RESEARCH ARTICLE | MARCH 11 2024

On determination of optimal coefficients of a quadrature formula of Hermite type in the Sobolev space $\widetilde{W}_2^{(m)}(T_1)$

Hakim Rustamov; Islomjon Jalolov; Bekzodjon Isomiddinov

AIP Conf. Proc. 3004, 060029 (2024)
https://doi.org/10.1063/5.0199865

On Determination of Optimal Coefficients of a Quadrature Formula of Hermite Type in the Sobolev Space $\tilde{W}_2^{(m)}(T_1)$

Hakim Rustamov,[a] Islomjon Jalolov,[b] and Bekzodjon Isomiddinov[c]

Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan.

[a] *Corresponding author: o_jalolov@mail.ru*
[b] *wera00@mail.ru*
[c] *rhsh0405@gmail.com*

Abstract. Numerous publications are devoted to quadrature formulas; they include the values of derivatives of integrable functions. When, besides the values of function f at points x on T_1, the values of its derivatives of some orders are also known, then naturally, with the correct use of all these data, a more accurate result can be expected than in the case of using only the values of the functions.
For the error functional of the quadrature formula of Hermite type for functions of class $\tilde{W}_2^{(m)}(T_1)$, the norms are found; an upper bound is obtained, and the optimal coefficients of the quadrature formula of Hermite type are determined for $p(x) = 1$ and $m = 4$ $(\alpha = 0, 1, 2, 3)$.

INTRODUCTION

It is known that the construction of quadrature formulas based on the methods of functional analysis was considered first in the studies by A.Sard [1] and S.M.Nikolskii [2], and the cubature formulas were considered by S.L.Sobolev [3]. S.L.Sobolev studied the problem of constructing optimal lattice formulas over the space $L_2^{(m)}(R^n)$ and reduced the finding of optimal coefficients to solving a discrete Wiener - Hopf problem (see [1]).

In the one-dimensional case, i.e. in the space $L_2^{(m)}(R)$, the continuous Wiener - Hopf problem was solved by Z.J.Jamolov (see [4]). Publications of many researchers were devoted to the construction of optimal interpolation, optimal quadrature and cubature formulas using the method proposed by S.L.Sobolev (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

In this connection, we consider the weighted quadrature formula of Hermite type

$$\int_{T_1} p(x) f(x)\, dx \approx \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} (-1)^{\alpha} c_{\lambda}^{(\alpha)} f^{(\alpha)}\left(x^{(\lambda)}\right), \tag{1}$$

in the Sobolev space $\tilde{W}_2^{(m)}(T_1)$, where $c_{\lambda}^{(\alpha)}$ and $x^{(\lambda)}$ are, respectively, the arbitrary coefficients and nodes of the quadrature formula (1), $f(x) \in \tilde{W}_2^{(m)}(T_1)$, T_1 is the one - dimensional torus, i.e. a circle of length equal to one, $p(x)$ is the weight function and α is the order of derivatives.

Definition 1. The space $\tilde{W}_2^{(m)}(T_1)$ is defined as a space of functions set on a one - dimensional torus T_1 and having all generalized derivatives of order m summable with a square in norm [1]

$$\left\| f | \tilde{W}_2^{(m)}(T_1) \right\|^2 = \left(\int_{T_1} f(x)\, dx \right)^2 + \sum_{k \neq 0} |2\pi k|^{2m} \left| \hat{f}_k \right|^2, \tag{2}$$

with the inner product

$$(f(x), \varphi(x)) = \int_{T_1} f^{(m)}(x)\, \varphi^{(m)}(x)\, dx + \left(\int_{T_1} f(x)\, dx \right) \left(\int_{T_1} \varphi(x)\, dx \right), \tag{3}$$

where $\hat{f}_k$ are the Fourier coefficients, i.e. $\hat{f}_k = \int_{T_1} f(x)\, e^{-2\pi i k x} dx$.

International Scientific and Practical Conference on "Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)"
AIP Conf. Proc. 3004, 060029-1–060029-12; https://doi.org/10.1063/5.0199865
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

The difference between the integral and the quadrature sum, i.e.

$$\int_{T_1} p(x) f(x)\,dx - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} (-1)^{\alpha} c_{\lambda}^{(\alpha)} f^{(\alpha)}\left(x^{(\lambda)}\right)$$

$$= \int_{T_1} \left[p(x)\,\varepsilon_{T_1}(x) - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} c_{\lambda}^{(\alpha)} \delta^{(\alpha)}\left(x - x^{(\lambda)}\right)\right] f(x)\,dx = < \ell_N^{(\alpha)}, f >,$$

is called the error of the quadrature formula (1), and this difference corresponds to the generalized function

$$\ell_N^{(\alpha)}(x) = p(x)\,\varepsilon_{T_1}(x) - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} c_{\lambda}^{(\alpha)} \delta^{(\alpha)}\left(x - x^{(\lambda)}\right), \tag{4}$$

and we call it the error functional of the quadrature formula (1). Here $\varepsilon_{T_1}(x)$ - is the characteristic function T_1, and $\delta(x)$ is the Dirac delta function.

STATEMENT OF THE PROBLEM

As it is known, the problem of estimating the error of a quadrature formula on functions of some space B is equivalent to calculating the value of the norm of error functional in the space B^*, conjugate to B, or, what is the same, to finding the extremal function for a given quadrature formula. To solve this problem, we took the space $\tilde{W}_2^{(m)}(T_1)$ as B.

The task of constructing optimal quadrature formulas over the Sobolev space $\tilde{W}_2^{(m)}(T_1)$ is to calculate the following quantity:

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\| = \inf_{c_{\lambda}^{(\alpha)}, x^{(\lambda)}} \sup_{\|f(x)\| \neq 0} \frac{\left| < \ell_N^{(\alpha)}, f > \right|}{\left\| f | \tilde{W}_2^{(m)}(T_1) \right\|}, \tag{5}$$

where $\tilde{W}_2^{(m)^*}(T_1)$ is a conjugate space to the space $\tilde{W}_2^{(m)}(T_1)$.

To estimate the error of the quadrature formula, it is necessary to solve the following problem.

Problem 1. Find the norm of the error functional (4) of the given quadrature formula.

Further, to construct the optimal quadrature formula, it is necessary to solve the following problem.

Problem 2. Find values $c_{\lambda}^{(\alpha)}$ and $x^{(\lambda)}$, such that equality (5) is satisfied.

In the present article, an optimal quadrature formula is constructed in the space $\tilde{W}_2^{(m)}(T_1)$ of periodic functions; the norm of the error functional of the constructed quadrature formula in the conjugated space $\tilde{W}_2^{(m)*}(T_1)$ is calculated, and an extremal function is found for this quadrature formula. For the error functional of the quadrature formula of Hermite type for functions of class $\tilde{W}_2^{(m)}(T_1)$, an upper bound is obtained and the optimal coefficients of the quadrature formula of Hermite type are found for $p(x) = 1$ and $m = 4\,(\alpha = 0, 1, 2, 3)$. Note that Problem 1 was solved for $p(x) = 1$ in [14] and Problem 2 was solved for $\alpha = 0$ in [15].

NORM AND EXTREMAL FUNCTION OF THE ERROR FUNCTIONAL OF WEIGHTED QUADRATURE FORMULAS OF HERMITE TYPE IN THE SOBOLEV SPACE PERIODIC FUNCTIONS

To find the norm of the error functional (4) in the space $\tilde{W}_2^{(m)^*}(T_1)$, its extremal function is used.

Definition 2. Function $\psi_{\ell}(x)$ is called an extremal function of functional $\ell_N^{(\alpha)}$, if the following equality holds

$$< \ell_N^{(\alpha)}, \psi_{\ell} > = \left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\| \cdot \left\| \psi_{\ell} | \tilde{W}_2^{(m)}(T_1) \right\|.$$

As the space $\tilde{W}_2^{(m)}(T_1)$ with the inner product (3) becomes Hilbert, then, based on the Riesz theorem on the general form of the linear continuous functional, there is a unique function $\psi_\ell \in \tilde{W}_2^{(m)}(T_1)$ for which

$$< \ell_N^{(\alpha)}(x), f(x) >=< \psi_\ell(x), f(x) > \text{ and } \left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\| = \left\| \psi_\ell | \tilde{W}_2^{(m)}(T_1) \right\|. \tag{6}$$

In particular, from (6) for $f = \psi_\ell$ we have

$$< \ell_N^{(\alpha)}, \psi_\ell >=< \psi_\ell, \psi_\ell >= \|\psi_\ell\|^2 = \|\psi_\ell\| \cdot \left\| \ell_N^{(\alpha)} \right\| = \left\| \ell_N^{(\alpha)} \right\|^2. \tag{7}$$

The following is true

Theorem 1. *The square of the norm of the error functional (4) of a weighted quadrature formula of Hermite type in the form (1) over the space $\tilde{W}_2^{(m)}(T_1)$ is*

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 = \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\left| \hat{p}_k - \sum\limits_{\alpha=0}^{m-1} \sum\limits_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i)^\alpha k^\alpha e^{2\pi i k x^{(\lambda)}} \right|^2}{k^{2m}}, \tag{8}$$

where $c_\lambda^{(\alpha)}$ are the coefficients, $x^{(\lambda)}$ are the nodes of the quadrature formula (1), and $\hat{p}_k$ are the Fourier coefficients of function $p(x)$.

Proof. It is known [3] that the following equation holds for a function $f \in \tilde{W}_2^{(m)}(T_1)$:

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi i k x} = \sum_k \hat{f}_k e^{2\pi i k x}, \tag{9}$$

where $\hat{f}_k = \int\limits_{T_1} f(x) e^{-2\pi i k x} dx$, i.e. Fourier coefficients.

Therefore, we have

$$< \ell_N^{(\alpha)}, f >=< \ell_N^{(\alpha)}, \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi i k x} >= \sum_{k=-\infty}^{\infty} \hat{f}_k < \ell_N^{(\alpha)}, e^{2\pi i k x} > = \sum_{k=-\infty}^{\infty} \hat{f}_k \hat{\ell}_k^{(\alpha)}$$

$$= \hat{f}_0 \hat{\ell}_0^{(\alpha)} + \sum_{k \neq 0} \hat{f}_k \hat{\ell}_k^{(\alpha)}. \tag{10}$$

Here $\hat{\ell}_k^{(\alpha)} = \int\limits_{T_1} \ell_N^{(\alpha)}(x) e^{2\pi i k x} dx$.

Now we sequentially calculate the value of the Fourier coefficients $\hat{\ell}_0^{(\alpha)}$ and $\hat{\ell}_k^{(\alpha)}$.

$$\hat{\ell}_0^{(\alpha)} = \int_{T_1} \ell_N^{(\alpha)}(x) dx \int_{T_1} [p(x) \varepsilon_{T_1}(x)] dx - \int_{T_1} \left[\sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \delta^{(\alpha)} \left(x - x^{(\lambda)} \right) \right] dx = \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)}. \tag{11}$$

$$\hat{\ell}_k^{(\alpha)} =< \ell_N^{(\alpha)}, e^{2\pi i k x} >=< p(x) \varepsilon_{T_1}(x) - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \delta^{(\alpha)} \left(x - x^{(\lambda)} \right), e^{2\pi i k x} >=< p(x) \varepsilon_{T_1}(x), e^{2\pi i k x} >$$

$$- < \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \delta^{(\alpha)} \left(x - x^{(\lambda)} \right), e^{2\pi i k x} >= \hat{p}_k - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i k)^\alpha e^{2\pi i k x^{(\lambda)}},$$

i.e.

$$\hat{\ell}_k^{(\alpha)} = \hat{p}_k - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i k)^\alpha e^{2\pi i k x^{(\lambda)}}. \tag{12}$$

Applying the Schwartz inequality to the right side of (10), we obtain the following estimate:

$$\left|< \ell_N^{(\alpha)}, f >\right| = \left|\hat{f}_0 \hat{\ell}_0^{(\alpha)} + \sum_{k\neq 0} \hat{f}_k \hat{\ell}_k^{(\alpha)}\right| \leq \left|\hat{f}_0 \hat{\ell}_0^{(\alpha)}\right|$$

$$+\left|\sum_{k\neq 0} \hat{f}_k \hat{\ell}_k^{(\alpha)} (2\pi k)^m \cdot \frac{1}{(2\pi k)^m}\right| \leq \left|\hat{f}_0 \hat{\ell}_0^{(\alpha)}\right| + \sum_{k\neq 0} \left|\hat{f}_k\right| \left|\hat{\ell}_k^{(\alpha)}\right| |(2\pi k)^m| \frac{1}{|(2\pi k)^m|}$$

$$\leq \left\{ \left|\hat{f}_0\right|^2 + \sum_{k\neq 0} \left|\hat{f}_k\right|^2 |2\pi k|^{2m} \right\}^{\frac{1}{2}} \cdot \left\{ \left|\hat{\ell}_0^{(\alpha)}\right|^2 + \sum_{k\neq 0} \frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{|2\pi k|^{2m}} \right\}^{\frac{1}{2}}$$

$$= \left\| f | \widetilde{W}_2^{(m)}(T_1) \right\| \cdot \left\{ \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}} \right\}^{\frac{1}{2}}. \tag{13}$$

With (2), (12), and (13), we obtain

$$\left\| \ell_N^{(\alpha)} | \widetilde{W}_2^{(m)*}(T_1) \right\|^2 \leq \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left| \hat{p}_k - \sum\limits_{\alpha=0}^{m-1} \sum\limits_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i)^\alpha k^\alpha e^{2\pi i k x^{(\lambda)}} \right|^2}{k^{2m}}. \tag{14}$$

Thus, we have obtained an upper estimate for the norm of the error functional $\ell_N^{(\alpha)}$, if we obtain a lower estimate for the norm of the error functional $\ell_N^{(\alpha)}$, then the assertion of the theorem follows from here.

There is some function from $\widetilde{W}_2^{(m)}(T_1)$, such that inequality (14) reaches equality.
Indeed, consider the following function $u(x)$

$$u(x) = \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} e^{2\pi k i x}}{k^{2m}}.$$

Let us calculate the value of functional $\ell_N^{(\alpha)}(x)$ for function $u(x)$

$$< \ell_N^{(\alpha)}, u > = < \ell_N^{(\alpha)}(x), \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} >$$

$$+ < \ell_N^{(\alpha)}(x), \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} e^{2\pi i k x}}{(2\pi)^{2m} k^{2m}} > = \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2$$

$$+ \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} \hat{\ell}_k^{(\alpha)}}{k^{2m}} = \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}}. \tag{15}$$

Now we prove the existence of function $u(x)$ and its norm in the space $\widetilde{W}_2^{(m)}(T_1)$, for this we will prove the validity of the following lemma.

Lemma 1. *The square of the norm of function $u(x)$ in the space $\tilde{W}_2^{(m)}(T_1)$ is:*

$$\left\| u|\tilde{W}_2^{(m)}(T_1) \right\|^2 = \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left| \hat{p}_k - \sum\limits_{\alpha=0}^{m-1}\sum\limits_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i)^\alpha k^\alpha e^{2\pi i k x^{(\lambda)}} \right|^2}{k^{2m}}. \tag{16}$$

Proof. Since equality (9) holds for all functions $f(x) \in \tilde{W}_2^{(m)}(T_1)$, then it follows that the equation below is true for the norm of $u(x)$

$$\left\| u|\tilde{W}_2^{(m)}(T_1) \right\|^2 = \left(\int_{T_1} u(x)\,dx \right)^2 + \sum_{k_1\neq 0} |2\pi k_1|^{2m} \left| \hat{u}_{k_1} \right|^2, \tag{17}$$

where $k_1 \in Z$ and $\hat{u}_{k_1}$ are the Fourier coefficients.

Thus, we calculate the norm of the function u in the space $\tilde{W}_2^{(m)}(T_1)$ using formula (17).

In (17), for each term, we perform separate calculations:

1. $\left(\int\limits_{T_1} u(x)\,dx \right)^2 = \left(\int\limits_{T_1} \left[\hat{p}_0 - \sum\limits_{\alpha=0}^{m-1}\sum\limits_{\lambda=1}^{N} c_\lambda^{(\alpha)} + \frac{1}{(2\pi)^{2m}} \sum\limits_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} e^{2\pi k i x}}{k^{2m}} \right] dx \right)^2$

$$= \left([\hat{p}_0 - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)}] \int_{T_1} dx + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} \int\limits_{T_1} e^{2\pi i k x} dx}{k^{2m}} \right)^2. \tag{18}$$

As $\int\limits_{T_1} e^{2\pi k x} dx = 0$ and $\int\limits_{T_1} dx = 1$, then (18) has the following form

$$\left(\int_{T_1} u(x)\,dx \right)^2 = \left([\hat{p}_0 - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)}] \right)^2. \tag{19}$$

2. Now we calculate the value of $\hat{u}_{k_1}$:

$$\hat{u}_{k_1} = \int_{T_1} u(x) e^{2\pi i k_1 x} dx = \int_{T_1} \left[\hat{p}_0 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} e^{2\pi k i x}}{k^{2m}} \right] e^{-2\pi i k_1 x} dx$$

$$= [\hat{p}_0 - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)}] \int_{T_1} e^{-2\pi i k x} dx + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)} \int\limits_{T_1} e^{-2\pi i k_1 x} e^{2\pi i k x} dx}{k^{2m}}. \tag{20}$$

It is evident that

$$\int_{T_1} e^{2\pi i (k-k_1) x} dx = \begin{cases} 1, \text{ if } k = k_1, \\ 0, \text{ if } k \neq k_1. \end{cases} \tag{21}$$

With (21), we obtain the following from (20)

$$\hat{u}_k = \hat{u}_{k_1} = \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\hat{\ell}_k^{(\alpha)}}{k^{2m}}. \tag{22}$$

Introducing (19) and (22) into the right side of (17), we obtain

$$\left\|u|\tilde{W}_2^{(m)}(T_1)\right\|^2 = \left|\hat{p}_0 - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)}\right|^2 + \sum_{k\neq 0}(2\pi)^{2m}k^{2m}\frac{\left|\ell_k^{(\alpha)}\right|^2}{(2\pi)^{4m}k^{4m}}. \tag{23}$$

Thus, after cancellations, it follows from (23) that

$$\left\|u|\tilde{W}_2^{(m)}(T_1)\right\|^2 = \left|\hat{p}_0 - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} c_\lambda^{(\alpha)}\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\ell_k^{(\alpha)}\right|^2}{k^{2m}}. \tag{24}$$

Considering (12), the proof of the lemma follows from (24).
Lemma 1 is proved.

Comparing the right sides of (14) and (24), we obtain

$$\left\|\ell_N^{(\alpha)}|\tilde{W}_2^{(m)^*}(T_1)\right\|^2 \leq \left\|u|\tilde{W}_2^{(m)}(T_1)\right\|^2. \tag{25}$$

Considering lemma 1 for the right-hand sides of (15), we have

$$<\ell_N^{(\alpha)}, u> = \left\|u|\tilde{W}_2^{(m)}(T_1)\right\| \cdot \left\|u|\tilde{W}_2^{(m)}(T_1)\right\|. \tag{26}$$

For the error of the quadrature formula (1) on functions $u(x)$, the following is true:

$$<\ell_N^{(\alpha)}, u> \leq \left\|\ell_N^{(\alpha)}|\tilde{W}_2^{(m)^*}(T_1)\right\| \cdot \left\|u|\tilde{W}_2^{(m)}(T_1)\right\|. \tag{27}$$

Substituting the right side of (26) into the left side of (27), we have

$$\left\|u|\tilde{W}_2^{(m)}(T_1)\right\| \cdot \left\|u|\tilde{W}_2^{(m)}(T_1)\right\| \leq \left\|\ell_N^{(\alpha)}|\tilde{W}_2^{(m)^*}(T_1)\right\| \cdot \left\|u|\tilde{W}_2^{(m)}(T_1)\right\|. \tag{28}$$

After cancellations, it follows from (28) that

$$\left\|\ell_N^{(\alpha)}|\tilde{W}_2^{(m)^*}(T_1)\right\| \geq \left\|u|\tilde{W}_2^{(m)}(T_1)\right\|. \tag{29}$$

From (25) and (29) we obtain

$$\left\|\ell_N^{(\alpha)}|\tilde{W}_2^{(m)^*}(T_1)\right\| = \left\|u|\tilde{W}_2^{(m)}(T_1)\right\|. \tag{30}$$

With (30), we can write the following:

$$<\ell_N^{(\alpha)}, u> = <u, u>. \tag{31}$$

Equation (31) testifies to the existence of $u(x) \in \tilde{W}_2^{(m)}(T_1)$ and thus it is an extremal function for the quadrature formula (1), i.e. $u = \psi_\ell \in \tilde{W}_2^{(m)}(T_1)$, for which the following equation holds

$$<\ell_N^{(\alpha)}, \psi_\ell> = \left\|\psi_\ell|\tilde{W}_2^{(m)}(T_1)\right\| \cdot \left\|\psi_\ell|\tilde{W}_2^{(m)}(T_1)\right\|. \tag{32}$$

Then (31) takes the following form

$$<\ell_N^{(\alpha)}(x), \psi_\ell(x)> = <\psi_\ell, \psi_\ell>. \tag{33}$$

This means that all conditions of the Riesz theorem are met.

The following theorem is true.

Theorem 2. *Equations (15), (31), and (32) confirm that*

$$u(x) = \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\hat{\ell}_k^{(\alpha)} e^{-2\pi k i x}}{k^{2m}}$$

is an extremal function for the quadrature formula (1) and $u \in \tilde{W}_2^{(m)}(T_1)$.

Thus, taking into account (23), (30) and the conditions of Lemma 1 for the square of the norm of the error functional of quadrature formula (1), we have

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 = \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\left| \hat{p}_k - \sum\limits_{\alpha=0}^{m-1} \sum\limits_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i)^\alpha k^\alpha e^{2\pi i x^{(\lambda)}} \right|^2}{k^{2m}}, \tag{34}$$

which is what was required to be proved.

Based on Theorem 1, the error functional of quadrature formula (1) for the class $\tilde{W}_2^{(m)}(T_1)$ functions has the following estimate

$$\left| < \ell_N^{(\alpha)}, f > \right| \leq \left\{ \left| \overset{\wedge}{f_0} \right|^2 + \sum_{k \neq 0} \left| \overset{\wedge}{f_k} \right|^2 |2\pi k|^{2m} \right\}^{\frac{1}{2}} \left\{ \left| \hat{p}_0 - \sum_{\alpha=0}^{m-1} \sum_{\lambda=1}^{N} c_\lambda^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \right.$$

$$\left. \times \sum_{k \neq 0} \frac{\left| \hat{p}_k - \sum\limits_{\alpha=0}^{m-1} \sum\limits_{\lambda=1}^{N} c_\lambda^{(\alpha)} (2\pi i)^\alpha k^\alpha e^{2\pi i x^{(\lambda)}} \right|^2}{k^{2m}} \right\}^{\frac{1}{2}}. \tag{35}$$

MINIMIZING THE NORM OF THE ERROR FUNCTIONAL OF A QUADRATURE FORMULA OF HERMITE TYPE IN THE PERIODIC SPACE $\tilde{W}_2^{(m)}(T_1)$

As seen in (34), the quality of the quadrature formula is characterized by the norm of the error functional and is the function of unknown coefficients and nodes. Therefore, for computational practice, it is appropriate to be able to calculate the norm of the error functional and estimate it. Finding the minimum of the norm of the error functional with respect to c_λ and $x^{(\lambda)}$ is the task of studying the function to an extremum. Values c_λ and $x^{(\lambda)}$, realizing this minimum, determine the best quadrature formula.

The main result of this study is

Theorem 3. *The optimal quadrature formula of Hermite type of the form (1) in the periodic space* $\tilde{W}_2^{(m)}(T_1)$*, for* $p(x) = 1$ *and* $m = 4$ $\left(\alpha = \overline{0,1,2},3\right)$ *has equidistant nodes* $x^{(\lambda)} = \frac{\lambda}{N}$, $\lambda = 1, 2, \ldots, N$ *and equal coefficients* $c_1 = c_2 = \cdots = c_N = c^0$, $c_1^{(1)} = c_2^{(1)} = \cdots = c_N^{(1)} = c^{0(1)}$ *and* $c_1^{(2)} = c_2^{(2)} = \cdots = c_N^{(2)} = c^{0(2)}$, $c_1^{(3)} = c_2^{(3)} = \cdots = c_N^{(3)} = c^{0(3)}$ *given by the following formulas*

$$c^0 = \frac{\sum\limits_{k \neq 0} \frac{1}{k^4}}{N \left[\sum\limits_{k \neq 0} \frac{1}{k^4} + \frac{1}{(2\pi)^8 N^8} \left(\sum\limits_{k \neq 0} \frac{1}{k^4} \sum\limits_{k \neq 0} \frac{1}{k^8} - \left(\sum\limits_{k \neq 0} \frac{1}{k^6} \right)^2 \right) \right]}, \; c^{0(1)} = 0 \;\; \text{and}$$

$$c^{0(2)} = \frac{\sum\limits_{k \neq 0} \frac{1}{k^6}}{(2\pi)^2 N^3 \left[\sum\limits_{k \neq 0} \frac{1}{k^4} + \frac{1}{(2\pi)^8 N^8} \left(\sum\limits_{k \neq 0} \frac{1}{k^4} \sum\limits_{k \neq 0} \frac{1}{k^8} - \left(\sum\limits_{k \neq 0} \frac{1}{k^6} \right)^2 \right) \right]}, \; c^{0(3)} = 0 \tag{36}$$

Proof. Since the optimization problem is considered for the quadrature formula (1) for $p(x)=1$, then if we assume that $m=4$ in equation (34), equation (34) takes the following form

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 = \left(1-\sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8}\cdot$$
$$\cdot \sum_{k\neq 0} \frac{\left| \sum\limits_{\lambda=1}^{N} c_\lambda e^{2\pi i x^{(\lambda)}} - (2\pi)^2 k^2 \sum\limits_{\lambda=1}^{N} c_\lambda^{(2)} e^{2\pi i x^{(\lambda)}} + (2\pi i)\, k \sum\limits_{\lambda=1}^{N} c_\lambda^{(1)} e^{2\pi i x^{(\lambda)}} + (2\pi i)^3 k^3 \sum\limits_{\lambda=1}^{N} c_\lambda^{(3)} e^{2\pi i x^{(\lambda)}} \right|^2}{k^8} \tag{37}$$

Let us now perform some transformations with the second term in equation (37).
Let $\sum\limits_{\beta=1}^{N} c_\beta \neq 0$, $\sum\limits_{\beta=1}^{N} c_\beta^{(1)} \neq 0$ and $\sum\limits_{\beta=1}^{N} c_\beta^{(2)} \neq 0$, $\sum\limits_{\beta=1}^{N} c_\beta^{(3)} \neq 0$, then after some transformations, we obtain from (37)

$$\left(1-\sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8}\cdot \sum_{k\neq 0}\left(\frac{1}{k^8} \left| \sum_{\lambda=1}^{N} c_\lambda e^{2\pi i x^{(\lambda)}} - (2\pi)^2 k^2 \sum_{\lambda=1}^{N} c_\lambda^{(2)} e^{2\pi i x^{(\lambda)}} \right.\right.$$
$$\left.\left. + \ (2\pi i) k \sum_{\lambda=1}^{N} c_\lambda^{(1)} e^{2\pi i x^{(\lambda)}} + (2\pi i)^3 k^3 \sum_{\lambda=1}^{N} c_\lambda^{(3)} e^{2\pi i x^{(\lambda)}} \right|^2 \right)$$

$$= \left(1-\sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8}\sum_{k\neq 0}\left(\frac{1}{k^8} \left| \left(\sum_{\beta=1}^{N} c_\beta\right) \sum_{\lambda=1}^{N} \frac{c_\lambda e^{2\pi i k x^{(\lambda)}}}{\sum\limits_{\beta=1}^{N} c_\beta} - (2\pi)^2 k^2 \left(\sum_{\beta=1}^{N} c_\beta^{(2)}\right) \sum_{\lambda=1}^{N} \frac{c_\lambda^{(2)} e^{2\pi i k x^{(\lambda)}}}{\sum\limits_{\beta=1}^{N} c_\beta^{(2)}} \right.\right.$$
$$\left.\left. + \ (2\pi i) k \left(\sum_{\beta=1}^{N} c_\beta^{(1)}\right) \sum_{\lambda=1}^{N} \frac{c_\lambda^{(1)} e^{2\pi i k x^{(\lambda)}}}{\sum\limits_{\beta=1}^{N} c_\beta^{(1)}} + (2\pi i)^3 k^3 \left(\sum_{\beta=1}^{N} c_\beta^{(3)}\right) \sum_{\lambda=1}^{N} \frac{c_\lambda^{(3)} e^{2\pi i k x^{(\lambda)}}}{\sum\limits_{\beta=1}^{N} c_\beta^{(3)}} \right|^2 \right) =$$

$$= \left(1-\sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8}\sum_{k\neq 0}\left(\frac{1}{k^8} \left| \left(\sum_{\beta=1}^{N} c_\beta\right) \sum_{\lambda=1}^{N} c'_\lambda e^{2\pi i k x^{(\lambda)}} - (2\pi)^2 k^2 \left(\sum_{\beta=1}^{N} c_\beta^{(2)}\right) \sum_{\lambda=1}^{N} c'^{(2)}_\lambda e^{2\pi i k x^{(\lambda)}} \right.\right.$$
$$\left.\left. + \ (2\pi i) k \left(\sum_{\beta=1}^{N} c_\beta^{(1)}\right) \sum_{\lambda=1}^{N} c'^{(1)}_\lambda e^{2\pi i k x^{(\lambda)}} + (2\pi i)^3 k^3 \left(\sum_{\beta=1}^{N} c_\beta^{(3)}\right) \sum_{\lambda=1}^{N} c'^{(3)}_\lambda e^{2\pi i k x^{(\lambda)}} \right|^2 \right), \tag{38}$$

where

$$c'_\lambda = \frac{c_\lambda}{\sum\limits_{\beta=1}^{N} c_\beta},\ c'^{(1)}_\lambda = \frac{c_\lambda^{(1)}}{\sum\limits_{\beta=1}^{N} c_\beta^{(1)}} \ \text{ and } \ c'^{(2)}_\lambda = \frac{c_\lambda^{(2)}}{\sum\limits_{\beta=1}^{N} c_\beta^{(2)}},\ c'^{(3)}_\lambda = \frac{c_\lambda^{(3)}}{\sum\limits_{\beta=1}^{N} c_\beta^{(3)}}. \tag{39}$$

It is evident that

$$\sum_{\lambda=1}^{N} c'_\lambda = 1,\ \sum_{\lambda=1}^{N} c'^{(1)}_\lambda = 1 \text{ and } \sum_{\lambda=1}^{N} c'^{(2)}_\lambda = 1\ ,\ \sum_{\lambda=1}^{N} c'^{(3)}_\lambda = 1. \tag{40}$$

With (39) and (40), equation (38) is rewritten in the following form

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 = \left(1 - \sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \left(\frac{1}{k^8} \left| \left(\sum_{\beta=1}^{N} c_\beta \right) \sum_{\lambda=1}^{N} c'_\lambda e^{2\pi i k x^{(\lambda)}} - (2\pi)^2 k^2 \left(\sum_{\beta=1}^{N} c_\beta^{(2)} \right) \right. \right.$$
$$\left. \left. \times \sum_{\lambda=1}^{N} c'^{(2)}_\lambda e^{2\pi i k x^{(\lambda)}} + i \left((2\pi) k \left(\sum_{\beta=1}^{N} c_\beta^{(1)} \right) \sum_{\lambda=1}^{N} c'^{(1)}_\lambda e^{2\pi i k x^{(\lambda)}} - (2\pi)^3 k^3 \left(\sum_{\beta=1}^{N} c_\beta^{(3)} \right) \sum_{\lambda=1}^{N} c'^{(3)}_\lambda e^{2\pi i k x^{(\lambda)}} \right) \right|^2 \right). \tag{41}$$

Hence

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 = \left(1 - \sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \left(\frac{1}{k^8} \left| \left(\left(\sum_{\beta=1}^{N} c_\beta \right) \sum_{\lambda=1}^{N} c'_\lambda e^{2\pi i k x^{(\lambda)}} - (2\pi)^2 k^2 \left(\sum_{\beta=1}^{N} c_\beta^{(2)} \right) \right. \right. \right.$$
$$\left. \left. \left. \times \sum_{\lambda=1}^{N} c'^{(2)}_\lambda e^{2\pi i k x^{(\lambda)}} \right)^2 + \left((2\pi) k \left(\sum_{\beta=1}^{N} c_\beta^{(1)} \right)^2 \left(\sum_{\lambda=1}^{N} c'^{(1)}_\lambda e^{2\pi i k x^{(\lambda)}} \right) - (2\pi)^3 k^3 \left(\sum_{\beta=1}^{N} c_\beta^{(3)} \right)^2 \left(\sum_{\lambda=1}^{N} c'^{(3)}_\lambda e^{2\pi i k x^{(\lambda)}} \right) \right)^2 \right| \right) \tag{42}$$

Denoting the left side of (42) by $\sum\limits_{\lambda=1}^{N} c_\lambda = x_1$, $\sum\limits_{\lambda=1}^{N} c_\lambda^{(1)} = x_2$, $\sum\limits_{\lambda=1}^{N} c_\lambda^{(2)} = x_3$ and $\sum\limits_{\lambda=1}^{N} c_\lambda^{(3)} = x_4$, after some transformations, equation (42) is rewritten as the polynomial of the second degree in x_1, x_2, x_3 and x_4

$$\left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 = \left(1 - \sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{1}{k^8} \left(\left(x_1 \sum_{\lambda=1}^{N} c'_\lambda e^{2\pi i k x^{(\lambda)}} - (2\pi)^2 k^2 x_3 \sum_{\lambda=1}^{N} c'^{(2)}_\lambda e^{2\pi i k x^{(\lambda)}} \right)^2 \right.$$
$$\left. + \left((2\pi) k x_2 \left(\sum_{\lambda=1}^{N} c'^{(1)}_\lambda e^{2\pi i k x^{(\lambda)}} \right) - (2\pi)^3 k^3 x_4 \left(\sum_{\lambda=1}^{N} c'^{(3)}_\lambda e^{2\pi i k x^{(\lambda)}} \right) \right)^2 \right). \tag{43}$$

Or

$$\begin{aligned} \left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 &= \left(1 - \sum_{\lambda=1}^{N} c_\lambda\right)^2 \\ &+ \frac{1}{(2\pi)^8} x_1^2 \sum_{k\neq 0} \frac{\left(\sum\limits_{\lambda=1}^{N} c'_\lambda e^{2\pi i k x^{(\lambda)}} \right)^2}{k^8} - \frac{1}{(2\pi)^6} 2 x_1 x_3 \sum_{k\neq 0} \frac{\sum\limits_{\lambda=1}^{N} c'^{(2)}_\lambda e^{2\pi i k x^{(\lambda)}} \sum\limits_{\lambda=1}^{N} c'_\lambda e^{2\pi i k x^{(\lambda)}}}{k^6} \\ &+ \frac{1}{(2\pi)^4} x_3^2 \sum_{k\neq 0} \frac{\left(\sum\limits_{\lambda=1}^{N} c'^{(2)}_\lambda e^{2\pi i k x^{(\lambda)}} \right)^2}{k^4} + \frac{1}{(2\pi)^6} x_2^2 \sum_{k\neq 0} \frac{\left(\sum\limits_{\lambda=1}^{N} c'^{(1)}_\lambda e^{2\pi i k x^{(\lambda)}} \right)^2}{k^6} \\ &- \frac{1}{(2\pi)^4} 2 x_2 x_4 \sum_{k\neq 0} \frac{\sum\limits_{\lambda=1}^{N} c'^{(1)}_\lambda e^{2\pi i k x^{(\lambda)}} \sum\limits_{\lambda=1}^{N} c'^{(3)}_\lambda e^{2\pi i k x^{(\lambda)}}}{k^4} + \frac{1}{(2\pi)^2} x_4^2 \sum_{k\neq 0} \frac{\left(\sum\limits_{\lambda=1}^{N} c'^{(3)}_\lambda e^{2\pi i k x^{(\lambda)}} \right)^2}{k^2}. \end{aligned} \tag{44}$$

With conditions (40) in equation (44), using the results given in [16, 17], we obtain

$$\begin{aligned} \left\| \ell_N^{(\alpha)} | \tilde{W}_2^{(m)^*}(T_1) \right\|^2 &= \left(1 - \sum_{\lambda=1}^{N} c_\lambda\right)^2 + \frac{1}{(2\pi N)^8} x_1^2 \sum_{k\neq 0} \frac{1}{k^8} - \frac{1}{(2\pi N)^6} 2 x_1 x_3 \sum_{k\neq 0} \frac{1}{k^6} \\ &+ \frac{1}{(2\pi N)^4} x_3^2 \sum_{k\neq 0} \frac{1}{k^4} + \frac{1}{(2\pi N)^6} x_2^2 \sum_{k\neq 0} \frac{1}{k^6} - \frac{1}{(2\pi N)^4} 2 x_2 x_4 \sum_{k\neq 0} \frac{1}{k^4} + \frac{1}{(2\pi N)^2} x_4^2 \sum_{k\neq 0} \frac{1}{k^2}. \end{aligned} \tag{45}$$

Here we took into account that the sums

$$\sum_{k\neq 0}\frac{\left|\sum\limits_{\lambda=1}^{N} c'_{\lambda} e^{2\pi i k x^{(\lambda)}}\right|^2}{k^8}, \sum_{k\neq 0}\frac{\left|\sum\limits_{\lambda=1}^{N} c'^{(2)}_{\lambda} e^{2\pi i k x^{(\lambda)}}\right|^2}{k^6}, \quad \text{and} \quad \sum_{k\neq 0}\frac{\left|\sum\limits_{\lambda=1}^{N} c'^{(1)}_{\lambda} e^{2\pi i k x^{(\lambda)}}\right|^2}{k^4}, \sum_{k\neq 0}\frac{\left|\sum\limits_{\lambda=1}^{N} c'^{(3)}_{\lambda} e^{2\pi i k x^{(\lambda)}}\right|^2}{k^2}$$

reach their minimum values, equal to

$$\frac{1}{N^8}\sum_{k\neq 0}\frac{1}{k^8}, \ \frac{1}{N^6}\sum_{k\neq 0}\frac{1}{k^6} \text{ and } \frac{1}{N^4}\sum_{k\neq 0}\frac{1}{k^4}, \ \frac{1}{N^2}\sum_{k\neq 0}\frac{1}{k^2}$$

respectively, when nodes $x^{(\lambda)}$ of the quadrature formula (1) are equidistant and all coefficients c'_{λ} are equal to each other, i.e.

$$c'_{\lambda} = \frac{1}{N}, c'^{(1)}_{\lambda} = \frac{1}{N}, c'^{(2)}_{\lambda} = \frac{1}{N}, c'^{(3)}_{\lambda} = \frac{1}{N} \text{ and } x^{(\lambda)} = \frac{\lambda}{N}, \lambda = \overline{1,N}. \tag{46}$$

Now we will consider the right side of (45) as a function of four variables x_1, x_2, x_3, x_4 and denote it by $y(x_1,x_2,x_3,x_4)$, i.e.

$$\begin{aligned} y(x_1,x_2,x_3,x_4) &= \left(1-\sum_{\lambda=1}^{N} c_{\lambda}\right)^2 + \frac{1}{(2\pi N)^8}x_1^2\sum_{k\neq 0}\frac{1}{k^8} - \frac{1}{(2\pi N)^6}2x_1x_3\sum_{k\neq 0}\frac{1}{k^6} \\ &+ \frac{1}{(2\pi N)^4}x_3^2\sum_{k\neq 0}\frac{1}{k^4} + \frac{1}{(2\pi N)^6}x_2^2\sum_{k\neq 0}\frac{1}{k^6} - \frac{1}{(2\pi N)^4}2x_2x_4\sum_{k\neq 0}\frac{1}{k^4} + \frac{1}{(2\pi N)^2}x_4^2\sum_{k\neq 0}\frac{1}{k^2} \end{aligned} \tag{47}$$

Then from the necessary condition for extremum, from (47) we obtain a system of equations with four unknowns x_1, x_2 and x_3, x_4.

$$\begin{aligned} y'_{x_1}(x_1,x_2,x_3,x_4) &= -2(1-x_1) + \frac{2}{(2\pi N)^8}x_1\sum_{k\neq 0}\frac{1}{k^8} - \frac{1}{(2\pi N)^6}2x_3\sum_{k\neq 0}\frac{1}{k^6} = 0 \\ y'_{x_2}(x_1,x_2,x_3,x_4) &= \frac{1}{(2\pi N)^6}2x_2\sum_{k\neq 0}\frac{1}{k^6} - \frac{1}{(2\pi N)^4}2x_4\sum_{k\neq 0}\frac{1}{k^4} = 0, \\ y'_{x_3}(x_1,x_2,x_3,x_4) &= -\frac{2}{(2\pi N)^6}x_1\sum_{k\neq 0}\frac{1}{k^6} + \frac{1}{(2\pi N)^4}2x_3\sum_{k\neq 0}\frac{1}{k^4} = 0, \\ y'_{x_4}(x_1,x_2,x_3,x_4) &= -\frac{2}{(2\pi N)^4}x_2\sum_{k\neq 0}\frac{1}{k^4} + \frac{1}{(2\pi N)^2}2x_4\sum_{k\neq 0}\frac{1}{k^2} = 0, \end{aligned} \tag{48}$$

After some simplifications, we obtain from (48)

$$\begin{cases} x_1\left(1+\frac{1}{(2\pi N)^8}\sum\limits_{k\neq 0}\frac{1}{k^8}\right) - \frac{1}{(2\pi N)^6}x_3\sum\limits_{k\neq 0}\frac{1}{k^6} = 1, \\ x_2\sum\limits_{k\neq 0}\frac{1}{k^6} = \frac{(2\pi N)^2}{\sum\limits_{k\neq 0}\frac{1}{k^6}}x_4\sum\limits_{k\neq 0}\frac{1}{k^4}, \\ x_4\left[(2\pi N)^2\sum\limits_{k\neq 0}\frac{1}{k^2} - \frac{(2\pi N)^2}{\sum\limits_{k\neq 0}\frac{1}{k^6}}\ _4\sum\limits_{k\neq 0}\frac{1}{k^4}\right] = 0, \\ x_3 = \frac{1}{(2\pi N^2)\sum\limits_{k\neq 0}\frac{1}{k^4}}x_1\sum\limits_{k\neq 0}\frac{1}{k^6}. \end{cases} \tag{49}$$

Solving system (49) and introducing some transformations, we successively determine x_1, x_2, x_3 and x_4 , i.e.

$$x_1 = \frac{\sum\limits_{k\neq 0} \frac{1}{k^4}}{\left[\sum\limits_{k\neq 0} \frac{1}{k^4} + \frac{1}{(2\pi)^8 N^8}\left(\sum\limits_{k\neq 0} \frac{1}{k^4} \sum\limits_{k\neq 0} \frac{1}{k^8} - \left(\sum\limits_{k\neq 0} \frac{1}{k^6}\right)^2\right)\right]},\ x_2 = 0$$

and

$$x_3 = \frac{\sum\limits_{k\neq 0} \frac{1}{k^6}}{(2\pi)^2 N^2 \left[\sum\limits_{k\neq 0} \frac{1}{k^4} + \frac{1}{(2\pi)^8 N^8}\left(\sum\limits_{k\neq 0} \frac{1}{k^4} \sum\limits_{k\neq 0} \frac{1}{k^8} - \left(\sum\limits_{k\neq 0} \frac{1}{k^6}\right)^2\right)\right]},\ x_4 = 0 \tag{50}$$

Let $c'_\lambda = \frac{1}{N}$, $c'^{(1)}_\lambda = \frac{1}{N}$ and $c'^{(2)}_\lambda = \frac{1}{N}$, $c'^{(3)}_\lambda = \frac{1}{N}$ $\left(\lambda = \overline{1,N}\right)$ then from (36) and (46), it follows that
$c_1 = c_2 = ... = c_N = c^0$, $c_1^{(1)} = c_2^{(1)} = ... = c_N^{(1)} = c^{0(1)}$ and $c_1^{(2)} = c_2^{(2)} = ... = c_N^{(2)} = c^{0(2)}$, $c_1^{(3)} = c_2^{(3)} = ... = c_N^{(3)} = c^{0(3)}$.
Hence

$$x_1 = \sum_{\lambda=1}^{N} c_\lambda = Nc^0,\ x_2 = \sum_{\lambda=1}^{N} c_\lambda^{(1)} = Nc^{0(1)} \text{ and } x_3 = \sum_{\lambda=1}^{N} c_\lambda^{(2)} = Nc^{0(2)},\ x_4 = \sum_{\lambda=1}^{N} c_\lambda^{(3)} = Nc^{0(3)}. \tag{51}$$

Substituting (50) into (51) we find the optimal coefficients of quadrature formulas of Hermite type in the following form (1)

$$c^0 = \frac{\sum\limits_{k\neq 0} \frac{1}{k^4}}{N\left[\sum\limits_{k\neq 0} \frac{1}{k^4} + \frac{1}{(2\pi)^8 N^8}\left(\sum\limits_{k\neq 0} \frac{1}{k^4} \sum\limits_{k\neq 0} \frac{1}{k^8} - \left(\sum\limits_{k\neq 0} \frac{1}{k^6}\right)^2\right)\right]},\ c^{0(1)} = 0 \tag{52}$$

and

$$c^{0(2)} = \frac{\sum\limits_{k\neq 0} \frac{1}{k^6}}{(2\pi)^2 N^3 \left[\sum\limits_{k\neq 0} \frac{1}{k^4} + \frac{1}{(2\pi)^8 N^8}\left(\sum\limits_{k\neq 0} \frac{1}{k^4} \sum\limits_{k\neq 0} \frac{1}{k^8} - \left(\sum\limits_{k\neq 0} \frac{1}{k^6}\right)^2\right)\right]},\ c^{0(3)} = 0 \tag{53}$$

which is what was required to be proved.
Note that similar problems were solved in [18, 19].

CONCLUSION

The quality of the quadrature formula is characterized by the norm of the error functional.

This formula is a function of unknown coefficients and nodes. Therefore, for computational practice, it is appropriate to be able to calculate the norm of the error functional and estimate it. Finding the minimum of the norm of the error functional with respect to c_λ and $x^{(\lambda)}$ is the task of investigating a function of many variables on an extremum.

Values c_λ and $x^{(\lambda)}$, realizing this minimum, determine the optimal formula. We considered the optimal quadrature formula to be the one in which, for a given number of nodes, the error functional has the minimum norm.

The following problems solved:

1. The norm of the error functional of quadrature formulas in the space B is calculated.
2. The optimal quadrature formula, i.e. quadrature formula with the minimum norm of the error functional in B is constructed.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the V.I.Romanovskiy Institute of Mathematics for making convenient research facilities. We much appreciate the reviewers for their thoughtful comments and efforts toward improving our manuscript.

REFERENCES

1. A. Sard., "Integral representations of remainders." Duke Math J **15**, 333–345 (1948).
2. S. M. Nikolsky, "On issue of estimates of approximations by quadrature formulas." Achievements in Mathematical Sciences **5. issue 2 (36)**, 165–177 (1950).
3. S. L. Sobolev, *Introduction to the theory of cubature formulas (in Russian).* (Nauka, Moscow., 1974) p. 808.
4. Z. J. Jamolov, G. N. Salikhov, and T. Kh. Sharipov, "Approximate integration of smooth functions," Mathematical analysis and related problems of mathematics , 37–60 (1978).
5. Kh. M. Shadimetov, "On calculation of coefficients of optimal quadrature formulas." DAN USSR **4** (1980).
6. M. D. Ramazanov and Kh. M. Shadimetov, "Weighted optimal cubature formulas in periodic Sobolev space." Reports of RAN. -Moscow **4(358)**, 453–455 (1999).
7. Kh. M. Shadimetov, *Lattice quadrature and cubature formulas in S.L.Sobolev spaces.*, Ph.D. thesis, Doctor of Phys.-Math. Sci. Tashkent (2002).
8. Kh. M. Shadimetov and A. R. Hayotov, "Calculation of coefficients of optimal quadrature formulas in space $W_2^{m,m-1}(0,1)$.(in russian)," Uzbek Mathematical Journal **3**, 80–98 (2004).
9. Kh. M. Shadimetov and A. R. Hayotov, "Optimal quadrature formulas with positive coefficients in $L_2^{(m)}(0,1)$ space," Journal of Computational and Applied Mathematics **235**, 1114–1128 (2011).
10. A.R.Hayotov, G.V.Milovanović, and Kh.M.Shadimetov, "On an optimal quadrature formula in the sense of Sard," Numerical Algorithms **57**, 487–510 (2011).
11. S.S.Babaev and A.R.Hayotov, "Optimal interpolation formulas in the space $W_2^{(m,m-1)}$," Calcolo **56**, doi.org/10.1007/s10092–019–0320–9 (2019).
12. S. Babaev, J. Davronov, A.Abdullaev, and S.Polvonov, "Optimal interpolation formulas exact for trigonometric functions," AIP Conference Proceedings **2781**, 020064 (2023), https://doi.org/10.1063/5.0144754.
13. A. Hayotov, S. Babaev, N.Olimov, and Sh.Imomova, "The error functional of optimal interpolation formulas in $W_{2,\sigma}^{(2,1)}$ space," AIP Conference Proceedings **2781**, 020044 (2023), https://doi.org/10.1063/5.0144752.
14. I. Jalolov, "On finding the extremal function and the norm of the error functional of quadrature formulas of Hermite type in the space of periodic Sobolev functions." Problems of Computational and Applied Mathematics. **4**, 89–94 (2016).
15. I. Jalolov, "The best quadrature formula over space $\tilde{W}_2^{(m)}[0,1]$." Questions of computational and applied mathematics. **77**, 80–92 (1985).
16. A. A. Jensykbaev, "On the best quadrature formula on class W^rL_p." DAN USSR **2**, 277–279 (1976).
17. A. A. Jensykbaev, "The best quadrature formula for some classes of periodic differentiable functions." Bulletin of Academy of Sciences of the USSR, series Math Sci. **5(41)**, 1110–1124 (1977).
18. A. R. Hayotov and S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in a Hilbert space." Problems of computational and applied mathematics **4(28)**, 73–84 (2020).
19. A. R. Hayotov, S. Jeon, and C.-O. Lee, "On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$." Journal of Computational and Applied Mathematics **372**, 112713 (2020).
20. N. E. Lushpay, "Best quadrature formulas on classes of differentiable periodic functions." Math. notes. **6. Issue. 4**, 475–480 (1969).
21. S. M. Nikolsky, *Quadrature formulas (in Russian).* (Nauka, Moscow., 1979) p. 256.
22. Ts. B. Shainzhurov, *Theory of cubature formulas in function spaces with norm dependent and its derivatives.*, Ph.D. thesis, Doctor of Phys.-Math. Sci. Novosibirsk.
23. Kh.M.Shadimetov and F.A.Nuraliev, "Optimization quadrature formulas with derivatives." Problems of computational and applied mathematics.-Tashkent **1**, 61–70 (2015).
24. A. R. Hayotov and S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in $W_2^{(m,m-1)}$ space," AIP Conference Proceedings **2365**, 020021 (2021), https://doi.org/10.1063/5.0057127.
25. A.R. Hayotov and U.N. Khayriev, "Construction of an Optimal Quadrature Formula in the Hilbert Space of Periodic Functions." Lobachevskii Journal of Mathematics. **11(43)**, 3151–3160 (2022).
26. A.R. Hayotov A.R. and U.N. Khayriev , "Optimal quadrature formulas in the space $\widetilde{W_2}^{(m,m-1)}$ of periodic functions." Vestnik KRAUNC. **3(40)**, 211–226 (2022), 10.26117/2079-6641-2022-40-3-211-226.