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MUNDARIJA *** СОДЕРЖАНИЕ *** CONTENTS

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SOLVABILITY OF AN INTEGRO DIFFERENTIAL HEAT EQUATION WITH NONLOCAL INITIAL–BOUNDARY CONDITION

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Abstract. *This paper studied a nonlocal initial-boundary problem for an integro – differential heat equation on the rectangular domain. First, we introduce a definition of a classical solution and then the original problem is reduced to an equivalent problem. After that, we investigate spectral problem for equivalent problem. Using the Fourier method, the equivalent problem is reduced with the nolinear Volterr – Fredgholm equivalent integral equation. Finally, the local existence and uniqueness result is proven.*

Keywords: *integro-differential equation, integral inequality, nonlocal initial and nonlocal boundary problem, integral equation.*

РАЗРЕШИМОСТЬ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО ТЕПЛОВОГО УРАВНЕНИЯ С НЕЛОКАЛЬНЫМ НАЧАЛЬНО-ГРАНИЧНЫМ УСЛОВИЕМ

Аннотация. *В данной статье изучалась нелокальная начально-краевая задача для интегро-дифференциального уравнения теплопроводности в прямоугольной области. Сначала мы вводим определение классического решения, а затем исходную задачу сводим к эквивалентной. После этого исследуем спектральную задачу для эквивалентной задачи. С помощью метода Фурье эквивалентная задача сводится к нелинейному эквивалентному интегральному уравнению Вольтерра–Фредгольма. Наконец, доказан локальный результат существования и единственности.*

Ключевые слова: *интегро-дифференциальное уравнение, интегральное неравенство, нелокальная начальная и нелокальная краевая задача, интегральное уравнение.*

NOLOKAL TARZIDAGI BOSHLANG’ICH - CHEGARA SHARTI BILAN INTEGRO DIFFERENSIAL ISSIQLIK TENGLAMASINING YECHISH QOBILIYATI

Annotatsiya. *Ushbu maqolada to'rtburchak sohada integr-differentsial issiqlik tenglamasi uchun nolokal boshlang'ich-chegaraviy masala o'rganilgan. Birinchidan, biz klassik yechimning ta'rifini kiritamiz va keyin qo’yilgan masalaga ekvivalent masalaga kelamiz. Shundan so'ng, biz ekvivalent masala uchun spektral masalani tekshiramiz. Furye usulidan foydalanib, ekvivalent masala nochiziqli Volterr - Fredgolm ekvivalent integral tenglamaga olib kelinadi. Nihoyat, masala yechimining mavjudligi va yagonaligi isbotlanadi.*

Kalit so'zlar: *integro-differensial tenglama, nolokal boshlang'ich va nolokal chegaraviy masala, integral tenglama, integral tengsizlik.*

Introduction and setting up the problem. During the past decades, the topic of integro-differential equations which are combination of differential and integral has attracted many scientists and researchers due to their applications in many areas; see, for example, [1-6]. The study of this type of problem is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics and life sciences can be modelled in this way. Specific problems occur in thermodynamics in thermoelasticity [7] [8] [9], heat transfer [10] [11] [12] and plasma physics [13]. The above-mentioned articles focus on the problems described in terms of parabolic equations. However, there are some problems dealing with the dynamics of the ground waters which are described in terms of hyperbolic equations [14].

Problems with nonlocal conditions for partial differential equations have been studied by many authors. Various statements of inverse problems ondetermination of thermal coefficient in one dimensional heat equation were studied in [15,16,17]. It is important tonote that in the papers [15,16] the time-dependent thermal coefficient is determined from nonlocal overdetermination condition data. Besides, in [19-21] the kernels of the integro-differential heat equations are determined in the case of nonlocal boundary conditions.

The paper is organized as follows. In Section 2, the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties are introduced by applying the Fourier method to the problem (1.1) - (1.3). In Section 3, the existence and uniqueness of the solutionof the auxiliary problem (1.1) - (1.3) is proved.

Let $T > 0, l > 0$ be fixed numbers and $D_{Tl} = \{(x,t): 0 < x < l, 0 < t \leq T\}$. Consider the problem of determining of function $u(x,t)$ such that they satisfy the equation

$$u_t - u_{xx} = \int_0^t k\,(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in D_{Tl}, \tag{1.1}$$

the nonlocal initial condition

$$u(x,0) + \delta u(x,T) = \varphi(x), \;\; x \in [0,l], \tag{1.2}$$

the boundary conditions

$$u_x(0,t) - hu(0,t) = \psi_1(t), \;\; u_x(l,t) + h\mathrm{u}(l,t) = \psi_2(t), \;\; t \in [0,T], \tag{1.3}$$

$$\left(\varphi'(0) - h\varphi(0) = \psi_1(0) + \delta\psi_1(T), \;\; \varphi'(l) + h\varphi(l) = \psi_2(0) + \delta\psi_2(T)\right) \tag{1.4}$$

here $\delta \geq 0, h > 0$ is a given number, $\varphi(x), \omega(x), k(t), \psi_1(t), \psi_2(t)$ are given functions of $x \in [0,l]$ and $t \in [0,T]$.

Definition. The function $u(x,t)$ is said to be a classical solution. Of problem (1.1) – (1.3), if the function $u(x,t) \in C^{2,1}(D_{Tl}) \cap C^{1,0}(\overline{D}_{Tl})$ satisfy the following conditions.

1) The function $u(x,t)$ and its derivatives u_t, u_x, u_{xx} a continuous in the domain D_{Tl}.
2) Equation (1.1) and conditions (1.2) – (1.3) are satisfied in the classical sense.

Let $C^m(0,l)$ be the class of m times continuously differentiable with all derivatives up to the m −th order (inclusive) in $(0,l)$ functions. In the case $m = 0$ this space coincides with the class of continuous functions. $C^{m,j}(D_{Tl})$ is the class of m times continuously differentiable with respect to x and j times continuously differentiable with respect to t all derivatives in the domain D_{Tl} functions.

2. Auxilary problem and spectral problem

If you require that the function

$$\nu(x,t) = \left((\alpha_1 x + \beta_1)\psi_1(t) + (\alpha_2 x + \beta_2)\psi_2(t)\right), \quad 0 < x < l, \;\; 0 < t < T,$$

satisfied the boundary conditions (1.3) of the boundary value problem (1.1), (1.2), (1.3), then the coefficients $\alpha_1, \beta_1, \alpha_2, \beta_2$ are determined uniquely:

$$\alpha_1 = \frac{1}{2+hl}, \; \beta_1 = -\frac{1+hl}{(2+hl)h}, \; \alpha_2 = \frac{1}{2+hl}, \; \beta_2 = \frac{1}{h(2+hl)}.$$

The solution to the boundary value problem (1.1), (1.2), (1.3) can be sought in the form

$$u(x,t) = \vartheta(x,t) + \nu(x,t), \quad 0 < x < l, \;\; 0 < t < T$$

where $\vartheta(x,t)$ is the new unknown function, and $\nu(x,t)$ has been defined. For function $\vartheta(x,t)$, we get the following problem

$$\vartheta_t = \vartheta_{xx} + \int_0^t k\,(t-\tau)\left(\vartheta(x,\tau) + \nu(x,\tau)\right)\mathrm{d}\tau - \nu_t(x,t) \quad 0 < x < l, \;\; 0 < t \leq T\,, \tag{2.1}$$

homogeneous boundary candition

$$\vartheta_x(0,t) - h\vartheta(0,t) = 0, \;\; \vartheta_x(l,t) + h\vartheta(l,t) = 0, \tag{2.2}$$

intial condition

$$\vartheta(x,0) + \delta\vartheta(x,T) = \varphi^*(x), \quad 0 \leq x \leq l, \tag{2.3}$$

where

$$\varphi^*(x) = \varphi(x) - [\nu(x,0) + \delta\nu(x,T)]. \tag{2.4}$$

We shall seek the $\vartheta(x,t)$ of the problem $(2.1 - 2.3)$ in the form

$$\vartheta(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t).$$

We arrive at the following Sturm-Liouville problem

$$X''(x) + \lambda^2 X(x) = 0, \quad 0 < x < l \tag{2.5}$$

$$X'(0) - hX(0) = 0, \tag{2.6}$$

$$X'(l) + hX(l) = 0. \tag{2.7}$$

From the general solution of equation (2.5)

$$X_n(x) = C_1 cos\lambda x + C_2 sin\lambda x,$$

satisfying the boundary condition (2.6), we obtain:

$$X(x,\lambda) = C_2\{\frac{h}{\lambda}cos\lambda x + sin\lambda x\} = C_2\overline{X}(x,\lambda). \tag{2.8}$$

Substituting (2.8) into the boundary condition (2.7) we obtain:

$$\{\frac{\partial\overline{X}(x,\lambda)}{\partial x} + h\overline{X}(x,\lambda)\}_{x=l} = C_2\{\frac{\partial\overline{X}(x,\lambda)}{\partial x} + h\overline{X}(x,\lambda)\}_{x=l} = 0,$$

since $C_2 \neq 0$, otherwise (2.8) would be a trivial solution, then

$$\{\frac{\partial\overline{X}(x,\lambda)}{\partial x} + h\overline{X}(x,\lambda)\}_{x=l} = 0 \tag{2.9}$$

After substituting an explicit expression

$$X(x,\lambda) = \frac{\lambda}{h}cos\lambda x + sin\lambda x$$

(2.9) is transformed into equation (2.8)

$$ctg\lambda l = \frac{1}{2}\left(\frac{\lambda}{h} - \frac{h}{\lambda}\right)$$

λ_n - are the roots of the equation and $X_n(x) = sin(\lambda_n x + \theta_n)$ - eigenfunctions of this boundary value problem

$$\theta_n = arctg\frac{\lambda_n}{h},$$

where $X_n(x)$ are the eigenfunctions of the boundary value problem. This equation can be approximately solved graphically [21].

The squared norm of the eigenfunction $X_n(x)$ is equal to

$$\| X_n(x) \|^2 = \int_0^l sin^2(\lambda_n x + \theta_n)dx = \frac{(\lambda_n^2 + h^2)l + 2h}{2(h^2 + \lambda_n^2)}.$$

3. Solvability of the auxiliary problem

The solution of equation (2.1) with the nonlocal initial condition (2.2) and the boundary conditions (2.3) satisfies the relation

$$\begin{aligned}\vartheta(x,t) = \Phi(x,t) - \int_0^T \int_0^l G_0\,(x,\xi,T+t-\tau)\int_0^\tau k\,(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha d\xi d\tau + \\ + \int_0^t \int_0^l G\,(x,\xi,t-\tau)\int_0^\tau k\,(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha d\xi d\tau,\end{aligned} \tag{3.1}$$

where

$$\Phi(x,t) = \int_0^l G_0\,(x,\xi,t)\varphi^*(\xi)d\xi - \int_0^T \int_0^l G_0\,(x,\xi,T+t-\tau)\int_0^\tau k\,(\alpha)\nu(\xi,\tau-\alpha)d\alpha d\xi d\tau -$$

$$-\int_0^t \int_0^l G\,(x,\xi,t-\tau)\int_0^\tau k\,(\alpha)\nu(\xi,\tau-\alpha)d\alpha d\xi d\tau + \int_0^T \int_0^l G_0\,(x,\xi,T+t-\tau)\nu_t(\xi,\tau)d\xi d\tau +$$

$$+\int_0^t \int_0^l G\,(x,\xi,t-\tau)\nu_t(\xi,\tau)d\xi d\tau,$$

$$G(x,\xi,t,\tau) = \sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)}{\| X_n \|}e^{-\lambda^2(t-\tau)},$$

$$G_0(x,\xi,T+t-\tau) = \sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)}{\| X_n \|}\frac{1}{1+\delta e^{-\lambda_n^2(T)}}e^{-\lambda_n^2(T+t-\tau)},$$

where $G(x,\xi,t,\tau)$ is the Green function of the initial-boundary problem for one-dimensional heat equation.

Remark 1. The integral of the Green function does not exceed C:

$$\int_0^l G_0\,(x,\xi,t)d\xi \leq \int_0^l G\,(x,\xi,t)d\xi \leq C, x \in [0,l], t \in [0,T], C = const.$$

Theorem 1. Let $\omega(t) \in C[0,T]$ and $k \geq 0$ be a real constant. Let $a(t,s), c(t,s) \in C[0,T]$; $a(t,s)$ are nondecreasing in t for each $s \in I$ and

$$\omega \le k + \int_0^t a\,(t,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds + \int_0^T a\,(t,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds \tag{3.2}$$

for $t \in I$. If

$$q(t) = \int_0^t a\,(t,s) \int_0^s c\,(s,\sigma) exp\left(\int_0^\sigma E\,(\sigma,\xi)d\xi\right) d\sigma ds < 1, \tag{3.3}$$

for $t \in I$, where

$$E(t,\xi) = a(t,\xi) \int_0^\xi c\,(\xi,\sigma)d\sigma, \tag{3.4}$$

for $(t,\xi) \in D$, then

$$\omega(t) \le \frac{k}{1-q(t)} exp\left(\int_0^t E\,(t,\xi)d\xi\right), \tag{3.5}$$

for $t \in I$.

Proof Let $k > 0$ and fix any $T \in I$, then for $0 \le t \le T$, from (3.2) we have

$$\omega \le k + \int_0^t a\,(T,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds + \int_0^T a\,(T,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds. \tag{3.6}$$

Define a function $z(t,T), t \in [\alpha,T]$ by the right hand side of (3.6). Then for $\omega(t) \le z(t,T); z(t,T) > 0, 0 < t \le T$;

$$z(t,T) = k + \int_0^t a\,(T,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds + \int_0^T a\,(T,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds \tag{3.7}$$

$$z(0,T) = k + \int_0^T a\,(T,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds, \tag{3.8}$$

and

$$z'(t,T) = a(T,t) \int_0^t c\,(t,\sigma)d\sigma \le a(T,t) \int_0^t c\,(t,\sigma)z(\sigma,T)d\sigma. \tag{3.9}$$

From (3.9) and using the fact that $z(t,T)$ is nondecreasing in t, it is easy to observe that

$$\frac{z'(t,T)}{z(t,T)} \le a(t,T) \int_0^t c\,(t,\sigma)d\sigma\,, \tag{3.10}$$

for $t \in [0,T]$. By setting $t = \xi$ in (3.10) and integrating it with respect to ξ from 0 to T we get

$$z(T,T) \le z(0,T) exp\left(\int_0^T a\,(T,\xi) \int_0^\xi c\,(\xi,\sigma)d\sigma d\xi\right). \tag{3.11}$$

Since T is arbitrary, from (3.11), (3.8) with T replaced by t we have for $t \in I$,

$$z(t,t) \le z(0,t) exp\left(\int_0^t E\,(t,\xi)d\xi\right), \tag{3.12}$$

$$z(0,t) = k + \int_0^t a\,(t,s) \int_0^s c\,(s,\sigma)\omega(\sigma)d\sigma ds. \tag{3.13}$$

Using (3.12) in $\omega(t) \le z(t)$ we get

$$\omega(t) \le z(0,t) exp\left(\int_0^t E\,(t,\xi)d\xi\right), \tag{3.14}$$

for $t \in I$. Using (3.14) on the right hand side of (3.13) and in view of (3.3), it is easy to observe that

$$z(0,t) \le \frac{k}{1 - \int_0^t a\,(t,s) \int_0^s c\,(s,\sigma) exp\left(\int_0^\sigma E\,(\sigma,\xi)d\xi d\sigma ds\right)}. \tag{3.15}$$

□

Theorem 2. *Let* $(t) \in C[0,T], \varphi^*(x) \in C[0,l], \left(\psi_1(t),\psi_2(t)\right) \in C^1[0,T]$, $\left(\varphi'(0) - h\varphi(0) = \psi_1(0) + \delta\psi_1(T),\ \varphi'(l) + h\varphi(l) = \psi_2(0) + \delta\psi_2(T)\right)$, $\delta \ge 0$, *are satisfied. Then there exists a number* $T^* \in (0,T)$ *such that there exists a unique solution* $\vartheta(x,t) \in C^{2,1}(D_{Tl}) \cap C^{1,0}(\overline{D}_{Tl})$ *of the integral equation (2.11), where* T^* *is satisfying following inequality*

$$C \parallel k \parallel T^2 e^{\parallel k \parallel T} < 1, \qquad \parallel k \parallel = \max_{t\in[0,T]} |k(t)|.$$

Proof. Denote by $\overline{\vartheta}(t)$ the maximum of the modulus of the function $u(x,t)$ over $x \in [0,l]$ for each fixed $t \in [0,T]$.

$$\overline{\vartheta}(t) := \max_{x\in[0,l]} |\vartheta(x,t)|.$$

Then we have the inequaliy

$$\overline{\vartheta}(t) \le \Phi_0 + \int_0^t C\int_0^\tau |k(\alpha)| \left|\overline{\vartheta}(\tau-\alpha)\right| d\alpha d\tau + \int_0^T C\int_0^\tau |k(\alpha)| \left|\overline{\vartheta}(\tau-\alpha)\right| d\alpha d\tau.$$

Hence, by the virtue of Theorem 2, we obtain the estimate for all T satisfying the inequality $C\ k \parallel T^2 e^{\|k\|T} < 1$:

$$|\vartheta| \le \frac{\Phi_0}{e^{-\|k\|T} - C \parallel k \parallel T^2}\ .$$

Let problem (1.1 – 1.4) have two solutions $\vartheta_1, \vartheta_2, \vartheta_1 \ne \vartheta_2$. We denote their differences by $W = \vartheta_1 - \vartheta_2$. We rewrite $(2.1) - (2.3)$ problem for W

$$W_t = W_{xx} + \int_0^t k\,(t-\tau)W(x,\tau)d\tau \quad 0 < x < l,\ \ 0 < t < T, \tag{3.16}$$

$$W_x(0,t) - hW(0,t) = 0,\ \ W_x(l,t) + hW(l,t) = 0,\ \ 0 < t < T, \tag{3.17}$$

$$W(x,0) + \delta W(x,T) = 0 \quad 0 < x < l. \tag{3.18}$$

Applying the formula (3.1) to problem $(3.16) - (3.18)$, we obtain the integral equation for determining W:

$$\begin{aligned} W(x,t) = &-\int_0^T\int_0^l G_0\,(x,\xi,T+t-\tau)\int_0^\tau k\,(\alpha)W(\xi,\tau-\alpha)d\alpha d\xi d\tau + \\ &+\int_0^t\int_0^l G\,(x,\xi,t-\tau)\int_0^\tau k\,(\alpha)W(\xi,\tau-\alpha)d\alpha d\xi d\tau. \end{aligned} \tag{3.19}$$

Denote by $\overline{W}(t)$ the maximum of the modulus of the function $W(x,t)$ over $x \in [0,l]$ for each fixed $t \in [0,T]$.

$$\overline{W}(t) := \max_{x\in[0,l]} |W(x,t)|.$$

Then we have the inequality

$$|\overline{W}(t)| \le \int_0^t C\int_0^\tau |k(\alpha)| \left|\overline{W}(\tau-\alpha)\right| d\alpha d\tau + \int_0^T C\int_0^\tau |k(\alpha)| \left|\overline{W}(\tau-\alpha)\right| d\alpha d\tau.$$

Which, by Theorem 2, implies that $|\overline{W}(t)| \le 0$ for $t \in [0,T]$. Hence, we also have $W(x,t) = 0$, in D_{Tl}, i.e. $\vartheta_1 = \vartheta_2$. The proof of the theorem is complete.

□

Conclusion. In this work, it is studied a nonlocal initial-boundary problem for an integro – differential heat equation on the rectangular domain. First, we are introduced a definition of a classical solution and then the original problem is reduced to an equivalent problem. After that, we investigated spectral problem for equivalent problem. Using the Fourier method, the equivalent problem is reduced with the nolinear Volterr – Fredgholm equivalent integral equation. Finally, the local existence and uniqueness result is proven.

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