\|_{C[\tau,T]}\le\rho\Big\}.$$

For given $(\overline{w},\overline{q})\in\widetilde{B}_{\rho,T}$, $w(t,x)$ is the solution to the problem

$$\begin{cases}w_{ttt}+w_{tt}+Aw_t+Aw+\overline{W}=0, & \Omega\times(\tau,T),\\ w(\tau,\cdot)=u(\tau,\cdot),\ w_t(\tau,\cdot)=u_t(\tau,\cdot),\ w_{tt}(\tau,\cdot)=u_{tt}(\tau,\cdot), & \Omega,\\ w=0, & \Gamma\times(\tau,T),\end{cases} \tag{52}$$

where $\overline{W}=-\overline{q}*A\overline{w}$.

Furthermore, q is solution of (49) in terms of w. Additionally, we have

$$\left[\begin{array}{ll}\overline{W}(t,x)\in C\left([\tau,T];D(A^{1/2})\right), & (i)\\ u(\tau,\cdot)\in D(A^{3/2}), & (ii)\\ u_t(\tau,\cdot)\in D(A), & (iii)\\ u_{tt}(\tau,\cdot)\in D(A^{1/2}). & (iiii)\end{array}\right.$$

By (35) the property (i) holds. According to Theorem 2 the functions $u(\tau,\cdot)$ the same as (8) at $t\in\{\tau,T\}$, then we can conclude that $u(\tau,\cdot)\in D(A^{3/2})$. Let us show the property (iii) and $(iiii)$ claim. Taking into account (8), we get

$$\|u_t(\tau,\cdot)\|^2_{D(A)}=\sum_{k=1}^\infty\lambda_k^2\left(u_k'(\tau)\right)^2$$

$$=\sum_{k=1}^\infty\lambda_k^2\Bigg[\frac{1}{\lambda_k+1}\left(-\lambda_k e^{-\tau}+\lambda_k\cos\left(\sqrt{\lambda_k}\tau\right)-\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}\tau\right)\right)u_{0k}$$

$$+\cos\left(\sqrt{\lambda_k}\tau\right)u_{1k}+\Psi_k'(\tau)u_{2k}-\int_0^\tau F_k(s)\Psi'(\tau-s)ds\Bigg]^2$$

$$\leq \delta_1 \left[\|u_0\|^2_{D(A^{3/2})} + \|u_1\|^2_{D(A)} + \|u_2\|^2_{D(A^{1/2})} + \tau^2 \|F\|^2_{D(A^{1/2})} \right].$$

As a result of simple mathematical calculations, we obtained

$$\|u_{tt}(\tau,\cdot)\|^2_{D(A^{1/2})} = \sum_{k=1}^{\infty} \lambda_k \left(u_k''(\tau)\right)^2$$

$$= \sum_{k=1}^{\infty} \lambda_k \left[\frac{\lambda_k}{\lambda_k+1} \left(e^{-\tau} - \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k}\tau\right) - \cos\left(\sqrt{\lambda_k}\tau\right)\right) u_{0k} \right.$$

$$\left. - \sin\left(\sqrt{\lambda_k}\tau\right) u_{1k} + \Psi_k''(\tau) u_{2k} - \int_0^\tau F_k(s) \Psi''(\tau-s) ds \right]^2$$

$$\leq \delta_2 \left[\|u_0\|^2_{D(A^{3/2})} + \|u_1\|^2_{D(A)} + \|u_2\|^2_{D(A^{1/2})} + \tau^2 \|F\|^2_{D(A^{1/2})} \right],$$

where δ_1, δ_2 are arbitrary real constants. Similarly, from (34) we have

$$\|w\|_{X_\tau^T} \leq C^* \|u(\tau,\cdot)\|_{D(A^{3/2})} +$$

$$+C^* \left[\|u_t(\tau,\cdot)\|_{D(A)} + \|u_{tt}(\tau,\cdot)\|_{D(A^{1/2})} + (T-\tau) \left\|\overline{F}\right\|_{C\left([\tau,T];D(A^{1/2})\right)} \right]. \quad (53)$$

According to (49) via (K1)-(K4), we have the following estimates

$$\|q\|_{C[\tau,T]} \leq m \left[\|h\|_{C^4[\tau,T]} + \|\eta\|_{L^2(\Omega)} \|u(\tau,\cdot)\|_{D(A^{3/2})} \right]$$

$$+m \left[\|\eta\|_{D(A^{1/2})} \|u_t(\tau,\cdot)\|_{D(A)} + \|\eta\|_{D(A^{1/2})} \|u_{tt}(\tau,\cdot)\|_{D(A^{1/2})} \right]$$

$$+ m\rho (T-\tau) \left[\|\eta\|_{L^2(\Omega)} \|u(\tau,\cdot)\|_{D(A^{3/2})} + \|\eta\|_{D(A^{1/2})} \|u_t(\tau,\cdot)\|_{D(A)} \right.$$

$$\left. + \|\eta\|_{D(A)} \|u_{tt}(\tau,\cdot)\|_{D(A^{1/2})} \right]$$

$$+m \left[\rho \|w\|_{X_\tau^T} \|\eta\|_{L^2(\Omega)} (T-\tau)^2 + \rho^2 \|w\|_{X_\tau^T} \|\eta\|_{L^2(\Omega)} (T-\tau)^3 \right], \quad (54)$$

where, C^* is a constant depending on the data $u(\tau,\cdot)$, $u_t(\tau,\cdot)$, $u_{tt}(\tau,\cdot)$, h, and η. Hence, by (53) and (54), we obtain

$$\|q\|_{C[\tau,T]} \leq C^* + C^*\rho (T-\tau) + C^*\rho (T-\tau)^2 \|w\|_{X_\tau^T} + C^*\rho^2 (T-\tau)^3 \|w\|_{X_\tau^T}. \quad (55)$$

We set $T-\tau \leq \tau$. From (53) and (55), we deduce that

$$\|Z(w,q)\|_{Y_\tau^T} \leq C^* + C^*\rho (T-\tau) + C^{*2}\rho (T-\tau)^2 + C^{*2}\rho^2 (T-\tau)^3$$

$$+C^{*2}\rho^3 (T-\tau)^4 + C^{*2}\rho^4 (T-\tau)^5 = C^* + C^*\zeta(\rho, T-\tau), \quad (56)$$

and by a mathematical calculation to (46), we get

$$\|Z(\overline{w}_1,\overline{q}_1)-Z(\overline{w}_2,\overline{q}_2)\|_{Y_\tau^T}$$

$$\le C^*\rho\,(T-\tau)^2\left((1+\rho)(1+C^*\rho)+1\right)\|(\overline{w}_1-\overline{w}_2,\overline{q}_1-\overline{q}_2)\|_{Y_\tau^T}. \tag{57}$$

If we choose $\rho > C^*$ larger, then we could get larger $T-\tau$ ta satisfy

$$C^*+C^*\zeta(\rho,T-\tau)\le\rho. \tag{58}$$

We can choose $T-\tau\le\tau$ to satisfy (57), which yields $Z\left(\widetilde{B}_{\rho,T}\right)\subset\widetilde{B}_{\rho,T}$ and

$$\|Z(\overline{w}_1,\overline{q}_1)-Z(\overline{w}_2,\overline{q}_2)\|_{Y_\tau^T}\le\frac{1}{2}\|(\overline{w}_1-\overline{w}_2,\overline{q}_1-\overline{q}_2)\|_{Y_\tau^T}. \tag{59}$$

for $T=2\tau$. Hence we prove that Z is contraction operator on $\widetilde{B}_{\rho,T}$ for $T=2\tau$.

Repeating the extension process limited times, we could obtain a solution $(u,g)\in X_0^T\times C[0,T]$ of the inverse problem (26) and (27) for any T. Lemma 2 shows that the inverse problem (26) and (27) is equivalent to our inverse problem. Consequently, the inverse problem (1), (2) also admits a unique solution (u,g) in the space $X_0^T\times C[0,T]$ for any T. This completes the proof of the theorem. □

Conclusion. In this paper we have studied the initial boundary (well-posed) problem and the inverse problem for an $N-$dimensional MGT integral-differential equation. First, assuming the existence of a solution to the well-posed problem, proved its uniqueness, then proved the solution's existence and obtained an a priori estimate for the solution. Then an equivalent lemma to the inverse problem was obtained, which was used to prove the global existence of the solution to the inverse problem.

Apparently, all the results of this article are correct in the case when the operator Δ in (1) is replaced by the more general operator A, where A is a self-adjoint differential operator, defined in the domain Ω, given by $A=\sum\limits_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right]-c(x)$, such that $a_{ij}(x)=a_{ji}(x)$, $\sum\limits_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\ge\alpha\sum\limits_{i=1}^{n}\xi_i^2$, $\alpha=const>0$. In addition, it is assumed $a_{ij}(x)$, $c(x)$ satisfy some conditions of smoothness and $c(x)\ge 0$ in Ω.

But, there are some open problems, i.e., when A is a generator or fractional Riesz operator case the problem (1) is still not solved. Our future research will focus on investigating such problems.

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References

1. A.E. Abouelregal, On Green and Naghdi thermoelasticity model without energy dissipation with higher order time differential and phase-lags, *J. Appl. Comput. Mech.* **6** (2019), 445–456.
2. ______________, A novel model of nonlocal thermoelasticity with time derivatives of higher order, *Math. Methods Appl. Sci.* **43** (2020), 6746–6760.
3. W. Al-Khulai and A. Boumenir, Reconstructing The Moore-Gibson-Thompson Equation, *Non. Dyn. Syst.* **7** (2020), 219–223.
4. M.A. Biot, Thermoelasticity and Irreversible Thermodynamics, *J. Appl. Phys.* **27** (1956), 240–253.
5. S. Boulaaras, Solvability of the Moore-Gibson-Thompson equation with viscoelastic memory term and integral condition via Galerkin method, *Fractals* **29**(5) (2021), 2140021. https://doi.org/10.1142/S0218348X21400211
6. S. Boulaaras, A. Choucha, D. Ouchenane, M. Abdalla, and A.J.M. Vazquez, Solvability of the Moore-Gibson-Thompson equation with viscoelastic memory type II and integral condition, *Disc. and Con. Dyn. Sys. - Series S* **16**(6) (2023), 1216–1241.
7. S. Boulaaras, S. Sriramulu, S. Arunachalam, A. Allahem, A. Alharbi, and T. Radwan, Chaos and stability analysis of the nonlinear fractional-order autonomous system, *Alexandria Engineering Journal* **118** (2025), 278–291.
8. S. Boulaaras, A. Zarai, and A. Dhraifia, Galerkin method for nonlocal mixed boundary value problem for the Moore-Gibson-Thompson equation with integral condition, *Math. Meth. App. Sci.* **42** (2019), 2664–2679.
9. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations,* Springer-Verlag, New York, Universitext, 2010.
10. S.K.R. Choudhuri, On A Thermoelastic Three-Phase-Lag Model, *J. Therm. Stress.* **30** (2007), 231–238.
11. F. Dell'Oro, I. Lasiecka, and V. Pata, The Moore-Gibson-Thompson equation with memory in the critical case, *Journal Differential Equations* **261** (2016), 4188–4222.
12. D.K. Durdiev and A.A. Rahmonov, A 2D Kernel Determination Problem in a Viscoelastic Porous Medium with a Weakly Horizontally Inhomogeneity, *Mathematical Methods in the Applied Sciences* **43**(15) (2020), 8776–8796.
13. D.K. Durdiev and Zh.D. Totieva, *Kernel Determination Problems in Hyperbolic Integro-Differential Equations,* Infosys Science Foundation Series, Springer, Berlin/Heidelberg/New York, 2023.
14. D.K. Durdiev and Zh.Zh. Zhumaev, Problem of determining the thermal memory of a conducting medium, *Differential Equations* **56**(6) (2020), 785–796.
15. L.C. Evans, *Partial differential equations,* Second Ed., Vol. 19, American Mathematical Society Providence, Providence, Rhode Island, 2010.
16. A.E. Green and K.A. Lindsay, Thermoelasticity, *J. Elast.* **2** (1972), https://doi.org/10.1007/BF00045689
17. A.E. Green and P.M. Naghdi, A re-examination of the basic postulates of thermomechanics, *Proc. R. Soc. Lond.* **432** (1991), 171–194.

18. ______, On undamped heat waves in an elastic solid, *J. Therm. Stress* **15** (1992), 253–264.
19. ______, Thermoelasticity without energy dissipation, *J. Elast.* **31** (1993), 189–208.
20. B. KALTENBACHER, I. LASIECKA, AND M. POSPIESZALSKA, Well-posedness and exponential decay of the energy in the nonlinear Jordan–Moore–Gibson–Thompson equation arising in high-intensity ultrasound, *Math. Models Methods Appl. Sci.* **22**(11) (2012), 195–207.
21. I. LASIECKA, Global solvability of Moore–Gibson–Thompson equation with memory arising in nonlinear acoustics, *J. Evol. Equ.* **17** (2017), 411–441.
22. I. LASIECKA AND X. WANG, General decay rate of Moore - Gibson - Thompson equation with memory-Part II, *Journal Differential Equations* **259**(12) (2015), 7610–7635.
23. ______, Moore Gibson Thompson equation with memory, part I, *ZAMP* **67**(17) (2016), DOI: 10.1007/s00033-015-0597-8
24. S. LIU AND R. TRIGGIANI, An inverse problem for a third order PDE arising in high-intensity ultrasound: Global uniqueness and stability by one boundary measurement, *Journal of Inverse and Ill-Posed Problems* **21**(6) (2013), 825–869.
25. ______, Inverse problem for a linearized Jordan-Moore-Gibson–Thompson equation, *New Prospects in Direct, Inverse and Control Problems for Evolution Equations* **10**(1) (2014), 305–351.
26. H. LORD AND Y. SHULMAN, A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids* **15** (1967), 299–309.
27. R. MARCHAND, T. MCDEVITT, AND R. TRIGGIANI, An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, *Math. Methods Appl. Sci.* **35**(15) (2012), 1896–1929.
28. A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, Berlin, 1983.
29. M. PELLICER AND J. SOLA-MORALES, Optimal scalar products in the Moore-Gibson-Thompson equation, *Evolution Equations and control Theory* **8** (2019), 203–220.
30. J.SH. SAFAROV AND D.K. DURDIEV, Inverse Problem for an Integro-Differential Equation of Acoustics, *Differential Equations* **54**(1) (2018), 134–142.
31. G.G. STOKES, An examination of the possible effect of the radiation of heat on the propagation of sound, *Philos. Mag. Ser.* **4**(1) (1851), 305–317.
32. D.Y. TZOU, *Macro-to-Microscale Heat Transfer: The Lagging Behavior,* 1st ed., Wiley, Hoboken, NJ, USA, 2014.
33. SH.K. VERMA AND S.M. BOULAARAS, Time-series price study of cryptocurrency: fractal dimensions and interpolation functions, *Fractals* (2025), 2540078.
34. S. ZOUATNIA, S. BOULAARAS, N.E. AMROUN, AND M.S. SOUID, Well-posedness and stability of coupled system of wave and strongly damped Petrovsky equations with internal fractional damping, *Afrika Matematika* **36**(22) (2025,) https://doi.org/10.1007/s13370-025-01238-4

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