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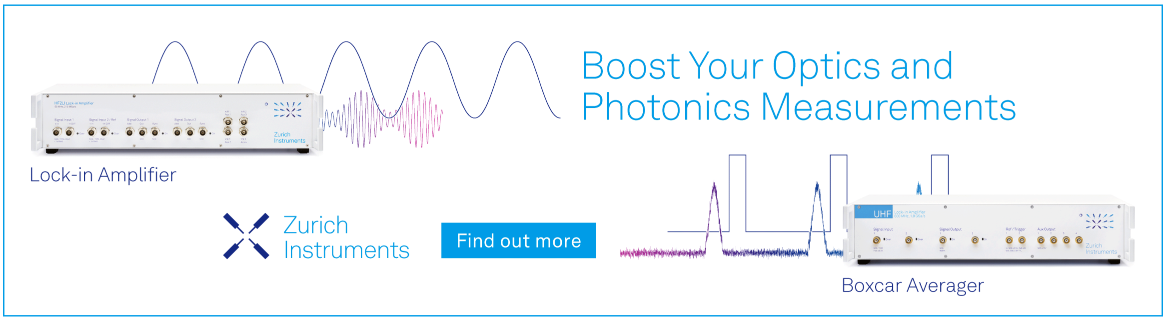

On an Optimal Method for the Approximate Solution of Singular Integral Equations

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Abstract. Many problems of science and engineering are naturally reduced to singular integral equations. Moreover plane problems are reduced to one dimensional singular integral equations. In the present paper, we develop an optimal algorithm for the approximate solution of one dimensional singular integral equations with the Cauchy kernel. Here we are engaged in finding the analytical form of the coefficients of the optimal quadrature formula. We apply these coefficients to an approximate solution of the Fredholm singular integral equation of the first kind. Thus, we show the possibility of solving singular integral equations with higher accuracy using the optimal quadrature formula based on the Sobolev method.

INTRODUCTION. STATEMENT OF THE PROBLEM

The study of various problems of mathematical physics, as well as specific problems from aerodynamics, electrodynamics, the theory of elasticity and other areas naturally reduces to singular integral equations. In this case, plane problems are reduced to solving the characteristic singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(x)}{x-t}\mathrm{d}x=\varphi(t), \quad t\in(-1,1). \tag{1}$$

Recall the definition of singular integrals in the sense of principal value of Cauchy.

Definition 1. The principal Cauchy value of the special integral $\int_a^b \frac{\varphi(x)}{x-t}dx$, $a<t<b$ is the limit

$$\lim_{\varepsilon\to 0}\left[\int_a^{t-\varepsilon}\frac{\varphi(x)}{x-t}dx+\int_{t+\varepsilon}^{b}\frac{\varphi(x)}{x-t}dx\right].$$

It is known that if a function φ on the segment $[a,b]$ satisfies the Holder condition with exponent α $(0<\alpha\leq 1)$ and coefficient A, i.e. if

$$|\varphi(x_1)-\varphi(x_2)|\leq A|x_1-x_2|^{\alpha},$$

then there exists the integral $\int_a^b \frac{\varphi(x)}{x-t}dx$, $a<t<b$.

Equation (1) has four complete analytical solutions corresponding to the values of parameter κ (see [1], pp. 49-50). In particular, for $\kappa=0$ the only solution of equation (1) in the class $h(1)$ is given by the formula

$$\phi(t)=-\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{\varphi(x)}{(x-t)}dx. \tag{2}$$

Thus, the solution of the singular integral equation of the form (1) can be reduced to calculation of the weighted singular integral (2). Therefore, the development of effective approximate methods for calculating singular integrals

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are of great applied importance and one of the actual problems of computational mathematics. Special techniques for constructing quadrature formulas uniformly approximating the integral (2) with respect to the variable t were proposed in Ph. Rabinowitz, S. Santi [2].

Also, the optimal quadrature formulas for numerical integration of integrals in Hilbert spaces were constructed and their errors were analysed [3, 4, 5, 6, 7].

Furthermore, optimal quadrature formulas for approximation of Fourier integrals were constructed in the many works (see, [8, 9, 10, 11, 12]).

In the works of V.V. Ivanov [13], optimization problem of calculation for singular integrals were considered. Study of these problems were continued in the works of B.G. Gabdulkhaev [14], I.V. Boykov [15], M.I. Isroilov, Kh.M. Shadimetov [16], Kh.M. Shadimetov [17], Kh.M. Shadimetov, A.R. Hayotov, D.M. Akhmedov [18, 19, 20, 21, 22].

While studying the method of discrete vortices, I. K. Lifanov constructed efficient methods for calculating the singular integral in the form (2) (see [1, 23]).

In the present paper, using the functional approach, we construct optimal quadrature formulas for the approximate calculation of the integral (2) in the space $L_2^{(3)}(-1,1)$.

We consider the following quadrature formula

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{\varphi(x)}{(x-t)}dx \cong \sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(x_\beta-1)+C_1[\beta]\varphi'(x_\beta-1)+C_2[\beta]\varphi''(x_\beta-1)\right), \tag{3}$$

here $-1<t<1$, $\varphi(x)$ is a function from the space $L_2^{(3)}(-1,1)$, $C_0[\beta],C_1[\beta],C_2[\beta]$ are coefficients, $x_\beta-1=h\beta-1$ are the nodes of quadrature formula (3), $[\beta]=h\beta$, $h=\frac{2}{N}$, $N=2,3,...,$ $n=0,1,2,...,m-1$.

Here $L_2^{(3)}(-1,1)$ is a Hilbert space of classes of all real valid functions φ defined on the interval $[-1,1]$ that differ by a polynomial of degree 2 and square integrable with derivative of order 3, i.e.

$$\|\varphi\|_{L_2^{(3)}(-1,1)}=\left(\int_{-1}^{1}\left(\varphi^{(3)}(x)\right)^2dx\right)^{\frac{1}{2}}.$$

The following difference is called *the error* of quadrature formula (3)

$$(\ell,\varphi)=\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{\varphi(x)}{(x-t)}\mathrm{d}x-\sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(x_\beta-1)+C_1[\beta]\varphi'(x_\beta-1)+C_2[\beta]\varphi''(x_\beta-1)\right)=\int_{-\infty}^{\infty}\ell(x)\varphi(x)\mathrm{d}x,$$

where ℓ is the error function of the formula (3) and has the form

$$\ell(x)=\sqrt{\frac{1+x}{1-x}}\frac{\varepsilon_{[-1,1]}(x)}{(x-t)}-\sum_{\beta=0}^{N}\left(C_0[\beta]\delta(x-x_\beta)-C_1[\beta]\delta'(x-x_\beta)+C_2[\beta]\delta''(x-x_\beta)\right). \tag{4}$$

Here $\varepsilon_{[-1,1]}(x)$ is the characteristic function of the interval $[-1,1]$, $\delta(x)$ is the Dirac delta-function.

Since the functional ℓ of the form (4) is defined on the space $L_2^{(3)}(-1,1)$, it belongs to the conjugate space $L_2^{(3)*}(-1,1)$, and satisfies the following equations (see [24])

$$(\ell,x^\alpha)=0 \quad \text{for} \quad \alpha=0,1,2. \tag{5}$$

The construction problem of optimal quadrature formulas of the form (3) in the sense of Sard [25] with the error functional (4) in the space $L_2^{(3)}(-1,1)$ for fixed nodes $x_\beta-1$ is to find the quantity

$$\|\overset{\circ}{\ell}|L_2^{(3)*}(-1,1)\|=\inf_{C_0[\beta],C_1[\beta],C_2[\beta]}\left(\sup_{\varphi,\|\varphi\|\neq 0}\frac{|(\ell,\varphi)|}{\|\varphi|L_2^{(3)}(-1,1)\|}\right). \tag{6}$$

This task has two parts:

Problem 1. Find the norm of the error functional ℓ of quadrature formulas of the form (3) in the space $L_2^{(3)*}(-1,1)$
Problem 2. Find the coefficients $C_0[\beta], C_1[\beta], C_2[\beta]$ which satisfy equality (6) when the nodes $x_\beta - 1$ are fixed.

If there are such coefficients $C_0[\beta] = \overset{\circ}{C_0}[\beta], C_1[\beta] = \overset{\circ}{C_1}[\beta], C_2[\beta] = \overset{\circ}{C_2}[\beta]$ that satisfy equality (6), they are called *optimal coefficients* and the corresponding formula is called *optimal quadrature formula*.

The rest of the paper is organized as follows. In Section 2, we give some auxiliary results and definitions that are used in solving above problems. In Section 3 we give the algorithm for construction of optimal quadrature formulas of the form (3).

AUXILIARY RESULTS

In this section we give some definitions and known results that we need to prove the main results.

Below mainly we use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given, for instance, in [24]. We give some definitions about functions of discrete argument.

Definition 2. The function $\varphi[\beta] = \varphi(h\beta)$ is *a function of the discrete argument* if it is given on some set of integer values β.

Definition 3. *The inner product* of two discrete functions $\varphi[\beta]$ and $\psi[\beta]$ is given by

$$[\varphi, \psi] = \sum_{\beta \in B} \varphi[\beta] \cdot \psi[\beta], \tag{7}$$

if the series on the right side of equality (7) converges absolutely.

Definition 4. *The convolution* of two functions $\varphi[\beta]$ and $\psi[\beta]$ of a discrete argument is called the inner product

$$\chi[\beta] = \varphi[\beta] * \psi[\beta] = [\varphi[\gamma], \psi[\beta - \gamma]] = \sum_{\gamma=-\infty}^{\infty} \varphi[\gamma]\psi[\beta - \gamma]. \tag{8}$$

In addition, in the calculations we need the discrete analogue $D_1(h\beta)$ of the differential operator d^2/dx^2, which is defined by the following formula (see [26])

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0. \end{cases} \tag{9}$$

Here are some properties of the discrete function $D_1(h\beta)$ (see [24]):

$$D_1(h\beta) * (h\beta)^\alpha = 0, \quad \alpha = 0, 1, \quad hD_1(h\beta) * \tfrac{|h\beta|}{2} = \delta_d(h\beta), \tag{10}$$

where $\delta_d(h\beta)$ is the discrete delta-function and $\delta_d(h\beta) = \begin{cases} 0, & \beta \neq 0, \\ 1, & \beta = 0. \end{cases}$

MAIN RESULTS

In the present paper, we solve this Problems 1 and 2 i.e., we calculate the norm of the error functional ℓ and minimize it by the coefficients $C_0[\beta], C_1[\beta], C_2[\beta]$ when the nodes $x_\beta - 1$ are fixed. For this we use the concept of extremal function of the error functional ℓ introduced by S. L. Sobolev [24].

The function ψ, for which the equality holds

$$(\ell, \psi) = \|\ell | L_2^{(3)*}\| \cdot \|\psi | L_2^{(3)}\|, \tag{11}$$

is called the *extremal function* for the error functional ℓ.

Since the space $L_2^{(3)}(-1,1)$ is a Hilbert space, by the Riesz theorem on the general form of a linear functional (see [27]) there exists a unique function $\psi_\ell \in L_2^{(3)}(-1,1)$ for which the equality holds

$$(\ell_N, \varphi) = <\psi_\ell, \varphi> \tag{12}$$

and $\|\ell\| = \|\psi_\ell\|$, here $<\psi_\ell, \varphi>$ is the inner product of two functions ψ_ℓ and φ from the space $L_2^{(3)}(-1,1)$.
Recall that the inner product $<\psi_\ell, \varphi>$ is defined as follows

$$<\psi_\ell, \varphi> = \int_{-1}^{1} \psi_\ell'''(x)\varphi'''(x)dx. \tag{13}$$

The extremal function $\psi_\ell(x)$ of the functional ℓ in the space $L_3^{(3)}(-1,1)$ was found by Sobolev [24] and has the form

$$\psi_\ell(x) = -\ell(x) * G_3(x) + P_2(x), \tag{14}$$

here

$$G_3(x) = \frac{x^5 \mathtt{sign(x)}}{240},$$

$$\mathtt{sign(x)} = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$ $P_2(x)$ is a polynomial of degree 2.

Since the space $L_2^{(3)}(-1,1)$ is a Hilbert space, then by the Riesz theorem on the general form of a linear functional and taking into account the definition of an extremal function, we have

$$(\ell, \psi_\ell) = \|\ell|L_2^{(3)*}(-1,1)\| \cdot \|\psi_\ell|L_2^{(3)}(-1,1)\| = \|\psi_\ell|L_2^{(3)}(-1,1)\|^2 = \|\ell|L_2^{(3)*}(-1,1)\|^2.$$

Where

$$\|\ell|L_2^{(3)*}(-1,1)\|^2 = (\ell, \psi_\ell) = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)dx.$$

Hence, using (4), (5), (14), for $\|\ell\|^2$ we obtain

$$\|\ell|L_2^{(3)*}(-1,1)\|^2 = \sum_{k=0}^{2}\sum_{\alpha=0}^{2}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k C_k[\gamma]C_\alpha[\beta]\frac{(h\beta - h\gamma)^{5-\alpha-k}\mathtt{sign}(h\beta - h\gamma)}{2(5-\alpha-k)!}$$

$$-2\sum_{\alpha=0}^{2}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{(x-(h\beta-1))^{5-\alpha}\mathtt{sign}(x-(h\beta-1))}{2(5-\alpha)!(x-t)}\mathtt{dx}$$

$$+\frac{1}{240}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1+y}{1-y}}\frac{(x-y)^5\mathtt{sign}(x-y)}{(x-t)(y-t)}\mathtt{dxdy}, \tag{15}$$

Now we minimize the norm (15) of the error functional of quadrature formulas under the condition (5). It should be noted that minimizing $\|\ell\|^2$ over all $C_0[\beta], C_1[\beta], C_2[\beta]$ $\beta = 0,1,2,...,N$ is a very difficult problem. Therefore, in this paper we propose a successive minimization of $\|\ell\|^2$ with respect to $C_0[\beta], C_1[\beta], C_2[\beta]$, i.e. first consider the norm $\|\ell\|^2$ will be minimized in $C_0[\beta]$, in the space $L_2^{(1)}(-1,1)$, and using the found optimal coefficients $\overset{\circ}{C}_0[\beta]$ the value

$\|\ell\|^2$ will be minimized in $C_1[\beta]$ in the space $L_2^{(2)}(-1,1)$, then putting the optimal coefficients $\overset{\circ}{C}_0[\beta]$ and $\overset{\circ}{C}_1[\beta]$ to the value $\|\ell_N\|^2$ will be minimized in $C_2[\beta]$ in the space $L_2^{(3)}(-1,1)$.

Next, we implement algorithm said the above.

To do this, we use the method of indefinite Lagrange multipliers.

We consider the auxiliary function

$$\Phi(C,\lambda)=\|\ell|L_2^{(3)*}(-1,1)\|^2+2\sum_{p=0}^{2}\lambda_p(\ell,x^p)$$

here $C=(C_0[0],C_0[1],...,C_0[N],C_1[0],C_1[1],...,C_1[N],C_2[0],C_2[1],...,C_2[N])$, $\lambda=(\lambda_0,\lambda_1,\lambda_2)$.

Consider in the space $L_2^{(1)}(-1,1)$, then $\|\ell\|^2$ depends only on $C_0[\beta]$.

Equating to zero the variation in $C_0[\beta]$ and λ_0 of $\Phi(C,\lambda)$, we obtain the following system of linear equations in $C_0[\beta]$ and λ_0

$$\begin{cases} \sum\limits_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)\mathtt{sign}(h\beta-h\gamma)}{2}+\lambda_0=f_0[\beta], \;\; \beta=0,1,2,...,N, \\ \sum\limits_{\gamma=0}^{N} C_0[\gamma]=g_0, \end{cases} \tag{16}$$

where

$$f_0[\beta]=\int_{-1}^{1}\sqrt{\tfrac{1+x}{1-x}}\,\frac{(x-(h\beta-1))\mathtt{sign}(x-(h\beta-1))}{2(x-t)}dx, \quad g_0=\int_{-1}^{1}\sqrt{\tfrac{1+x}{1-x}}\,\frac{1}{(x-t)}dx.$$

We note that. The obtained system (16) was solved in the work [16] and the following theorems were proved

Theorem 1. *Among all quadrature formulas of the form*

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,\frac{\varphi(x)}{(x-t)}dx\cong\sum_{\beta=0}^{N}C_0[\beta]\varphi(x_\beta-1),$$

with the error functional

$$\ell(x)=\sqrt{\frac{1+x}{1-x}}\,\frac{\varepsilon_{[-1,1]}(x)}{(x-t)}-\sum_{\beta=0}^{N}C_0[\beta]\delta(x-x_\beta),$$

in the space $L_2^{(1)}(-1,1)$*, there is a unique quadrature formula whose coefficients are defined by the equalities*

$$\begin{aligned} C_0[0] &= h^{-1}\left[f_0[1]-\tfrac{\pi}{2}(2+t-h)\right], \\ C_0[\beta] &= h^{-1}\left[f_0[\beta-1]-2f_0[\beta]+f_0[\beta+1]\right], \quad \beta=\overline{1,N-1}, \\ C_0[N] &= h^{-1}\left[f_0[N-1]+\tfrac{\pi}{2}(t+h)\right], \end{aligned}$$

here

$$f_0[\beta] \;=\; -(2+t-h\beta)\mathtt{arcsin}(h\beta-1)+\sqrt{1-(h\beta-1)^2}+(t-h\beta+1)\sqrt{\tfrac{1+t}{1-t}}\mathtt{ln}\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|.$$

Further, in the space $L_2^{(2)}(-1,1)$. In this case the value $\|\ell\|^2$ depends on $C_0[\beta]$ and $C_1[\beta]$, then using the solution $C_0[\beta]$ and λ_0 of the system (16) equating to zero the variation in $C_1[\beta]$ and λ_1 of $\Phi(C,\lambda)$, we obtain

$$\begin{cases} \sum\limits_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^2\mathtt{sign}(h\beta-h\gamma)}{4} - \sum\limits_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)\mathtt{sign}(h\beta-h\gamma)}{2} + \lambda_1 = f_1[\beta], \\ \beta = 0,1,2,...,N, \\ \sum\limits_{\gamma=0}^{N}\Big(C_0[\gamma](h\gamma-1)+C_1[\gamma]\Big) = g_1, \end{cases} \tag{17}$$

where

$$f_1[\beta] = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{(x-(h\beta-1))^2\mathtt{sign}(x-(h\beta-1))}{4(x-t)}dx, \tag{17a}$$

$$g_1 = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{x}{(x-t)}dx. \tag{17b}$$

In the space $L_2^{(3)}(-1,1)$. Here $C_0[\beta]$ and $C_1[\beta]$ are known. Then equating to zero the variation in $C_2[\beta]$ and λ_2 of $\Phi(C,\lambda)$, we obtain

$$\begin{cases} \sum\limits_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^3\mathtt{sign}(h\beta-h\gamma)}{12} - \sum\limits_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)^2\mathtt{sign}(h\beta-h\gamma)}{4} + \sum\limits_{\gamma=0}^{N} C_2[\gamma]\frac{(h\beta-h\gamma)\mathtt{sign}(h\beta-h\gamma)}{2} + \lambda_2 = f_2[\beta], \\ \beta = 0,1,2,...,N, \\ \sum\limits_{\gamma=0}^{N}\Big(C_0[\gamma](h\gamma-1)^2+2C_1[\gamma](h\gamma-1)+2C_2[\gamma]\Big) = g_2, \end{cases} \tag{18}$$

where

$$f_2[\beta] = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{(x-(h\beta-1))^3\mathtt{sign}(x-(h\beta-1))}{12(x-t)}dx, \tag{18a}$$

$$g_2 = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{x^2}{(x-t)}dx. \tag{18b}$$

Next, we find the optimal coefficients $\overset{\circ}{C_2}[\gamma]$, $\gamma = 0,1,2,...,N$ and the unknowns $\overset{\circ}{\lambda_2}$, which are solution of system (18).

Now we solve the system (17). Here we use Theorem 1.

Let us rewrite system (17) in the following form

$$\begin{cases} \sum\limits_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)\mathtt{sign}(h\beta-h\gamma)}{2} - \lambda_1 = F_1[\beta], \quad \beta = \overline{0,N}, \\ \sum\limits_{\gamma=0}^{N} C_1[\gamma] = g_1 - \sum\limits_{\gamma=0}^{N} C_0[\gamma](h\gamma-1), \end{cases} \tag{19}$$

here

$$F_1[\beta] = f_1[\beta] + \sum_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^2\mathtt{sign}(h\beta-h\gamma)}{4}, \tag{20}$$

$f_1[\beta]$ and g_1 are determined with formulas (17a) and (17b).

Let $C_1[\beta]=0$ at $\beta<0$ and $\beta>N$.

We introduce the following notation

$$v(h\beta)=C_1[\beta]*\frac{|h\beta|}{2}, \tag{21}$$

$$u(h\beta)=v(h\beta)-\lambda_1. \tag{22}$$

Using properties (10) of the operator $D_1(h\beta)$ from (9) and (22) we obtain

$$C_1[\beta]=hD_1(h\beta)*u(h\beta). \tag{23}$$

But for calculate the convolution (23) we need to define the function $u(h\beta)$ for all integer values of β. For $\beta=0,1,2,...,N$, from (19) we have $u(h\beta)=F_1[\beta]$. Therefore, it suffices for us to define the function $u(h\beta)$ when $\beta<0$ and $\beta>N$.

Next, we define the form $u(h\beta)$ for $\beta\leq 0$ and $\beta\geq N$. From (21) using (19) with $\beta\leq 0$ and $\beta\geq N$ we have

$$v(h\beta)=\begin{cases} -\frac{h\beta}{2}\left(g_1-\sum\limits_{\gamma=0}^{N}C_0[\gamma](h\gamma-1)\right)-\lambda_1^-, & \beta\leq 0,\\ \frac{h\beta}{2}\left(g_1-\sum\limits_{\gamma=0}^{N}C_0[\gamma](h\gamma-1)\right)+\lambda_1^-, & \beta\geq N.\end{cases}$$

Taking into account the last equality, from (22) for $u(h\beta)$ we obtain

$$u(h\beta)=\begin{cases} -\frac{h\beta}{2}\left(g_1-\sum\limits_{\gamma=0}^{N}C_0[\gamma](h\gamma-1)\right)+a_0^-, & \beta\leq 0,\\ F_1[\beta], & 0\leq\beta\leq N,\\ \frac{h\beta}{2}\left(g_1-\sum\limits_{\gamma=0}^{N}C_0[\gamma](h\gamma-1)\right)+a_0^+, & \beta\geq N,\end{cases} \tag{24}$$

here a_0^-, a_0^+ are unknowns and

$$a_0^-=-\lambda_1-\lambda_1^-,\quad a_0^+=-\lambda_1+\lambda_1^-. \tag{25}$$

If we find the unknowns a_0^-, a_0^+, then from (25) we have

$$\lambda_1=-\tfrac{1}{2}(a_0^-+a_0^+),\quad \lambda_1^-=\tfrac{1}{2}(a_0^+-a_0^-). \tag{26}$$

The unknowns a_0^-, a_0^+ can be found from (24). Then we obtain the explicit form of the function $u(h\beta)$ and from (23) we find the optimal coefficients $C_1[\beta]$. Moreover, from (26) there are λ_1. Thus, the problem is completely resolved.

The following theorem is true.

Theorem 2. *The optimal coefficients of the quadrature formulas of the form*

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{\varphi(x)}{(x-t)}dx\cong\sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(x_\beta-1)+C_1[\beta]\varphi'(x_\beta-1)\right),$$

with the error functional

$$\ell(x)=\sqrt{\frac{1+x}{1-x}}\frac{\varepsilon_{[-1,1]}(x)}{(x-t)}-\sum_{\beta=0}^{N}\left(C_0[\beta]\delta(x-x_\beta)-C_1[\beta]\delta'(x-x_\beta)\right),$$

in the space $L_2^{(2)}(-1,1)$ *are determined by the formulas*

$$C_1[0]=h^{-1}\left[f_1[1]+\tfrac{h}{2}f_0[1]+\tfrac{h}{4}\pi(t+2)-\tfrac{\pi}{4}(t^2+3t+3.5)\right],$$
$$C_1[\beta]=h^{-1}\left[f_1[\beta-1]-2f_1[\beta]+f_1[\beta+1]-\tfrac{h}{2}\left(f_0[\beta-1]-2f_0[\beta]+f_0[\beta+1]\right)\right.$$
$$\left.+\tfrac{h^2}{2}\left(2\sum_{\gamma=0}^{\beta}C_0[\gamma]-\pi\right)\right],\quad \beta=1,2,...,N-1$$
$$C_1[N]=h^{-1}\left[f_1[N-1]-\tfrac{h}{2}f_0[N-1]+\tfrac{h}{4}\pi t+\tfrac{\pi}{4}(t^2-t-0.5)\right],$$

where

$$f_1[\beta]=-\tfrac{1}{2}\left[\left(2(h\beta-1)-\tfrac{1}{2}(h\beta+2t+1)\right)\sqrt{1-(h\beta-1)^2}+\left((h\beta-1)^2+\tfrac{1}{2}(2t^2+2t+1)-2(h\beta-1)(t+1)\right)\times\right.$$
$$\left.\times\arcsin(h\beta-1)+(t-(h\beta-1))^2\sqrt{\tfrac{1+t}{1-t}}\ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\right].$$

and $C_0[\beta],\ \beta=0,1,...,N$ *are defined in Theorem 1.*

Proof. From (24) for $\beta=0$ and $\beta=N$ immediately get a_0^- and a_0^+

$$a_0^-=F_1[0],\qquad a_0^+=F_1[N]-g_1+\sum_{\gamma=0}^{N}C_0[\gamma](h\gamma-1).$$

This means that we have obtained an explicit form of the function $u(h\beta)$.

Further, using (24) from (23) and calculating the convolution $hD_1(h\beta)*u(h\beta)$ for $\beta=\overline{0,N}$ we obtain

$$C_1[\beta]=hD_1(h\beta)*u(h\beta)=h\sum_{\gamma=-\infty}^{\infty}D_1(h\beta-h\gamma)u(h\gamma)$$

$$=h\left[\sum_{\gamma=1}^{\infty}D_1(h\beta+h\gamma)\left(\frac{h\gamma}{2}\left(g_1-\sum_{\gamma=0}^{N}C_0[\gamma]\cdot(h\gamma-1)\right)+a_0^-\right)\right.$$

$$+\sum_{\gamma=0}^{\infty}D_1(h\beta-h\gamma)F_1[\gamma]$$

$$\left.+\sum_{\gamma=1}^{\infty}D_1(h(N+\gamma)-h\beta)\left(\frac{2+h\gamma}{2}\left(g_1-\sum_{\gamma=0}^{N}C_0[\gamma]\cdot(h\gamma-1)\right)+a_0^+\right)\right].$$

Hence, taking into account (9) for $\beta=\overline{0,N}$, we have

$$C_1[0]=h^{-1}\left[F_1[1]-F_1[0]+\frac{h}{2}\left(g_1-\sum_{\alpha=0}^{N}C_0[\alpha]\cdot(h\alpha-1)\right)\right],$$

$$C_1[\beta]=h^{-1}\left[F_1[\beta-1]-2F_1[\beta]+F_1[\beta+1]\right],\ \ \beta=1,2,...,N-1,$$

$$C_1[N]=h^{-1}\left[F_1[N-1]-F_1[N]+\frac{h}{2}\left(g_1-\sum_{\alpha=0}^{N}C_0[\alpha]\cdot(h\alpha-1)\right)\right].$$

Hence, using (9) and formula (10), taking into account (17a) and (17b), after some calculations, we arrive at the expressions for the coefficients $C_1[\beta], \beta = 0,1,2,...,N$ which are given in the statement of the theorem.

Theorem 2 is proved.

Next we solve the system (18). Here we use Theorems 1 and 2.

Then, the system of equations (18) is rewritten as follow

$$\begin{cases} \sum\limits_{\gamma=0}^{N} C_2[\gamma]\frac{(h\beta-h\gamma)\mathtt{sign}(h\beta-h\gamma)}{2} + \lambda_2 = F_2[\beta], \;\; \beta = 0,1,2,...,N, \\ \sum\limits_{\gamma=0}^{N} C_2[\gamma] = \frac{1}{2}\left(g_2 - \sum\limits_{\gamma=0}^{N}\left(C_0[\gamma](h\gamma-1)^2 + 2C_1[\gamma](h\gamma-1)\right)\right), \end{cases} \tag{27}$$

where

$$F_2[\beta] = f_2[\beta] + \sum_{\gamma=0}^{N} C_1[\gamma]\frac{(h\beta-h\gamma)^2\mathtt{sign}(h\beta-h\gamma)}{4} - \sum_{\gamma=0}^{N} C_0[\gamma]\frac{(h\beta-h\gamma)^3\mathtt{sign}(h\beta-h\gamma)}{12},$$

$f_2[\beta]$ and g_2 are determined with formulas (18a) and (18b).

Calculations for the system (17) are also done for system (27). And the following holds.

Theorem 3. *The optimal coefficients of the quadrature formulas of the form (3) with the error functional*

$$\ell(x) = \sqrt{\frac{1+x}{1-x}}\frac{\varepsilon_{[-1,1]}(x)}{(x-t)} - \sum_{\beta=0}^{N}\left(C_0[\beta]\delta(x-x_\beta) - C_1[\beta]\delta'(x-x_\beta) + C_2[\beta]\delta''(x-x_\beta)\right),$$

in the space $L_2^{(3)}(-1,1)$ are determined by the formulas

$$C_2[0] = h^{-1}\left[f_2[1] + \frac{h}{2}f_1[1] + \frac{h^2}{12}f_0[1] - \frac{h^2\pi}{24}(t+2) + \frac{h\pi}{8}(t^2+3t+\frac{7}{2}) - \frac{\pi}{24}(2t^3+8t^2+13t+12)\right],$$

$$C_2[\beta] = h^{-1}\Bigg[f_2[\beta-1] - 2f_2[\beta] + f_2[\beta+1] - \frac{h}{2}(f_1[\beta-1] - 2f_1[\beta] + f_1[\beta+1]) - \frac{h^2}{12}\left(f_0[\beta-1] - 2f_0[\beta] + f_0[\beta+1]\right)$$
$$+\frac{h^3}{4}\left(2\sum_{\gamma=0}^{\beta} C_0[\gamma](2\gamma-2\beta-1) + \pi(1+2\beta)\right) + \frac{h^2}{2}\left(2\sum_{\gamma=0}^{\beta} C_1[\gamma] - \pi(2+t)\right)\Bigg], \;\; \beta = 1,2,...,N-1,$$

$$C_2[N] = h^{-1}\left[f_2[N-1] - \frac{h}{2}f_1[N-1] + \frac{h^2}{12}f_0[N-1] + \frac{h^2}{24}\pi t + \frac{h\pi}{8}(t^2-t-0.5) + \frac{\pi}{24}(2t^3-4t^2+t+2)\right],$$

where

$$f_2[\beta] = \frac{1}{12}\Bigg[\left(\frac{11}{3}(h\beta-1)^2 - 5(h\beta-1)(t+1) + \frac{1}{3}(6t^2+6t+4)\right)\sqrt{1-(h\beta-1)^2} + \Bigg(-2(t-h\beta+1)^3 - 6(h\beta-1)^2$$
$$+3(h\beta-1)(2t+1) - 2t^2 - t - 1\Bigg)\mathtt{arcsin}(h\beta-1) + (t-(h\beta-1))^2\sqrt{\frac{1+t}{1-t}}\mathtt{ln}\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg].$$

and $C_0[\beta]$, $C_1[\beta]$, $\beta = 0,1,...,N$ are defined satisfy in Theorem 1 and 2.

Theorem 3 is proved similarly as Theorem 2.

CONCLUSION

In conclusion, we note that in the Sobolev space $L_2^{(3)}(-1,1)$ an optimal quadrature formula for the approximate solution of singular integral equations with the Cauchy kernel is constructed. Here we have found analytic forms for the coefficients of the constructed optimal quadrature formula. The method of indefinite Lagrange multipliers and a discrete analogue of the differential operator $\frac{d^2}{dx^2}$ are used. Explicit formulas for optimal coefficients are obtained. We can apply these coefficients to an approximate solution of the Fredholm singular integral equation of the first kind. The possibility of solving singular integral equations with higher accuracy using the optimal quadrature formula based on the Sobolev method is shown.

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