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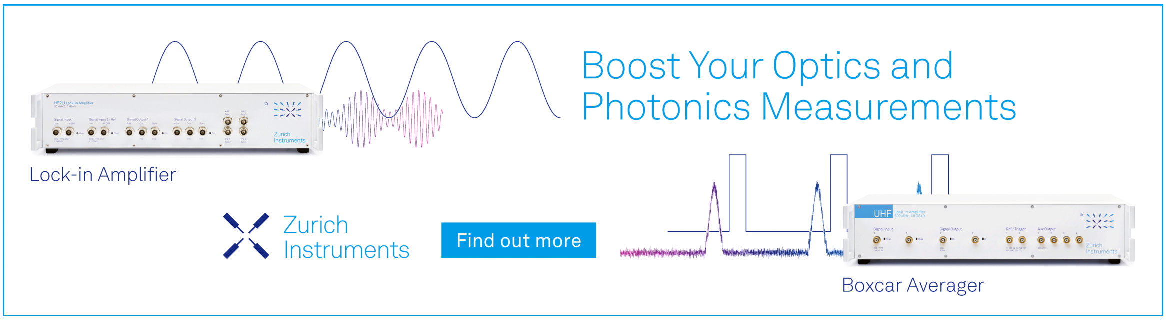

Construction of Natural L Spline in $W_{2,\sigma}^{(2,1)}$ Space

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Abstract. The present paper is devoted to construction of a natural L spline in the Hilbert space $W_{2,\sigma}^{(2,1)}(0,1)$. Here, the interpolation spline is chosen so that its norm becomes minimum in this space and satisfies the interpolation conditions. This work is consecrated to constructing polynomial and exponential L-spline. Explicit formulas for the coefficients of interpolation L-spline in the $W_{2,\sigma}^{(2,1)}(0,1)$ space are obtained.

INTRODUCTION AND THE STATEMENT OF THE PROBLEM

A typical approximation problem is an interpolation problem. The classical method for solving it is to construct an interpolation polynomial. However, polynomials have several disadvantages, such as an apparatus for approximating functions with singularities and functions with not too much smoothness. It is proved that the sequence of Lagrange interpolation polynomials constructed for a specific continuous function by equidistant nodes does not tend to this function as the degree of the polynomial increases. Therefore, in practice, splines are used instead of constructing a high-degree interpolation polynomial to approximate the function sufficiently well. The study of two problems - the interpolation of functions by splines and the optimal approximation of linear functionals, where exact solutions are spline functions - led to the formation of algebraic and variational directions in the theory of splines. In the algebraic order, splines are considered as smooth piecewise polynomial functions. In the variational direction, splines are Hilbert or Banach spaces elements that minimise certain functionals. Then the properties of these solutions are studied, i.e. questions of existence, uniqueness, a convergence of splines, and algorithms for their construction.

We present some results obtained by the theory of L -splines. Early contributions to L-spline theory include the work of Ahlberg, Nilson, and Walsh [1], as well as Schultz and Varga (see [2] p. 459). These works concentrated on natural L-splines, which appear as solutions to the corresponding best interpolation problems. In them, the order of approximation of generalised splines was investigated for the first time.

A considerable number of works are devoted to the theory of splines, e.g., J. Ahlberg et al. [1], Ignatev and Pevniy [3], Korneichuk et al. [4], Laurent [5], Subbotin, and Stechkin [6], Bezhaev and Vasilenko [7], Arcangeli et al. [8], Attea [9], Berlinet and Thomas-Agnan [10], Bojanov et al. [11], de Boor [12], Mastroianni, Milovanović [13], Shadimetov [14, 15, 16], Hayotov [17, 18], Boltaev [19, 20, 21], Akhmedov [22, 23] and other latest researches [24].

L.L. Schumaker [2], p. 407, introduced the space of natural hyperbolic splines. Also, in [25], it is shown that hyperbolic ones can be treated as an example of L-splines.

The present work is devoted to constructing polynomial and exponential L-splines. Here we obtain explicit formulas for the coefficients of L-splines in $W_{2,\sigma}^{(2,1)}(0,1)$ space.

We consider the Hilbert space $W_{2,\sigma}^{(2,1)}(0,1)$ equipped with the norm

$$\|\varphi\|_{W_{2,\sigma}^{(2,1)}} = \left(\int_0^1 (\varphi'' + \sigma\cdot\varphi')^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{1}$$

and $\int_0^1 (\varphi'' + \sigma\cdot\varphi')^2 \mathrm{d}x < \infty$. Note that equality (1) is a seminorm. Indeed, $\|\varphi\| = 0$ if and only if $\varphi(x) = a\cdot e^{-\sigma x} + b$. Here, a and b are unknown real constants, and σ is a non-zero real parameter.

The optimal interpolation formula was constructed in $W_{2,\sigma}^{(2,1)}$ space. Also, the discrete analogue of the differential operator $\frac{d^2}{dx^2} - \sigma^2$ and the extremal function of an error functional of the optimal interpolation formula were found that is needed for the construction process of the optimal interpolation formula(see, [26]). The extremal function of the error functional of the optimal interpolation formula was determined in the space $W_{2,\sigma}^{(2,1)}$ we are investigating [27].

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As in this work, variational methods were also used to construct quadrature formulas. The optimal quadrature formulas for numerical integration of integrals in some Hilbert spaces were constructed, and their errors were analysed [28, 29, 30, 31, 32, 33, 34, 35, 36]. Furthermore, optimal quadrature formulas for approximation Fourier integral were constructed in the many works (see, [37, 38, 39, 40, 41]).

We consider the following interpolation problem.

Problem 1. Find a function $S_2(x) \in W_{2,\sigma}^{(2,1)}(0,1)$ that gives a minimum to the norm (1) and satisfies the interpolation conditions $S_2(x_\beta) = \varphi(x_\beta)$, $\beta = 0,1,\dots,N$, where $x_\beta \in [0,1]$ are interpolation nodes and $\varphi(x_\beta)$ are given values of a function.

In pursuance of Theorem 2.2 in [42], page 45, we obtain the following analytical representation of the interpolation spline S_2:

$$S_2(x) = \sum_{\gamma=0}^{N} C_\gamma G_2(x - x_\gamma) + ae^{-\sigma x} + b, \tag{2}$$

here C_γ ($\gamma = 0,1,\dots,N$) are real numbers. And

$$G_2(x) = \frac{\mathrm{sign}(x)}{4\sigma^3}(-2\sigma x + e^{\sigma x} - e^{-\sigma x}) \tag{3}$$

is fundamental solution of the operator $\frac{d^4}{dx^4} - \sigma^2 \frac{d^2}{dx^2}$, i.e., the solution of the equation

$$\left(\frac{d^4}{dx^4} - \sigma^2 \frac{d^2}{dx^2} \right) G_2(x) = \delta(x),$$

where $\delta(x)$ is Dirac's delta-function.

It should be noted that equality (2) was considered in the works [3, 5, 8]. It is known that (see, for example, [3, 5, 8]) a solution S_2 of the form (2) of Problem 1 exists and unique for $N \geq 1$, and the coefficients C_γ, a and b of the interpolation spline $S_2(x)$ are defined by the system of $(N+3)$ linear equations

$$\sum_{\gamma=0}^{N} C_\gamma G_2(x_\beta - x_\gamma) + a \cdot e^{-\sigma x_\beta} + b = \varphi(x_\beta), \;\; \beta = 0,1,\dots,N, \tag{4}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-\sigma x_\gamma} = 0, \tag{5}$$

$$\sum_{\gamma=0}^{N} C_\gamma = 0. \tag{6}$$

The main goal of the following sections is to solve Problem 1, i.e., we solve system (4)-(6) for equally spaced nodes $x_\beta = h\beta$, $\beta = 0,1,\dots,N$, $h = 1/N$ and find analytical formulas for the coefficients C_γ, a and b of the spline $S_2(x)$.

The structure of the paper is as follows. Section 2 gives an algorithm for calculating the coefficients of the interpolation spline, and some properties of the discrete argument functions needed for this implementation are mentioned. Section 3 is devoted to the calculation of optimal coefficients using the algorithm which is given in Section 2.

ALGORITHM FOR COMPUTATION OF COEFFICIENTS OF THE INTERPOLATION SPLINE (2)

We use Sobolev's method [43, 44] to find optimal quadrature formula coefficients in $L_2^{(m)}$ space.

Below we use the notion of discrete argument functions and operations on them. The theory of discrete argument functions is given in [43, 44]. For completeness, we give some definitions of functions of discrete argument.

Definition 1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 2. *The inner product* of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta),\psi(h\beta)]=\sum_{\beta=-\infty}^{\infty}\varphi(h\beta)\cdot\psi(h\beta),$$

if the series on the right-hand side absolutely converges.

Definition 3. *The convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta)*\psi(h\beta)=[\varphi(h\gamma),\psi(h\beta-h\gamma)]=\sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\beta-h\gamma).$$

Now we turn to our problem.

Suppose that $C_\beta=0$ when $\beta<0$ and $\beta>N$. Using properties of discrete argument functions (see, [43, 44]) we rewrite the system (4)–(6) in the convolution form

$$G_2(h\beta)*C_\beta+a\cdot e^{-\sigma h\beta}+b=\varphi(h\beta),\quad \beta=0,1,\dots,N, \tag{7}$$

$$\sum_{\gamma=0}^{N}C_\gamma e^{-\sigma h\gamma}=0, \tag{8}$$

$$\sum_{\gamma=0}^{N}C_\gamma=0. \tag{9}$$

The problem of finding the solution of the system (7)-(9) is equivalent to Problem 1.

Further, instead of C_β we input the following functions

$$v_2(h\beta)=G_2(h\beta)*C_\beta, \tag{10}$$

$$u_2(h\beta)=v_2(h\beta)+a\cdot e^{-\sigma h\beta}+b. \tag{11}$$

In this problem, C_β can be represented by $u_2(h\beta)$ as follows

$$C_\beta=D_2(h\beta)*u_2(h\beta). \tag{12}$$

Thus, if we find $u_2(h\beta)$, then from (12) we can obtained the coefficients C_β. To calculate the convolution (12), the values of the function $u_2(h\beta)$ in all integers β are needed. From equality (7), we get $u_2(h\beta)=\varphi(h\beta)$ when $h\beta\in[0,1]$. Now, we need the representation of the function $u_2(h\beta)$ when $\beta<0$ and $\beta>N$. Since, $C_\beta=0$ for $h\beta\notin[0,1]$, then we have $C_\beta=D_2(h\beta)*u_2(h\beta)=0$, $h\beta\notin[0,1]$.

Now we calculate the convolution $v_2(h\beta)=G_2(h\beta)*C_\beta$ for $\beta\le 0$ and $\beta\ge N$.

Taking into account (3), (4), (5) and (6), we have

$$\begin{aligned} v_2(h\beta)&=\sum_{\gamma=-\infty}^{\infty}C_\gamma\cdot G_2(h\beta-h\gamma)\\ &=\sum_{\gamma=0}^{N}C_\gamma\frac{\operatorname{sign}(h\beta-h\gamma)}{4\sigma^3}(-2\sigma(h\beta-h\gamma)+e^{\sigma(h\beta-h\gamma)}-e^{-\sigma(h\beta-h\gamma)}).\end{aligned}$$

Firstly, we consider for $\beta\le 0$

$$\begin{aligned} v_2(h\beta)&=\frac{h\beta}{2\sigma^2}\sum_{\gamma=0}^{N}C_\gamma-\frac{h}{2\sigma^2}\sum_{\gamma=0}^{N}C_\gamma\gamma-\frac{e^{\sigma h\beta}}{4\sigma^3}\sum_{\gamma=0}^{N}C_\gamma e^{-\sigma h\gamma}+\frac{e^{-\sigma h\beta}}{4\sigma^3}\sum_{\gamma=0}^{N}C_\gamma e^{\sigma h\gamma}\\ &=\frac{-h}{2\sigma^2}\sum_{\gamma=0}^{N}C_\gamma\gamma+\frac{e^{-\sigma h\beta}}{4\sigma^3}\cdot\sum_{\gamma=0}^{N}C_\gamma e^{\sigma h\gamma}.\end{aligned}$$

Similarly, in the case $\beta\ge N$ we get the following expression

$$v_2(h\beta)=\frac{h}{2\sigma^2}\sum_{\gamma=0}^{N}C_\gamma\gamma-\frac{e^{-\sigma h\beta}}{4\sigma^3}\cdot\sum_{\gamma=0}^{N}C_\gamma e^{\sigma h\gamma}.$$

Now we denote

$$a^- = a + \frac{1}{4\sigma^3}\sum_{\gamma=0}^{N} C_\gamma e^{\sigma h\gamma}, \quad b^- = b - \frac{h}{2\sigma^2}\sum_{\gamma=0}^{N} C_\gamma \gamma,$$
$$a^+ = a - \frac{1}{4\sigma^3}\sum_{\gamma=0}^{N} C_\gamma e^{\sigma h\gamma}, \quad b^+ = b + \frac{h}{2\sigma^2}\sum_{\gamma=0}^{N} C_\gamma \gamma.$$

Consequently, we get the following form of the function $u_2(h\beta)$

$$u_2(h\beta) = \begin{cases} e^{-\sigma h\beta} a^- + b^-, & \beta \le 0, \\ \varphi(h\beta), & 0 \le \beta \le N, \\ e^{-\sigma h\beta} a^+ + b^+, & \beta \ge N. \end{cases} \tag{13}$$

From the last expression, we get the system of two linear equations

$$\begin{array}{ll} \text{for } \beta = 0, & a^- + b^- = \varphi(0), \\ \text{for } \beta = N, & e^{-\sigma} a^+ + b^+ = \varphi(1), \end{array}$$

and we obtain

$$b^- = \varphi(0) - a^-,$$
$$b^+ = \varphi(1) - e^{-\sigma} a^+.$$

Thus, we rewrite expression (13) in the form

$$u_2(h\beta) = \begin{cases} e^{-\sigma h\beta} a^- + \varphi(0) - a^-, & \beta \le 0, \\ \varphi(h\beta), & 0 \le \beta \le N, \\ e^{-\sigma h\beta} a^+ + \varphi(1) - e^{-\sigma} a^+, & \beta \ge N, \end{cases} \tag{14}$$

here a^- and a^+ are unknown constants. If we find a^- and a^+ then we determine the function $u_2(h\beta)$.

THE COEFFICIENTS OF THE INTERPOLATION SPLINE

Problem 2. Find a function $u_2(h\beta)$ satisfying the following equation

$$D_2(h\beta) * u_2(h\beta) = 0, \ \ h\beta \notin [0,1].$$

Where $D_2(h\beta)$ is the discrete analogue of the differential operator $\frac{d^4}{dx^4} - \sigma^2 \frac{d^2}{dx^2}$. The discrete argument function $D_2(h\beta)$ satisfies the next equation

$$D_2(h\beta) * G_2(h\beta) = \delta_d(h\beta), \tag{15}$$

where $\delta_d(h\beta) = \begin{cases} 0, & \beta \ne 0, \\ 1, & \beta = 0 \end{cases}$ is the discrete delta function.

Theorem 1. *The discrete analogue $D_2(h\beta)$ of the differential operator $\frac{d^4}{dx^4} - \sigma^2 \frac{d^2}{dx^2}$ satisfying* (15) *has the form*

$$D_2(h\beta) = K \begin{cases} A_1 \lambda_1^{|\beta|-1}, & |\beta| \ge 2, \\ -e^{\sigma h} + A_1, & |\beta| = 1, \\ \frac{A_1}{\lambda_1} + (1 + e^{\sigma h})^2 + P e^{\sigma h}, & \beta = 0, \end{cases} \tag{16}$$

where

$$A_1 = \frac{e^{\sigma h}\lambda_1^4 - (e^{\sigma h}+1)^2 \lambda_1^3 + 2(e^{2\sigma h} + e^{\sigma h} + 1)\lambda_1^2 - (e^{\sigma h}+1)^2 \lambda_1 + e^{\sigma h}}{\lambda_1^2 - 1},$$

$$K = \frac{2\sigma^3}{2\sigma h e^{\sigma h} + 1 - e^{2\sigma h}},$$

$$P=\frac{-2\sigma h-2\sigma he^{2\sigma h}-2+2e^{2\sigma h}}{2\sigma he^{\sigma h}+1-e^{2\sigma h}},$$

$$\lambda_1=\frac{\sigma h(e^{2\sigma h}+1)-e^{2\sigma h}+1-(e^{\sigma h}-1)\sqrt{(e^{\sigma h}+1)^2(\sigma h)^2-2\sigma h(e^{2\sigma h}-1)}}{-e^{2\sigma h}+2\sigma he^{\sigma h}+1},$$

$|\lambda_1|<1$.

Theorem 2. *The discrete analogue $D_2(h\beta)$ of the differential operator $\frac{d^4}{dx^4}-\sigma^2\frac{d^2}{dx^2}$ is satisfies the following equalities*

$$\begin{array}{ll} 1)\ D_2(h\beta)*e^{\sigma h\beta}=0, & 2)\ D_2(h\beta)*e^{-\sigma h\beta}=0,\\ 3)\ D_2(h\beta)*1=0, & 4)\ D_2(h\beta)*(h\beta)=0.\end{array}$$

Theorem 1 and 2 were proved in the works [45].

Now, we solve Problem 2 using equations (14) and (16)

$$\begin{aligned} D_2(h\beta)*u_2(h\beta)&=\sum_{\gamma=-\infty}^{\infty}D_2(h\beta-h\gamma)\cdot u_2(h\gamma)=\sum_{\gamma=-\infty}^{-1}D_2(h\beta-h\gamma)\cdot u_2(h\gamma)\\ &+\sum_{\gamma=0}^{N}D_2(h\beta-h\gamma)\cdot u_2(h\gamma)+\sum_{\gamma=N+1}^{\infty}D_2(h\beta-h\gamma)\cdot u_2(h\gamma)\\ &=\sum_{\gamma=1}^{\infty}D_2(h\beta+h\gamma)\cdot[a^-\cdot e^{\sigma h\gamma}+\varphi(0)-a^-]+\sum_{\gamma=0}^{N}D_2(h\beta-h\gamma)\cdot\varphi(h\gamma)\\ &+\sum_{\gamma=1}^{\infty}D_2(h(N+\gamma-\beta))\cdot[a^+\cdot e^{-\sigma h(N+\gamma)}+\varphi(1)-e^{-\sigma}a^+].\end{aligned}$$

Then, for $\beta=-1$ and by Theorem 1, Theorem 2 and last expression we obtain the equation

$$\begin{aligned} &a^-\left[(e^{\sigma h}-1)(Pe^{\sigma h}+1+e^{\sigma h})+\frac{A_1}{\lambda_1}\left(\frac{e^{\sigma h}}{1-\lambda_1e^{\sigma h}}-\frac{1}{1-\lambda_1}\right)\right]+a^+A_1e^{-\sigma}\lambda_1^{N+1}\left(\frac{1}{e^{\sigma h}-\lambda_1}-\frac{1}{1-\lambda_1}\right)\\ &=-\varphi(0)\left(Pe^{\sigma h}+1+e^{2\sigma h}+\frac{A_1}{\lambda_1(1-\lambda_1)}\right)-\frac{\varphi(1)A_1\lambda_1^{N+1}}{1-\lambda_1}-A_1\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma).\end{aligned}\tag{17}$$

Similarly, for $\beta=N+1$ we get the following equation

$$\begin{aligned} &a^-\left[A_1\lambda_1^{N+1}\left(\frac{e^{\sigma h}}{1-\lambda_1e^{\sigma h}}-\frac{1}{1-\lambda_1}\right)\right]+a^+\left[e^{-\sigma}\left((1-e^{\sigma h})(P+1+e^{\sigma h})+\frac{A_1}{\lambda_1}\left(\frac{1}{e^{\sigma h}-\lambda_1}-\frac{1}{1-\lambda_1}\right)\right)\right]\\ &=-\frac{\varphi(0)A_1\lambda_1^{N+1}}{1-\lambda_1}-\varphi(1)\left(1+e^{2\sigma h}+Pe^{\sigma h}+\frac{A_1}{\lambda_1(1-\lambda_1)}\right)-A_1\lambda_1^{N}\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}\varphi(h\gamma).\end{aligned}\tag{18}$$

Thereby we have got a system of linear equations (17) and (18), and we rewrite it as follow

$$\begin{aligned} B_1a^-+B_2a^+&=T_1,\\ B_3a^-+B_4a^+&=T_2.\end{aligned}\tag{19}$$

Here

$$
\begin{aligned}
B_1 &= (e^{\sigma h}-1)(Pe^{\sigma h}+1+e^{\sigma h})+\frac{A_1}{\lambda_1}\left(\frac{e^{\sigma h}}{1-\lambda_1 e^{\sigma h}}-\frac{1}{1-\lambda_1}\right),\\
B_2 &= A_1 e^{-\sigma}\lambda_1^{N+1}\left(\frac{1}{e^{\sigma h}-\lambda_1}-\frac{1}{1-\lambda_1}\right),\\
T_1 &= -\varphi(0)\left(Pe^{\sigma h}+1+e^{2\sigma h}+\frac{A_1}{\lambda_1(1-\lambda_1)}\right)-\frac{\varphi(1)A_1\lambda_1^{N+1}}{1-\lambda_1}-A_1\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma),\\
B_3 &= A_1\lambda_1^{N+1}\left(\frac{e^{\sigma h}}{1-\lambda_1 e^{\sigma h}}-\frac{1}{1-\lambda_1}\right),\\
B_4 &= e^{-\sigma}\left((1-e^{\sigma h})(P+1+e^{\sigma h})+\frac{A_1}{\lambda_1}\left(\frac{1}{e^{\sigma h}-\lambda_1}-\frac{1}{1-\lambda_1}\right)\right),\\
T_2 &= -\frac{\varphi(0)A_1\lambda_1^{N+1}}{1-\lambda_1}-\varphi(1)\left(1+e^{2\sigma h}+Pe^{\sigma h}+\frac{A_1}{\lambda_1(1-\lambda_1)}\right)-A_1\lambda_1^{N}\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}\varphi(h\gamma).
\end{aligned}
$$

We find a^- and a^+ from the system of equations (19)

$$
a^- = \frac{T_1B_4-T_2B_2}{B_1B_4-B_2B_3},\ \ a^+ = \frac{T_2B_1-T_1B_3}{B_1B_4-B_2B_3}. \tag{20}
$$

Thus, the discrete function $u_2(h\beta)$ is fully determined from (14).
Now we find C_β from the equality $C_\beta = D_2(h\beta) * u_2(h\beta)$, for $0 \le \beta \le N$

$$
C_\beta = \sum_{\gamma=-\infty}^{\infty} D_2(h\beta - h\gamma)\cdot u_2(h\gamma).
$$

From here for $\beta = 0$

$$
C_0 = \sum_{\gamma=-\infty}^{\infty} D_2(-h\gamma)\cdot u_2(h\gamma) = \sum_{\gamma=-\infty}^{-1} D_2(h\gamma)\cdot u_2(h\gamma)+\sum_{\gamma=0}^{N} D_2(h\gamma)\cdot u_2(h\gamma)+\sum_{\gamma=N+1}^{\infty} D_2(h\gamma)\cdot u_2(h\gamma)
$$

$$
= \sum_{\gamma=1}^{\infty} D_2(h\gamma)\cdot u_2(-h\gamma)+\sum_{\gamma=0}^{N} D_2(h\gamma)\cdot u_2(h\gamma)+\sum_{\gamma=1}^{\infty} D_2(h(\gamma+N))\cdot u_2(h(\gamma+N)).
$$

So

$$
\begin{aligned}
C_0 = K\Bigg[& a^-\left(e^{\sigma h}-e^{2\sigma h}+\frac{A_1 e^{\sigma h}}{1-\lambda_1 e^{\sigma h}}-\frac{A_1}{1-\lambda_1}\right)+\varphi(0)\left(\frac{A_1}{1-\lambda_1}+1+e^{\sigma h}+e^{2\sigma h}+Pe^{\sigma h}\right)\\
&-\varphi(h)e^{\sigma h}+\frac{A_1}{\lambda_1}\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma)+a^+\left(\frac{e^{-\sigma}A_1\lambda_1^N}{e^{\sigma h}-\lambda_1}-\frac{e^{-\sigma}A_1\lambda_1^N}{1-\lambda_1}\right)+\varphi(1)\frac{A_1\lambda_1^N}{1-\lambda_1}\Bigg].
\end{aligned}
$$

For simplicity, we determine

$$
M_1 = a^-\left(\frac{A_1 e^{\sigma h}}{1-\lambda_1 e^{\sigma h}}-\frac{A_1}{1-\lambda_1}\right)+\frac{\varphi(0)A_1}{1-\lambda_1},
$$

and

$$
N_1 = a^+\left(\frac{e^{-\sigma}A_1}{e^{\sigma h}-\lambda_1}-\frac{e^{-\sigma}A_1}{1-\lambda_1}\right)+\frac{\varphi(1)A_1}{1-\lambda_1}.
$$

Thereby

$$C_0 = K\left[M_1+\lambda_1^N N_1+a^-(e^{\sigma h}-e^{2\sigma h})+\varphi(0)\left(1+e^{\sigma h}+e^{2\sigma h}+Pe^{\sigma h}\right)-\varphi(h)e^{\sigma h}+\frac{A_1}{\lambda_1}\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma)\right]. \quad (21)$$

Now we calculate the coefficients for $\beta=1,2,\dots,N-1$

$$C_\beta=\sum_{\gamma=1}^{\infty}D_2(h\beta+h\gamma)\cdot u_2(-h\gamma)+\sum_{\gamma=0}^{N}D_2(h\beta-h\gamma)\cdot u_2(h\gamma)+\sum_{\gamma=1}^{\infty}D_2(h(\gamma+N-\beta))u_2(h(\gamma+N))$$

$$=K\cdot\left(A_1\lambda_1^{\beta-1}\sum_{\gamma=1}^{\infty}\lambda_1^{\gamma}\left(a^-(e^{\sigma h\gamma}-1)+\varphi(0)\right)+\sum_{\gamma=0}^{\beta-2}A_1\cdot\lambda_1^{\beta-\gamma-1}\cdot\varphi(h\gamma)+(-e^{\sigma h}+A_1)\varphi(h(\beta-1))\right.$$

$$+\left((1+e^{\sigma h})^2+Pe^{\sigma h}+\frac{A_1}{\lambda_1}\right)\cdot\varphi(h\beta)+(-e^{\sigma h}+A_1)\varphi(h(\beta+1))$$

$$\left.+\sum_{\gamma=\beta+2}^{N}A_1\lambda_1^{\gamma-\beta-1}\varphi(h\gamma)+A_1\lambda_1^{N-\beta-1}\sum_{\gamma=1}^{\infty}\lambda_1^{\gamma}\cdot(a^+\cdot(e^{-\sigma h(N+\gamma)}-e^{-\sigma})+\varphi(1))\right)$$

$$=K\left[\lambda_1^{\beta}\left(A_1a^-\left(\frac{e^{\sigma h}}{1-e^{\sigma h}\lambda_1}\frac{1}{1-\lambda_1}\right)+\varphi(0)\frac{A_1}{1-\lambda_1}\right)-e^{\sigma h}(\varphi(h(\beta-1))+\varphi(h(\beta+1)))\right.$$

$$+\varphi(h\beta)((1+e^{\sigma h})^2+Pe^{\sigma h})+\sum_{\gamma=0}^{N}A_1\lambda_1^{|\beta-\gamma|-1}\cdot\varphi(h\gamma)$$

$$\left.+\lambda^{N-\beta}\left(A_1a^+e^{-\sigma}\left(\frac{1}{e^{\sigma h}-\lambda_1}-\frac{1}{1-\lambda_1}\right)+\varphi(1)\frac{A_1}{1-\lambda_1}\right)\right].$$

Accordingly

$$C_\beta=K\left[\lambda_1^{\beta}M_1+\lambda_1^{N-\beta}N_1-e^{\sigma h}\left(\varphi(h(\beta-1))+\varphi(h(\beta+1))\right)+\varphi(h\beta)\left((1+e^{\sigma h})^2+Pe^{\sigma h}\right)\right.$$
$$\left.+\frac{A_1}{\lambda_1}\sum_{\gamma=0}^{N}\lambda_1^{|\beta-\gamma|}\cdot\varphi(h\gamma)\right]. \quad (22)$$

Finally, we calculate for $\beta=N$

$$C_N=\sum_{\gamma=1}^{\infty}D_2(h\beta+h\gamma)u_2(-h\gamma)+\sum_{\gamma=0}^{N}D_2(h\beta-h\gamma)u_2(h\gamma)+\sum_{\gamma=1}^{\infty}D_2(h(N+\gamma-\beta))u_2(h(N+\gamma))$$

$$=K\left[A_1\lambda_1^N a^-\left(\frac{e^{\sigma h}}{1-\lambda_1e^{\sigma h}}-\frac{1}{1-\lambda_1}\right)+\frac{\varphi(0)A_1\lambda_1^N}{1-\lambda_1}+((1+e^{\sigma h})^2+Pe^{\sigma h})\varphi(1)+A_1\lambda_1^{N-1}\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}\varphi(h\gamma)\right.$$

$$\left.-e^{\sigma h}(\varphi(1-h)+a^+e^{-\sigma}(e^{-\sigma h}-1)+\varphi(1))+e^{-\sigma}a^+A_1\left(\frac{1}{e^{\sigma h}-\lambda_1}-\frac{1}{1-\lambda_1}\right)+\varphi(1)\frac{A_1}{1-\lambda_1}\right].$$

Thus, we obtain

$$C_N = K\left[\lambda_1^N M_1 + N_1 + (1+e^{\sigma h}+e^{2\sigma h}+Pe^{\sigma h})\varphi(1) + A_1\lambda_1^{N-1}\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}\varphi(h\gamma) - e^{\sigma h}(\varphi(1-h)+a^+e^{-\sigma}(e^{-\sigma h}-1))\right]. \tag{23}$$

Consequently, the following is proven.

Theorem 3. *Coefficients of the optimal interpolation formulas* (2) *with equally spaced nodes* $x_\beta = h\beta$ *in the space* $W_{2,\sigma}^{(2,1)}(0,1)$ *have the following form*

$$C_0 = K\left[M_1 + \lambda_1^N N_1 + a^-(e^{\sigma h}-e^{2\sigma h}) + \varphi(0)\left(1+e^{\sigma h}+e^{2\sigma h}+Pe^{\sigma h}\right) - \varphi(h)e^{\sigma h} + \frac{A_1}{\lambda_1}\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma)\right],$$

$$C_\beta = K\left[\lambda_1^\beta M_1 + \lambda_1^{N-\beta}N_1 - e^{\sigma h}\left(\varphi(h(\beta-1))+\varphi(h(\beta+1))\right) + \varphi(h\beta)\left((1+e^{\sigma h})^2+Pe^{\sigma h}\right) + \frac{A_1}{\lambda_1}\sum_{\gamma=0}^{N}\lambda_1^{|\beta-\gamma|}\varphi(h\gamma)\right],$$

$\beta = 1,2,\ldots,N-1,$

$$C_N = K\left[\lambda_1^N M_1 + N_1 + (1+e^{\sigma h}+e^{2\sigma h}+Pe^{\sigma h})\varphi(1) + A_1\lambda_1^{N-1}\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}\varphi(h\gamma) - e^{\sigma h}(\varphi(1-h)+a^+e^{-\sigma}(e^{-\sigma h}-1))\right]$$

here a^- *and* a^+ *are known from the expression* (20)*, and*

$$M_1 = a^-\left(\frac{A_1e^{\sigma h}}{1-\lambda_1e^{\sigma h}} - \frac{A_1}{1-\lambda_1}\right) + \frac{\varphi(0)A_1}{1-\lambda_1},$$

$$N_1 = a^+\left(\frac{e^{-\sigma}A_1}{e^{\sigma h}-\lambda_1} - \frac{e^{-\sigma}A_1}{1-\lambda_1}\right) + \frac{\varphi(1)A_1}{1-\lambda_1}.$$

CONCLUSION

Thus, polynomial and exponential L-spline was constructed in this work. A system of linear equations was obtained for the coefficients of the interpolation spline formula. The discrete analogue of the fourth-order differential operator depending on the σ parameter was used to solve this system of linear equations. The explicit form for the coefficients of L-spline (2) in $W_{2,\sigma}^{(2,1)}(0,1)$ space were obtained.

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