Numerical solution of hyperbolic equations using the finite element method for modeling dynamic systems

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ABSTRACT

In this study, the state of the art in solving hyperbolic equations using the finite element method has been analyzed in order to achieve accurate and stable numerical solutions. Various shape functions and variational formulations have been investigated to assess their influence on the accuracy and stability of computational algorithms. The advantage of hybrid finite-element schemes in modeling wave processes in heterogeneous media has been identified. The effect of time-step selection and mesh adaptation on solution convergence and computational cost has been examined. An optimal relationship between the order of approximation and the degree of mesh refinement has been determined to improve accuracy under limited computational resources. It has been established that the proposed algorithm ensures a reliable numerical representation of a wide range of wave dynamics and continuum mechanics problems. A justification for the applicability of optimization methods for finite element method parameters in the solution of multidimensional hyperbolic problems has been provided for further integration into industrial and research software packages.

Keywords: hyperbolic equations, finite element method, wave processes, adaptive mesh, numerical modeling

1. INTRODUCTION

Hyperbolic partial differential equations appear in numerous fields, including solid mechanics, acoustics, aerodynamics, electronics, and quantum physics. They are integral to modeling wave propagation, shock formation, and other dynamic phenomena. The need for accurate, stable, and efficient numerical methods for such partial differential equations has grown significantly with advances in computational technology.

The finite element method is one of the most universal and powerful approaches for discretizing differential equations [1-2]. Historically, finite element method was predominantly developed for elliptic and parabolic problems. However, over the last few decades, various modifications and extensions have been introduced to handle hyperbolic partial differential equations effectively [3-5].

This article focuses on investigating how finite element method can be adapted and optimized for solving hyperbolic equations, with particular emphasis on wave processes. Our approach covers the selection of an appropriate variational formulation, shape functions, mesh types, time discretization schemes, and adaptive techniques for balancing accuracy and computational cost. Numerical experiments in one and two dimensions demonstrate the efficacy of the proposed methodology.

The main goal of this research is to develop, justify, and test an finite element method-based algorithm for solving multidimensional hyperbolic problems, primarily those involving wave dynamics, in a way that can be scaled to large problem sizes and complex physical domains.

2. MATERIALS AND METHODS

In this section, we present the hyperbolic equations under consideration, their variational formulation, the specifics of finite element method implementation, as well as the software tools and computational resources used.

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2.1 Problem Statement

Hyperbolic equations can generally be represented by the classical wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + f(x,t), \tag{1}$$

Where $u = u(x,t)$ is the unknown function (e.g., displacement in mechanics or pressure in acoustics), Δ is the Laplacian operator, c is the wave propagation speed (potentially dependent on position or other physical parameters), and $f(x,t)$ is an external source term.

The initial conditions are given by

$$u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = \vartheta_0(x) \tag{2}$$

While boundary conditions (e.g., Dirichlet or Neumann) are imposed on $\partial\Omega$.

2.2 Variational Formulation of Hyperbolic Equations

Finite element method is typically based on a weak (variational) formulation of the partial differential equation. For simplicity, consider a one-dimensional hyperbolic problem with Dirichlet boundary conditions. The weak form is:

$$\int_{\Omega} \frac{\partial^2 u}{\partial t^2} \omega d\,\Omega = -\int_{\Omega} c^2 \nabla u * \nabla \omega d\,\Omega + \int_{\Omega} f(x,t)\omega d\,\Omega \tag{3}$$

Where ω is an arbitrary test function from the same function space as uuu. For higher-dimensional domains, the integral is taken over Ω with appropriate initial and boundary conditions.

2.3 Spatial Approximation via the Finite Element Method

Spatial approximation via the finite element method involves several key steps. First, the domain Ω is discretized into a finite number of elements, which can be line segments in one dimension, triangles or quadrilaterals in two dimensions, and tetrahedra or hexahedra in three dimensions. In each element, a set of basis, or shape, functions $\phi_i(x)$ is defined, which may be linear, quadratic, or of higher order. The approximate solution u_h is then expressed as a linear combination of these basis functions. Next, the local equations corresponding to each element are assembled into a global system that relates the solution values at the mesh nodes. Throughout this process, careful attention to stability is crucial, especially since wave-dominated phenomena often require satisfying conditions such as the Courant–Friedrichs–Lewy condition for time discretization, among others [6-7].

2.4 Time Discretization

Time discretization for approximating $\frac{\partial^2 u}{\partial t^2}$ involves several schemes. Explicit schemes, such as central differences, are simpler to implement but impose stringent constraints on the time-step size due to Courant–Friedrichs–Lewy-type stability conditions. Implicit schemes, including Newmark's method and certain symplectic integrators, allow for larger time steps but require solving large systems of equations at each step, which increases computational costs[8]. Hybrid schemes, or semi-implicit approaches, seek to balance accuracy and computational efficiency. The choice of the appropriate scheme depends on factors such as the problem's dimension, the geometry of the domain, and the specific wave phenomena encountered.

2.5 Adaptive Mesh and A Posteriori Error Estimation

For problems with wave fronts or shock waves—where the solution has sharp gradients—an adaptive mesh refinement strategy is highly beneficial. In this study, we employ an a posteriori error estimation method to automatically refine the mesh in regions with high error, thus improving local accuracy without excessively increasing the global number of elements [7, 14].

2.6 Software Implementation and Computational Resources

All numerical experiments were performed on a server with multi-core processors (about 3 GHz) and 64 GB of RAM. The algorithms were implemented in C++ with libraries such as Deal.II and PETSc. Python scripts and visualization software (Paraview, Gnuplot) were used for data processing and graphical analysis.

3. RESULTS

This section presents the results of numerical experiments in one and two dimensions, illustrating the efficiency of the proposed finite element approach for solving hyperbolic equations.

3.1 One-Dimensional Wave Problem

Consider the classical vibrating string equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in [0, L], t > 0 \tag{4}$$

With the initial conditions $u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = g(x)$, and boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{5}$$

Mesh Discretization: A uniform mesh with N finite elements was tested using linear and quadratic shape functions.

Time Approximation: Both an explicit (central difference) scheme and the implicit Newmark scheme were employed.

Quadratic shape functions exhibited higher accuracy, especially in representing high-frequency modes of the vibrating string. The implicit Newmark scheme remained stable for relatively larger time-step sizes, whereas the explicit scheme necessitated a smaller Δt due to the Courant–Friedrichs–Lewy constraint [13].

3.2 Two-Dimensional Membrane Vibration

We next examine wave propagation in a rectangular membrane $\Omega = [0, L_x] \times [0, L_y]$ governed by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{6}$$

The initial perturbation was localized around the center (a pulse-shaped disturbance), with either reflective or absorbing (Perfectly Matched Layer) boundary conditions. A high-accuracy spectral solution was used as a reference [3,4].

Accuracy: Using linear elements led to noticeable numerical dispersion around the wavefront, while higher-order elements significantly reduced dispersion errors.

Adaptation: Implementing adaptive refinement around the wavefront improved accuracy without excessively increasing the total number of elements, highlighting the benefits of adaptive strategies.

These results confirm that the proposed finite element method algorithm, combined with a posteriori error estimation and mesh refinement, yields accurate and efficient solutions for two-dimensional wave problems.

3.3 Additional analyses and applications

While the primary focus has been on linear wave equations and classical boundary conditions, further analyses reveal multiple avenues for advancing and applying these methods in broader contexts.

Extension to Nonlinear Hyperbolic Equations

Real-world wave phenomena often involve nonlinear effects, particularly in materials with strain-rate-dependent properties or in systems exhibiting large deformations. Nonlinear hyperbolic equations may arise from:

Nonlinear Material Constitutive Laws: For instance, stress-strain relationships in hyperelastic or viscoelastic materials lead to additional nonlinear terms in the governing partial differential equations.

Geometrical Nonlinearity: In plates, shells, or membranes undergoing large displacements, mid-surface stretching can introduce nonlinear curvature terms.

Fluid-Structure Interaction: Coupling between fluid flow and structural deformation can lead to hyperbolic systems that merge acoustic or aerodynamic wave propagation with elastic or even plastic deformation.

In such cases, implicit time-integration schemes often become more attractive due to their robustness. However, each time step involves iteratively solving nonlinear equations (e.g., via Newton–Raphson iterations). This increases computational demands but may still be more efficient than overly restrictive Courant–Friedrichs–Lewy -based time-step constraints in explicit methods.

Multi-Physics and Complex Boundary Conditions

Wave phenomena frequently intersect with other physical effects, such as thermal conduction (parabolic partial differential equations) or electromagnetic fields (also hyperbolic partial differential equations in Maxwell's equations). Coupled multi-physics problems can be approached with a partitioned or monolithic strategy:

Partitioned Approach: Solve each physical field separately and exchange interface boundary conditions or source terms at each time step.

Monolithic Approach: Combine all partial differential equations into a single larger system and solve simultaneously.

Complex boundary conditions—such as frequency-dependent absorption or frequency-selective interfaces—also demand careful numerical treatment. Special boundary or interface elements can be developed to handle, for example, wave reflection and transmission in layered media, ensuring minimal numerical reflection at artificial boundaries.

High-Performance Computing Considerations

As problem size increases (e.g., three-dimensional domains with millions of elements), parallelization becomes essential. Domain decomposition techniques, such as Schwarz methods or Balancing Domain Decomposition by Constraints, can be implemented to distribute the computational load across multiple processors or nodes. Key considerations include:

Load Balancing: Ensuring that each processor receives roughly the same amount of computational work.

Scalable Solvers: Using iterative solvers (e.g., GMRES, CG) with appropriate preconditioners that scale well with problem size.

GPU Acceleration: For explicit time stepping, certain linear-algebra operations can be offloaded to GPUs, which are particularly effective at vectorized operations.

Error Estimation and Adaptive Strategies

Adaptive mesh refinement remains a central topic for hyperbolic problems. Unlike elliptic problems, where errors usually spread out more smoothly, wave and shock features create localized, traveling regions of high error. Thus, dynamic refinement and coarsening are needed:

Dynamic Load Balancing: In parallel environments, when mesh elements are frequently refined or coarsened, the computational load changes. An efficient dynamic load-balancing strategy ensures minimal inter-processor data transfer.

Goal-Oriented Adaptivity: Sometimes, the aim is not to achieve a uniform global error target but rather to accurately capture specific quantities of interest (e.g., maximum displacement at a boundary, stress intensity near an interface). Goal-oriented methods drive refinement in regions that most affect these target quantities.

Industrial and Research Software Integration

Implementing finite element method -based hyperbolic solvers in large-scale, commercial software (e.g., ANSYS, ABAQUS) or open-source platforms (e.g., Deal.II, FEniCS) involves:

Modular Solver Design: A separate hyperbolic partial differential equation module that can interface with existing mesh-handling, linear-algebra, and post-processing libraries.

Verification and Validation (V&V): Benchmarking against analytical solutions and experimental data where available.

User-Defined Subroutines: Enabling end-users to introduce custom material laws, boundary conditions, and source terms.

Such integration broadens the impact of advanced finite element method techniques, making them readily available for engineers and researchers in various fields.

Potential for Machine Learning Integration.

Recent advances in machine learning offer avenues to speed up or enhance hyperbolic partial differential equation solvers:

Parameter Inference: Neural networks can approximate constitutive parameters or wave speeds from sparse experimental data.

Surrogate Modeling: Reduced-order models built with machine learning can provide near-real-time solutions for complex wave propagation scenarios.

Adaptive Time Stepping: Learning-based algorithms can predict where and when the solution requires finer resolution in time, potentially optimizing the overall computational cost.

Although such methods are still in the early stages of research for hyperbolic problems, they hold promise for future high-fidelity simulations and real-time decision-making.

4. DISCUSSION

Numerical experiments highlight that certain key choices determine the effectiveness of the finite element method in solving hyperbolic problems. The order of approximation plays a crucial role, with high-order elements being advantageous for smooth wave solutions such as harmonic vibrations. In cases involving discontinuities like shock fronts, a promising strategy combines high-order elements in smooth regions with mesh refinement in areas of steep gradients. Stability considerations are also important; while explicit schemes are simpler, they require small time steps to maintain stability, whereas implicit schemes can handle larger time steps but demand solving large linear systems at each step. Therefore, the choice of scheme depends on the problem's specific requirements, grid size, and available computational resources. Adaptive methods that involve a posteriori error estimation coupled with local mesh refinement significantly improve both accuracy and computational efficiency, especially for wave fronts and other localized phenomena. For large-scale problems, parallelization using algorithms such as MPI and Open MP is essential to effectively distribute elements and nodes across multiple computing cores. The design of data structures and domain decomposition strategies greatly influences performance and scalability. Looking ahead, extending these methods to nonlinear wave problems, multi-physics systems, and complex geometries remains an active area of research. Additionally, deep learning approaches can be integrated for parameter tuning and to reduce computational times in large-scale simulations.

Thus, a comprehensive approach that integrates variational formulation, appropriate element choice, time-integration scheme, and adaptive mesh refinement is validated as an effective solution strategy for wide-ranging wave propagation problems.

5. CONCLUSION

This article presented the results of using the finite element method to solve hyperbolic equations that typically arise in wave-type phenomena. We demonstrated that careful selection of the variational formulation, high-order shape functions, time-integration scheme, and adaptive refinement strategies significantly benefits the accuracy and efficiency of the solutions. Numerical tests in one and two dimensions confirmed the method's ability to accurately capture wave fronts, reduce errors, and maintain computational feasibility.

The additional analyses discussed in Section 4 underscore the wide applicability of finite element method -based hyperbolic solvers to complex, real-world problems, including nonlinear partial differential equations, multi-physics systems, and challenges posed by large-scale parallel computing. Integrating advanced error estimators and machine learning approaches further enhances the potential of these methods to address both classical and emerging wave propagation scenarios. Consequently, the proposed approach can be extended to larger-scale, possibly nonlinear problems involving complex domains and diverse physical properties, offering a robust foundation for future research and industrial applications.

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