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The Error Functional of Optimal Interpolation Formulas in $W_{2,\sigma}^{(2,1)}$ Space

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Abstract. The present paper is devoted to construction of an optimal interpolation formula $\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta \cdot \varphi(x_\beta)$. The difference between function and sum is estimated by the norm of the error functional. For calculating the norm we use an extremal function for the error functional ℓ. We find the extremal function of the error functional.

INTRODUCTION AND THE STATEMENT OF THE PROBLEM

Assume we are given a table of values $\varphi(x_\beta)$, $\beta = 0, 1, \dots, N$ of a function φ at the points $x_\beta \in [0,1]$. It is required to approximate the function by another more simple function P_φ with interpolation conditions, i.e.

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x)\varphi(x_\beta),$$
$$\varphi(x_\beta) = P_\varphi(x_\beta), \;\; \beta = 0, 1, \dots, N. \quad (1)$$

Here, $C_\beta(x)$ and $x_\beta(\in [0,1])$ are *the coefficients* and *the nodes* of the interpolation formula (1), respectively. There are many results about optimal interpolation formulas, see [1, 2, 3, 4, 5] for a recent articles. Furthermore, optimal quadrature formulas for approximation Fourier integral were constructed in hot research topics, see [6, 7, 8, 9, 10] and their errors were evaluated in this space with $\sigma = 1$.

By $W_{2,\sigma}^{(2,1)}(0,1)$ we denote the class of all functions φ defined on $[0,1]$ which possess an absolutely continuous $(m-1)$th derivative on $[0,1]$ and whose m-th derivative is in $L_2(0,1)$. The class $W_{2,\sigma}^{(2,1)}(0,1)$ under the inner product

$$\langle \varphi, \psi \rangle_{W_{2,\sigma}^{(2,1)}} = \int_0^1 \left(\varphi''(x) + \sigma\varphi'(x)\right)\left(\psi''(x) + \sigma\psi'(x)\right) dx$$

is a Hilbert space.

Here, we consider the norm

$$\|\varphi\|_{W_{2,\sigma}^{(2,1)}} = \left\{ \int_0^1 (\varphi'' + \sigma\varphi')^2 dx \right\}^{\frac{1}{2}}, \quad (2)$$

where $\sigma \in \mathbb{R}$, $\sigma \neq 0$.

The difference $\varphi - P_\varphi$ is called the error of the interpolation formula (1). The value of this error at a point $z \in [0,1]$ is a linear functional on the space $W_{2,\sigma}^{(2,1)}(0,1)$ and it has the following form

$$(\ell, \varphi) \;=\; \varphi(z) - \sum_{\beta=0}^{N} C_\beta(z)\varphi(x_\beta) = \int_{-\infty}^{\infty} \left(\delta(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x-x_\beta) \right) dx \quad (3)$$

where $\delta(x)$ is Dirac's delta-function and using the formula for the inner product, we have the following

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$$\ell(x,z)=\delta(x-z)-\sum_{\beta=0}^{N}C_{\beta}(z)\delta(x-x_{\beta}), \tag{4}$$

which is *the error functional* of the interpolation formula (1) and it belongs to the space $W_{2,\sigma}^{(2,1)*}(0,1)$. We note that $W_{2,\sigma}^{(2,1)*}(0,1)$ is the conjugate space to the space $W_{2,\sigma}^{(2,1)}(0,1)$.

By the Cauchy-Schwarz inequality the absolute value of the error (3) is estimated as follows

$$|(\ell,\varphi)|\le\|\ell\|_{W_{2,\sigma}^{(2,1)*}}\|\varphi\|_{W_{2,\sigma}^{(2,1)}}.$$

EXTREMAL FUNCTION

For finding the explicit form of the norm of the error functional ℓ in the space $W_{2,\sigma}^{(2,1)}$ we use its extremal function. The function ψ_ℓ from $W_{2,\sigma}^{(2,1)}(0,1)$ space is called the extremal function (see [11, 12]) for the error functional ℓ if the following equality is fulfilled

$$(\ell,\psi_\ell)=\|\ell\|_{W_{2,\sigma}^{(2,1)*}}\cdot\|\psi_\ell\|_{W_{2,\sigma}^{(2,1)}}.$$

According to the Riesz theorem, any linear continuous functional ℓ in a Hilbert space is represented in the form of an inner product. Since $W_{2,\sigma}^{(2,1)}(0,1)$ is the Hilbert space then for any function φ from this space, we have

$$(\ell,\varphi)=\langle\psi_\ell,\varphi\rangle_{W_{2,\sigma}^{(2,1)}}. \tag{5}$$

Then using integral form of the right-hand side of (5) and using integration by part, we get

$$\langle\varphi,\psi_\ell\rangle=\int_0^1\left(\varphi''(x)+\sigma\varphi'(x)\right)\left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)dx$$

$$=\int_0^1\varphi''(x)\ \left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)dx+\sigma\int_0^1\varphi'(x)\left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)dx$$

$$=\varphi'(x)\left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)|_0^1-\int_0^1\varphi'(x)\left(\psi_\ell'''(x)+\sigma\psi_\ell''(x)\right)dx+\int_0^1\varphi'(x)(\sigma\psi_\ell''(x)+\sigma^2\psi_\ell'(x))dx$$

$$=\varphi'(x)\left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)|_0^1-\int_0^1\varphi'(x)\left(\psi_\ell'''(x)-\sigma^2\psi_\ell'(x)\right)dx$$

$$=\varphi'(x)\left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)|_0^1-\varphi(x)\left(\psi_\ell'''(x)-\sigma^2\psi_\ell'(x)\right)|_0^1+\int_0^1\varphi(x)\left(\psi_\ell^{IV}(x)-\sigma^2\psi_\ell''(x)\right)dx$$

$$=\int_0^1\varphi(x)(\psi_\ell^{IV}(x)-\sigma^2\psi_\ell''(x))dx=(\ell,\varphi).$$

From here we get the following boundary value problem

$$\left(\psi_\ell''(x)+\sigma\psi_\ell'(x)\right)|_0^1=0, \tag{6}$$

$$\left(-\psi_\ell'''(x)+\sigma^2\psi_\ell'(x)\right)|_0^1=0, \tag{7}$$

$$\psi_\ell^{IV}(x)-\sigma^2\psi_\ell''(x)=\ell(x). \tag{8}$$

Now we solve equation (8) under the conditions (6) and (7). Corresponding homogeneous equation to (8) is

$$\psi_\ell^{IV}(x)-\sigma^2\psi_\ell''(x)=0. \tag{9}$$

The characteristic equation for (9) is

$$a^4-\sigma^2a^2=0$$

and it has roots $a_1=a_2=0$ and $a_{3,4}=\pm\sigma$.

Hence the general solution for equation (9) is

$$d_0+d_1x+d_2e^{-\sigma x}+d_3e^{\sigma x},$$

where d_0, d_1, d_2 and d_3 are real numbers and a particular solution of equation (8) is

$$\ell(x)+G_2(x).$$

Here $G_2(x)$ is fundamental solution of the differential operator $\frac{d^4}{dx^4}-\sigma^2\frac{d^2}{dx^2}$, i.e. satisfies the following equation

$$\frac{d^4}{dx^4}G_2(x)-\sigma^2\frac{d^2}{dx^2}G_2(x)=\delta(x). \tag{10}$$

We apply the rule of finding fundamental solution [13]. For this, we change operator $\frac{d}{dx}$ to parameter p and we get corresponding polynomial $P(p)$

$$P(p)=p^4-\sigma^2p^2=p^2(p-\sigma)(p+\sigma).$$

Now we expand the fraction $\frac{1}{P(p)}$ to elementary fractions

$$\frac{1}{P(p)}=\frac{1}{p^2(p-\sigma)(p+\sigma)}=\frac{A}{p^2}+\frac{B}{p}+\frac{C}{p-\sigma}+\frac{D}{p+\sigma}=\frac{1}{2\sigma^3}\left(-\frac{2\sigma}{p^2}+\frac{1}{p-\sigma}-\frac{1}{p+\sigma}\right).$$

From this we have the following fundamental solution

$$G_2(x)=\frac{\mathrm{sign}(x)}{4\sigma^3}(-2\sigma x+e^{\sigma x}-e^{-\sigma x}). \tag{11}$$

Then the general solution of equation (8) is in the following form

$$\psi_\ell(x)=(G_2*\ell)(x)+d_2e^{-\sigma x}+d_3e^{\sigma x}+d_1x+d_0. \tag{12}$$

The function $\psi_\ell(x)$ is uniquely defined in $W_{2,\sigma}^{(2,1)}$ space, we can find it from conditions (6) and (7). For this, we need some order derivatives of $\psi_\ell(x)$, i.e.

$$\psi_\ell'(x)=\ell(x)*\left(\frac{\mathrm{sign}(x)}{4\sigma^3}\left(-2\sigma+\sigma e^{\sigma x}+\sigma e^{-\sigma x}\right)\right)+d_3\sigma e^{\sigma x}-d_2\sigma e^{-\sigma x}+d_1,$$

$$\psi_\ell''(x)=\ell(x)*\left(\frac{\mathrm{sign}(x)}{4\sigma^3}\left(\sigma^2e^{\sigma x}-\sigma^2e^{-\sigma x}\right)\right)+d_3\sigma^2e^{\sigma x}+d_2\sigma^2e^{-\sigma x},$$

$$\psi_\ell'''(x) = \ell(x) * \left(\frac{\text{sign}(x)}{4\sigma^3}\left(\sigma^3 e^{\sigma x} + \sigma^3 e^{-\sigma x}\right)\right) + d_3\sigma^3 e^{\sigma x} - d_2\sigma^3 e^{-\sigma x}.$$

Then taking into account (6), we have

$$\psi_\ell''(x) + \sigma\psi_\ell'(x) = \ell(x) * \left(\frac{\text{sign}(x)}{4\sigma^3}\left(\sigma^2 e^{\sigma x} - \sigma^2 e^{-\sigma x}\right)\right) + d_3\sigma^2 e^{\sigma x} + d_2\sigma^2 e^{-\sigma x}$$

$$+\ell(x) * \left(\frac{\text{sign}(x)}{4\sigma^3}\left(-2\sigma^2 + \sigma^2 e^{\sigma x} + \sigma^2 e^{-\sigma x}\right)\right) + d_3\sigma^2 e^{\sigma x} - d_2\sigma^2 e^{-\sigma x} + d_1\sigma$$

$$= \ell(x) * \left(\frac{\text{sign}(x)}{4\sigma^3}\left(2\sigma^2 e^{\sigma x} - 2\sigma^2\right)\right) + 2d_3\sigma^2 e^{\sigma x} + d_\ell\sigma$$

$$= \left(\ell(y), \frac{\text{sign}(x-y)}{2\sigma}\left(e^{\sigma(x-y)} - 1\right)\right) + 2d_3\sigma^2 e^{\sigma x} + d_1\sigma.$$

Putting boundary conditions (6) on the resulting equation

$$\begin{cases} \left(\ell(y), \frac{\text{sign}(1-y)}{2\sigma}\left(e^{\sigma}e^{-\sigma y} - 1\right)\right) + 2d_3\sigma^2 e^{\sigma} + d_1\sigma = 0, & \text{for } x = 1, \\ \left(\ell(y), \frac{\text{sign}(-y)}{2\sigma}\left(e^{-\sigma y} - 1\right)\right) + 2d_3\sigma^2 + d_1\sigma = 0, & \text{for } x = 0. \end{cases} \tag{13}$$

and then taking into account (7), we obtain

$$\psi_\ell'''(x) - \sigma^2\psi_\ell'(x) = \ell(x) * \left(\frac{\text{sign}(x)}{4\sigma^3}\left(\sigma^3 e^{\sigma x} + \sigma^3 e^{-\sigma x}\right)\right) + d_3\sigma^3 e^{\sigma x}$$

$$-\ell(x) * \left(\frac{\text{sign}(x)}{4\sigma^3}\left(-2\sigma^3 + \sigma^3 e^{\sigma x} + \sigma^3 e^{-\sigma x}\right)\right) - d_3\sigma^3 e^{\sigma x} + d_2\sigma^3 e^{-\sigma x} - d_1\sigma^2$$

$$= \ell(x) * \left(\frac{\text{sign}(x)}{2}\right) - d_1\sigma^2 = \left(\ell(y), \frac{\text{sign}(x-y)}{2}\right) - d_1\sigma^2.$$

Thereby, we have the following system of equations for $x = 1$ and $x = 0$

$$\begin{cases} \left(\ell(y), \frac{\text{sign}(1-y)}{2}\right) - d_1\sigma^2 = 0, \\ \left(\ell(y), \frac{\text{sign}(-y)}{2}\right) - d_1\sigma^2 = 0, \end{cases} \Rightarrow$$

$$\begin{cases} \frac{1}{2}(\ell(y), 1) - d_1\sigma^2 = 0, \\ -\frac{1}{2}(\ell(y), 1) - d_1\sigma^2 = 0, \end{cases} \Rightarrow \begin{cases} d_1 = 0, \\ (\ell(y), 1) = 0. \end{cases} \tag{14}$$

Now, from system (13), we get

$$\begin{cases} \frac{e^{\sigma}}{2\sigma}\left(\ell(y), e^{-\sigma y}\right) - \frac{1}{2\sigma}(\ell(y), 1) + 2d_3\sigma^2 e^{\sigma} + d_1\sigma = 0, \\ -\frac{1}{2\sigma}\left(\ell(y), e^{-\sigma y}\right) + \frac{1}{2\sigma}(\ell(y), 1) + 2d_3\sigma^2 + d_1\sigma = 0, \end{cases}.$$

Taking into account (14), we get

$$\begin{cases} \frac{e^{\sigma}}{2\sigma}\left(\ell(y), e^{-\sigma y}\right) + 2d_3\sigma^2 e^{\sigma} = 0, \\ -\frac{1}{2\sigma}\left(\ell(y), e^{-\sigma y}\right) + 2d_3\sigma^2 = 0, \end{cases} \Rightarrow \begin{cases} d_3 = 0, \\ \left(\ell(y), e^{-\sigma y}\right) = 0, \end{cases}$$

defined as $a = d_2, d = d_0$ from equation (12), we rewrite (12) with determined coefficients of the expression

$$\psi_\ell(x) = (G_2 * \ell)(x) + ae^{-\sigma x} + d. \tag{15}$$

Thus, the following theorem is proved.

Theorem 1 *The solution of the boundary value problem* (6)-(8) *has the following form*

$$\psi_\ell(x) = \ell(x) * G_2(x) + ae^{-\sigma x} + d,$$

where $G_2(x)$ *is defined by* (11).

CONCLUSION

In this work, the extremal function of the error functional of the optimal interpolation formula (1) was determined in $W_{2,\sigma}^{(2,1)}(0,1)$ space.

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