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Transient Processes during Nonstationary Flow of an Elastic-Viscous Liquid in a Flat Channel

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Abstract—The paper considers the solution to the problem of unsteady flow of an elastic-viscous fluid in a flat channel under the influence of a constant pressure gradient based on the generalized Maxwell model. By solving the problem, formulas for velocity distribution, fluid flow, and other hydrodynamic quantities are determined. Based on the formulas found, transient processes during unsteady flow of an elastic-viscous fluid in a flat channel are analyzed. Based on the results of the analysis, it is shown that transient processes under the influence of the Deborah number, which determines the elasticity properties of a fluid in an elastic-viscous flow, are fundamentally different from the transient process in a Newtonian fluid. At the same time, it is discovered that the process of transition of the characteristics of an elastic-viscous fluid from an unsteady state to a stationary state at small values of the Deborah number are almost not different from the processes of transition of a Newtonian fluid. When the Deborah number exceeds unity, it is established that the process of transition of an elastic-viscous fluid from an unsteady state to a stationary state is a wave-type change, in contrast to the transition process of a Newtonian fluid, and the transition time is several times longer than the transition time of a Newtonian fluid. It is also revealed that disturbances may occur during the transient process. This disturbance, occuring in the unsteady flow of an elastic-viscous fluid, will be stabilized by mixing a Newtonian fluid into it. That is, the instantaneous maximum increase in the velocity of the viscoelastic fluid as a result of an increase in the concentration of the Newtonian fluid is normalized. The implementation of this property is important in technical and technological processes, in preventing technical failures or malfunctions.

Keywords: viscoelastic fluid, unsteady flow, longitudinal velocity, fluid flow, steady flow

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INTRODUCTION

Studying the nonstationary flow of a viscoelastic fluid in a flat channel under action of a constant pressure gradient is of practical interest. In particular, we observe a nonstationary flow in engineering processes in the pipeline transport of a viscous and/or viscoelastic liquid at launching and stopping modes of working elements (mechanisms). In these cases the variations in the velocity, flow rate, and other hydrodynamic characteristics of the flow essentially differ from the characteristics of usual flows of a Newtonian liquid. However, such flows are widely used in various technological processes, in the chemical technology, in biomechanics, and in acoustics.

Works [1–6] are devoted to studying the nonstationary flows of Newtonian and non-Newtonian liquids, in particular, elastoviscous liquids in pipes and channels. For the first time, the nonstationary flow of a viscous incompressible liquid in a cylindrical pipe was investigated in works by Gromeka [7, 8]. In these works he determined the variations in the velocity, flow rate, and tangent shear stress at the wall. Using these formulas, we can determine the time of settling the hydrodynamic quantities in the flow of a viscous liquid in a cylindrical pipe.

Nonstationary pulsating flows of viscous and elastoviscous liquids in a circular cylindrical pipe of infinite length under the action of a harmonic variable pressure gradient were studied in work [9]. The solution to the problem leads to the calculation formulas for determining the velocity and flow rate of liquid. Numerical calculations demonstrated that in a pulsating flow at lower values of the dimensionless frequency of vibrations, the velocity, flow rate, and other hydrodynamic parameters are settled slowly from the initial zero state for comparatively high frequencies of vibrations and are close to the parameters of a nonpulsating flow. In an oscillating flow, with high values of the frequency of vibrations, these parameters are settled almost instantaneously.

In [10] Hassan et al. analyzed nonstationary viscoelastic flows of Oldroyd-B liquid through a pipe of circular cross-section. Liquid moves under the action of a time-dependent pressure gradient in the three cases listed below:

(1) The pressure gradient varies with time following an exponential law.

(2) The pressure gradient pulsates.

(3) The pressure gradient is constant.

The formulas for the distribution of the velocity, flow rate, and other hydrodynamic characteristics of the flow were obtained.

Using the Maxwell model, we examined the problem of nonstationary vibrating flow of a viscoelastic liquid in a flat channel [11, 12]. We obtained the formulas for determining the dynamic and frequency characteristics. Using numerical experiments, we investigated the effect of the frequency of vibrations and relaxation properties of liquid on the tangent shear stress at the wall. We demonstrated that the viscoelastic properties of liquid, as well as its acceleration are limiting factors for using the quasi-stationary approach. In works [13, 14] Shul'man and Khusid solved the problems of nonstationary flows of a viscoelastic liquid in long pipes of annular cross-section. They considered the rheological characteristics of liquid to be independent of the strain rate, this condition transforms the problem to the linear ones, and they found its analytic solution by means of the Laplace transform. In particular, the calculations were given for a liquid with constant properties corresponding to a Newtonian liquid, where the development of velocity profiles has a diffusion character. The velocity and tangent stresses grow monotonically to their stationary values. The elasticity of liquid provides the development of its flow with the wave-like character.

Laminary vibrating flows of viscoelastic Maxwell and Oldroyd-B liquids were studied in work [15], where Casanellas and Ortin demonstrated many interesting peculiarities absent in the flows of Newtonian liquids.

In work [16] Ding and Jian investigated the electrokinetic flow of viscoelastic liquids in a flat channe under the action of a vibrating pressure gradient. The assumed that the motion of liquid occurs laminarly and unidirectionally, which is why the liquid moves in the linear mode. They assume the surface potentials small; therefore, the Poisson–Boltzmann equation can be linearized. The resonant behavior occurs in the flow when the elastic properties of the Maxwell liquid prevail. The resonant phenomenon is strengthened by the electrokinetic effects, and, in addition to that, the efficiency of transforming the electrokinetic energy is facilitated.

In work [17] Kuz'min considered a mathematical model of the motion of a nonlinearly viscous liquid with a slip boundary condition, where the mathematical model of non-Newtonian liquids is subjected to the principle of behavior objectivity of the considered materials. In addition to that, the considered problem with slip boundary conditions is essentially different from the classical no-slip conditions.

In the above mentioned works, researchers mainly study the velocity field of liquid at different modes of variation in the pressure gradient. The variation in the maximum axial velocity, flow rate, and tangent shear stress at the wall arising at the motion of a nonstationary flow was studied relatively poorly.

In the current work we investigate the nonstationary flow of a viscoelastic liquid on a generalized Maxwell model in a flat channel under the action of a constant pressure gradient. We determine the calculation formulas for the distribution of the axial velocity, flow rate, and tangent shear stress at the wall. We analyze the transient processes in the launching flow mode of a viscoelastic liquid.

1. PROBLEM FORMULATION AND SOLUTION METHOD

Let us consider the problems of the nonstationary flow of a viscoelastic incompressible liquid between two fixed parallel planes stretching both ways to infinity. We denote the distance between the walls by $2h$. The Ox axis is directed horizontally at the middle of the channel along the flow. The Oy axis is directed perpendicularly to the Ox axis. The flow of a viscoelastic liquid occurs symmetrically with respect to the channel axis. The differential equation of motion of a viscoelastic incompressible liquid in stresses has the form [1, 6]

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}-\frac{\partial \tau}{\partial y}, \tag{1}$$

where u is the axial velocity, p is the pressure, ρ is the density, τ is the tangent stress, t is time. The rheological equations of state of liquid are taken in the form of the generalized Maxwell equation [15, 18]

$$\tau=\tau_s+\tau_p,\ \tau_s=-\eta_s\frac{\partial u}{\partial y},\ \ \lambda\frac{\partial \tau_p}{\partial_t}+\tau_p=-\eta_p\frac{\partial u}{\partial y}; \tag{2}$$

here, λ is the relaxation time, τ_s is the tangent stress of Newtonian liquid, τ_p is the tangent stress of Maxwell liquid, τ is the tangent stress of solution, η_s is the dynamic viscosity of Newtonian liquid, and η_p is the dynamic viscosity of Maxwell liquid. We represent the relation of the dynamic viscosities by the equalities [14, 19]

$$\eta=\eta_s+\eta_p,$$

where η is the dynamic viscosity of the solution. By substituting (2) in the equation of motion (1) for the velocity of liquid, we obtain

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}=-\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}+\eta_s\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}+\eta_p\frac{\partial^2 u}{\partial y^2}. \tag{3}$$

To solve Eq. (3), we need to formulate the initial and boundary conditions. We assume that up to time instance $t=0$ the liquid is at rest. Since the time instance $t=0$ the liquid moves due to a positive constant pressure gradient $\frac{\partial p}{\partial x}=\text{const}$.

In this case the initial condition and the no-slip boundary condition at the wall are given by

$$u=0 \quad \text{at } t=0,\ \ u=0 \quad \text{at } y=h. \tag{4}$$

The flow occurs symmetrically with respect to the channel axis; therefore,

$$\frac{\partial u}{\partial y}=0 \quad \text{at } y=0. \tag{5}$$

Performing the Laplace–Carson transform, that is, passing from the original to the image in Eq. (3) and in the boundary conditions (4) and (5), we get

$$\frac{d^2\overline{u}(y,s)}{dy^2}+\frac{i^2\rho s}{\eta\eta^*(s)}\overline{u}(y,s)=\frac{1}{\eta\eta^*(s)}\frac{d\overline{p}(x)}{dx}.$$

Here,

$$\eta^*(s)=\left(\frac{\eta_s}{\eta}+\frac{\eta_p}{\eta}\frac{1}{1+s\lambda}\right)=\left(X+Z\frac{1}{1+s\lambda}\right).$$

The fundamental solutions to the equation without the right-hand side are the cosine and sine functions $\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)$ and $\sin\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)$, and the particular solution to Eq. (3) with the right-hand side is a constant $-\frac{1}{\rho s}\frac{d\overline{p}(x)}{dx}$.

Thus, the general solution to Eq. (3) is given by

$$\overline{u(y,s)}=C\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)+D\sin\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)-\frac{1}{\rho s}\frac{d\overline{p}(x)}{dx}.$$

Using the boundary condition (5), we find the constant D that is zero, and, to determine the constant C, we use the boundary condition (4). As a result, for the image of the velocity we get

$$\overline{u}(y,s)=\frac{1}{\rho s}\left(-\frac{d\overline{p}(x)}{dx}\right)\left(1-\frac{\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)}{\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)}\right).$$

Using the formula for inversion of the Laplace–Carson transform, for the veloctiy of motion of liquid we obtain the following integral expression:

$$u(y,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{st}\frac{1}{\rho s}\left(-\frac{dp(x)}{dx}\right)\left(1-\frac{\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)}{\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)}\right)\frac{ds}{s}. \tag{6}$$

To compute integral (6) over the complex variable, we need to establish the residues under the integral sign. By equating the denominator to zero and taking into account that the roots of cosine are real values, we find

$$s=0 \quad \text{and} \quad s=-v\frac{s_{1,2,n}}{h^2};$$

here, $s_{1,2,n}$ are the solutions to the transcendent equation $\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)=0$. All poles are simple; therefore, we can use the decomposition of the meromorphic function into simple fractions in the form

$$\frac{F_1(s)}{F_2(s)}=\frac{1}{\rho s}\left(-\frac{dp}{dx}\right)\frac{\left(\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)-\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)\right)}{\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)}\frac{e^{st}}{s}=\frac{C_0}{s}+\sum_{i=1}^{2}\frac{C_{i,n}}{s-s_{i,n}}, \tag{7}$$

$$F_1(s)=\left(-\frac{dp}{dx}\right)\left(\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)-\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)\right)e^{st},\quad F_2(s)=\rho s^2\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right).$$

To determine the residue C_0, we must multiply both sides of Eq. (7) by s and then tending s to zero:

$$C_0=\lim_{s\to 0}\frac{sF_1(s)}{F_2(s)}=\lim_{s\to 0}\frac{s\left(-\frac{dp}{dx}\right)\left(\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)-\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}y\right)\right)}{\rho s^2\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right)}$$

$$=\frac{1}{2\eta}\left(-\frac{\partial p}{\partial x}\right)h^2\left(1-\frac{y^2}{h^2}\right);$$

here, $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$.

To determine the residue $C_{i,n}$, we need to multiply (7) by the difference $s-s_{in}$ and tend s to s_{in}, taking into account that

$$C_{in}=\lim_{s\to s_{in}}\frac{F_1(s)}{\frac{F_2(s)-F_2(s_{in})}{s-s_{in}}}=\frac{F_1(s_{in})}{F_2'(s_{in})}. \tag{8}$$

Now, we need to find the value s_{in}; for this purpose, we use the equation

$$\cos\left(i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h\right) = 0,$$

the solution to which is given by

$$i\sqrt{\frac{\rho s}{\eta\eta^*(s)}}h = \frac{2n+1}{2}\pi. \tag{9}$$

Here, taking into account that $\eta^*(s) = \left(X + Z\frac{1}{1+s\lambda}\right)$, we bring Eq. (9) to the form

$$\frac{\rho s}{\eta\eta^*(s)}h^2 = -\frac{(2n+1)^2}{4}\pi^2 \quad \text{or} \quad s = -\frac{(2n+1)^2}{4}\pi^2\frac{\nu}{h^2}\left(X + Z\frac{1}{1+s\lambda}\right). \tag{10}$$

Hence, we can easily bring Eq. (10) to the quadratic equation

$$\mathrm{De}\,\overline{s}^2 - \overline{s}(1 + X\,\mathrm{De}\,a_0^2) + a_0^2 = 0; \tag{11}$$

here, $a_0 = \frac{2n+1}{2}\pi, \mathrm{De} = \frac{\lambda\nu}{h^2}$. The quadratic equation (10) has two roots: real different roots, real equal ones, or complex conjugate ones. We will consider all these cases in the numerical calculation and discussions.

Now, taking into account $s = s_{in}$, we find

$$C_{in} = \lim_{s\to s_{in}}\frac{F_1(s)}{\dfrac{F_2(s) - F_2(s_{in})}{s - s_{in}}} = \frac{F_1(s_{in})}{F_2'(s_{in})},$$

where the numerator can be easily computed:

$$\begin{aligned} F_1(s_{in}) &= \left(-\frac{dp}{dx}\right)\left(\cos\left(i\sqrt{\frac{\rho s_{in}}{\eta\eta^*(s_{in})}}h\right) - \cos\left(i\sqrt{\frac{\rho s_{in}}{\eta\eta^*(s_{in})}}y\right)\right)e^{s_{in}t} \\ &= \left(-\frac{dp}{dx}\right)\left(-\cos\left(\frac{(2n+1)}{2}\pi\frac{y}{h}\right)\right)e^{s_{in}t}, \end{aligned} \tag{12}$$

$$\cos\left(i\sqrt{\frac{\rho s_{in}}{\eta\eta^*(s_{in})}}h\right) = 0.$$

The denominator is computed using the derivative with respect to s:

$$\begin{aligned} F_2'(s) &= 2\rho s\cos\left(ih\sqrt{\frac{\rho s}{\eta\eta^*(s)}}\right) - \rho s^2\sin\left(ih\sqrt{\frac{\rho s}{\eta\eta^*(s)}}\right)\left(ih\sqrt{\frac{\rho s}{\eta\eta^*(s)}}\right)_s' \\ &= -\rho s^2(-1)^n\left(ih\sqrt{\frac{\rho s}{\eta\eta^*(s)}}\right)_s' = -\rho s^2(-1)^n\frac{ih}{\sqrt{\nu}}\left(\sqrt{\frac{s}{\eta^*(s)}}\right)_s', \end{aligned} \tag{13}$$

where $\nu = \frac{\eta}{\rho}$. The derivative $\left(\sqrt{\frac{s}{\eta^*(s)}}\right)_s'$ from formula (13) is computed as follows:

$$\left(\sqrt{\frac{s}{\eta^*(s)}}\right)_s' = \frac{1}{2\left(\sqrt{\frac{s}{\eta^*(s)}}\right)}\left(\frac{s}{\eta^*(s)}\right)_s' = \frac{1}{2\left(\sqrt{\frac{s}{\eta^*(s)}}\right)}\left(\frac{\eta^*(s) - s(\eta^*(s))_s'}{\eta^{*2}(s)}\right).$$

Because $\frac{ih}{\sqrt{\nu}}\sqrt{\frac{s}{\eta^*(s)}} = \frac{2n+1}{2}\pi$, we have

$$\left(\sqrt{\frac{s}{\eta^*(s)}}\right)'_s = \frac{ih}{(2n+1)\pi\sqrt{\nu}}\left(\frac{\eta^*(s) - s\left(\eta^*(s)\right)'_s}{\eta^{*2}(s)}\right). \tag{14}$$

Taking into account that $\eta^*(s) = -\frac{4s}{(2n+1)^2\pi^2}\frac{h^2}{\nu}$, relation (14) transforms to

$$\begin{aligned}\left(\sqrt{\frac{s}{\eta^*(s)}}\right)'_s &= \frac{ih}{(2n+1)\pi\sqrt{\nu}}\left(\frac{\eta^*(s) - s\left(\eta^*(s)\right)'_s}{\eta^{*2}(s)}\right)\\ &= \frac{ih}{(2n+1)\pi\sqrt{\nu}}\frac{(2n+1)^4\pi^4}{16s^2}\frac{\nu^2}{h^4}\left(\eta^*(s) - s\left(\eta^*(s)\right)'_s\right).\end{aligned} \tag{15}$$

Having found derivative (15), we take into account (13) and find the formula for computing the denominator of (8):

$$\begin{aligned}F_2'(s) &= -\rho s^2(-1)^n\frac{ih}{\sqrt{\nu}}\left(\sqrt{\frac{s}{\eta^*(s)}}\right)'_s\\ &= -\rho s^2(-1)^n\frac{ih}{\sqrt{\nu}}\frac{ih}{(2n+1)\pi\sqrt{\nu}}\frac{(2n+1)^4\pi^4}{16s^2}\frac{\nu^2}{h^4}\left(\eta^*(s) - s\left((\eta^*(s)\right)'_s\right)\\ &= \rho(-1)^n\frac{(2n+1)^3\pi^3}{16}\frac{\nu}{h^2}\left(\eta^*(s) - s\left((\eta^*(s)\right)'_s\right).\end{aligned} \tag{16}$$

Subsituting instead of $\eta^*(s)$ its value

$$\eta^*(s) = \left(X + Z\frac{1}{1+s\lambda}\right), \quad \eta^{*\prime}(s) = \frac{-\lambda Z}{(1+s\lambda)^2},$$

$$\eta^*(s) - s\eta^{*\prime}(s) = \left(X + Z\frac{1}{1+s\lambda}\right) - \frac{-s\lambda Z}{(1+s\lambda)^2} = \frac{1 + 2\lambda s + \lambda^2 s^2 X}{(1+s\lambda)^2}$$

into (16) and taking into account that $s = -\frac{\nu}{h^2}\bar{s}$, $\mathrm{De} = \frac{\lambda\nu}{h^2}$, we get

$$\begin{aligned}F_2'(\bar{s}) &= \rho(-1)^n\frac{(2n+1)^3\pi^3}{16}\frac{\nu}{h^2}\left(\eta^*(s) - s\left((\eta^*(s)\right)'_s\right)\\ &= (-1)^n\frac{(2n+1)^3\pi^3}{16}\frac{\eta}{h^2}\frac{1 - 2\,\mathrm{De}\,\bar{s} + \mathrm{De}^2\,\bar{s}^2 X}{(1 - s\,\mathrm{De})^2}.\end{aligned} \tag{17}$$

Substituting the values $F_1(\bar{s}), F_2'(\bar{s})$ from (12), (17) into (8) and replacing $\bar{s}$ into $\bar{s}_{in}$, we obtain the final values for C_{in}:

$$C_{in} = \frac{F_1(s_{in})}{F_2'(s_{in})} = \sum_{n=1}^{\infty}\sum_{i=1}^{2}\frac{\left(-\frac{dp}{dx}\right)\left(-\cos\left(\frac{(2n+1)}{2}\pi\frac{y}{h}\right)e^{-\frac{\nu}{h^2}\bar{s}_{in}t}\right)}{(-1)^{n+1}\frac{(2n+1)^3\pi^3}{16}\frac{\eta}{h^2}\frac{1 - 2\,\mathrm{De}\,\bar{s}_{in} + \mathrm{De}^2\,\bar{s}_{in}^2 X}{(1 - s_{in}\,\mathrm{De})^2}}$$

$$= \frac{16h^2}{\eta}\left(-\frac{\partial p}{\partial x}\right)\sum_{n=1}^{\infty}\sum_{i=1}^{2}\frac{(-1)^n\cos\left(\frac{(2n+1)}{2}\pi\frac{y}{h}\right)e^{-\frac{\nu}{h^2}\bar{s}_{in}t}.}{(2n+1)^3\pi^3\frac{1 - 2\,\mathrm{De}\,\bar{s}_{in} + \mathrm{De}^2\,\bar{s}_{in}^2 X}{(1 - s_{in}\,\mathrm{De})^2}}.$$

By summing the values C_o and C_{in} and substituting them into (7), we find the solution to Eq. (3) as

$$u(y,t)=\frac{h^2}{2\eta}\left(-\frac{dp(x)}{dx}\right)\left[\left(1-\frac{y^2}{h^2}\right)+32\sum_{n=1}^{\infty}\sum_{i=1}^{2}\frac{(-1)^n\cos\left(\left(\frac{2n+1}{2}\right)\pi\frac{y}{h}\right)e^{-\frac{\nu}{h^2}\bar{s}_{in}t}}{(2n+1)^3\pi^3\frac{1-2\,\mathrm{De}\,\bar{s}_{in}+\mathrm{De}^2\,\bar{s}_{in}^2X}{(1-s_{in}\,\mathrm{De})^2}}\right], \tag{18}$$

$$\frac{u(0,t)}{u_0}=1+32\sum_{n=1}^{\infty}\sum_{i=1}^{2}\frac{(-1)^n e^{-\frac{\nu}{h^2}\bar{s}_{in}t}}{(2n+1)^3\pi^3\frac{1-2\,\mathrm{De}\,\bar{s}_{in}+\mathrm{De}^2\,\bar{s}_{in}^2X}{(1-\bar{s}_{in}\,\mathrm{De})^2}}; \tag{19}$$

here, $u_0=\frac{h^2}{2\eta}\left(-\frac{dp(x)}{dx}\right)$. Expression (19) points out that as t tends to infinity, the distribution of the veocity becomes parabolic:

$$u(y,t)=\frac{h^2}{2\eta}\left(-\frac{dp(x)}{dx}\right)\left(1-\frac{y^2}{h^2}\right).$$

Thus, the solution to the problem about the stationary motion of liquid between parallel walls is obtained from the solution to the problem about the nonstationary motion as t tends to infinity.

2. NUMERICAL CALCULATIONS AND DISCUSSION

Using formulas (18) and (19), we performed numerical calculations on settling the nonstationary processes of the hydrodynamic characteristics at nonstationary flow of a viscoelastic liquid. For convenience of comparison of settling of the Newtonian liquid and settling of the viscoelastic liquid, we first study the Newtonian liquid. The formulas for the calculation of Newtonian liquid are obtained from formulas (18) and (19) at $\lambda=0$, $\eta^*(s)=0$, $X=1$, $Z=0$; in this case the solution to the transcendent equation is given by

$$\cos\left(i\sqrt{\frac{\rho s}{\eta}}h\right)=0,\quad \left(i\sqrt{\frac{\rho s}{\eta}}h\right)=\frac{2n+1}{2}\pi,\quad s=-\frac{\nu}{h^2}\bar{s},\quad \bar{s}=\frac{(2n+1)^2}{4}\pi^2.$$

Using these expressions from formulas (18) and (19), we can find the calculation formulas for studying the Newtonian liquid

$$u(y,t)=\frac{h^2}{2\eta}\left(-\frac{dp(x)}{dx}\right)\left[\left(1-\frac{y^2}{h^2}\right)+32\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}\cos\left(\left(\frac{2n+1}{2}\right)\pi\frac{y}{h}\right)e^{-\frac{\nu}{h^2}\frac{(2n+1)^2}{4}\pi^2t}\right],$$

$$\frac{u(0,t)}{u_{0\max}}=1+32\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^3\pi^3}e^{-\frac{\nu}{h^2}\frac{(2n+1)^2}{4}\pi^2t}. \tag{20}$$

Based on formula (20), we carried out the numerical calculations showing the variation in the maximum velocity to the maximum velocity of the stationary flow depending on time (Fig. 1). We see that the relative maximum velocity at nonstationary flow of Newtonian liquid, depending on time, monotonously increases to a value corresponding to the stationary flow.

Information about the process of transition of a Newtonian liquid from the nonstationary flow mode to the stationary one is given in many literature sources [1, 13]. However, the processes of transition of a viscoelastic liquid from the nonstatoinary flow to the stationary one are understudied. Works [13, 14] are also devoted to nonstationary motions of a viscoelastic liquid in some annular pipes, the solution to which is based on the finite difference method. Below, we provide an analysis of the problem solved on the basis of the generalized two-liquid Maxwell model. For this purpose, we use formulas (18) and (19). One of the physical properties of the flow of a viscoelastic liquid is that at the initial time instance the velocity reaches its maximum value and then transits to the stage of monotone decrease and stationary flow. Based on for-

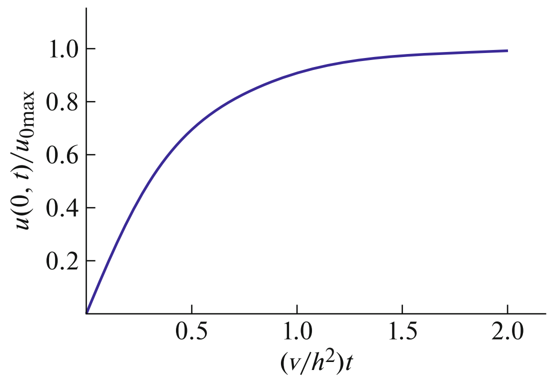

Fig. 1. Time variation in the ratio of the maximum velocity to the maximum velocity of a stationary profile at the nonstationary flow of the Newtonian liquid.

mulas (18) and (19), we analyze the process of transition from the nonstationary state to the stationary one in a flat channel of a viscoelastic liquid on the basis of the Maxwell model. In this case the solution to the quadratic Eq. (11) determines the roots $s_{1n,2n}$ from (18) and (19). It is well-known that the quadratic Eq. (11) has two roots: real different roots, real equal roots, or complex conjugate ones:

$$s_{1n} = \frac{1 + X\,\mathrm{De}\,a_0^2 + \sqrt{1 - 2\,\mathrm{De}\,a_0^2(X-2) + X^2\,\mathrm{De}^2\,a_0^4}}{2\,\mathrm{De}}, \tag{21}$$

$$s_{2n} = \frac{1 + X\,\mathrm{De}\,a_0^2 - \sqrt{1 - 2\,\mathrm{De}\,a_0^2(X-2) + X^2\,\mathrm{De}^2\,a_0^4}}{2\,\mathrm{De}}. \tag{22}$$

For the roots of a quadratic equation to be real, we need the following: the discriminant in formulas (21) and (12) $(1 - 2\,\mathrm{De}\,a_0^2(X-2) + X^2\,\mathrm{De}^2\,a_0^4)$ is zero or larger than zero. In this case the roots are real; therefore, the solutions to Eqs. (18) and (19) are represented as

$$u(y,t) = \frac{h^2}{2\eta}\left(-\frac{dp(x)}{dx}\right)\left[\left(1 - \frac{y^2}{h^2}\right) + 32\sum_{n=0}^{\infty}\sum_{i=1}^{2}\frac{(-1)^{n+1}\cos\left(\left(\frac{2n+1}{2}\right)\pi\frac{y}{h}\right)e^{-\frac{\nu}{h^2}\bar{s}_{in}t}}{(2n+1)^3\pi^3\,\dfrac{1 - 2\,\mathrm{De}\,\bar{s}_{in} + \mathrm{De}^2\,\bar{s}_{in}^2 X}{(1 - s_{in}\,\mathrm{De})^2}}\right., \tag{23}$$

$$\frac{u(0,t)}{u_0} = 1 + 32\sum_{n=0}^{\infty}\sum_{i=1}^{2}\frac{(-1)^{n+1}e^{-\frac{\nu}{h^2}\bar{s}_{in}t}}{(2n+1)^3\pi^3\,\dfrac{1 - 2\,\mathrm{De}\,\bar{s}_{in} + \mathrm{De}^2\,\bar{s}_{in}^2 X}{(1 - \bar{s}_{in}\,\mathrm{De})^2}}. \tag{24}$$

In this case the solutions (23) and (24) have the same form as the solutions (18) and (19). This case corresponds to the lower values of the Deborah number. Analysis of formulas (23) and (24) is given in Fig. 2.

In Fig. 2 we see that in this case the processes of nonstationarity in the flow of a viscoelastic liquid is almost not different from the processes of nonstationarity in a Newtonian liquid. In this case, instead of the nonstationarity of the flow of a viscoelastic liquid, we can accept the process of settling the nonstationary flow of a Newtonian liquid. Now, we consider the case when the roots of the quadratic Eq. (11) consist of complex conjugate roots. This case is observed when the discriminant of Eq. (11) is below zero. This case corresponds to large values of the Deborah number. The solution to the quadratic equation is determined as follows:

$$s_{1n} = \frac{1 + X\,\mathrm{De}\,a_0^2 + i\sqrt{2\,\mathrm{De}\,a_0^2(2-X) - X^2\,\mathrm{De}^2\,a_0^4 - 1}}{2\,\mathrm{De}} = a + bi,$$

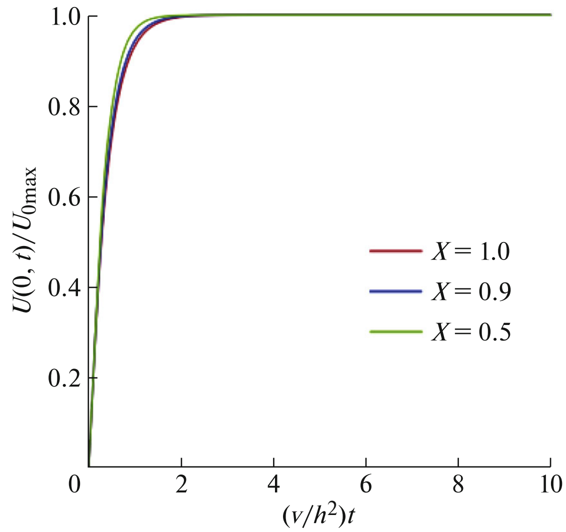

Fig. 2. Time variation in the ratio of the maximum velocity to the maximum velocity of a stationary profile at the nonstationary flow of the viscoelastic liquid (when the Deborah number is De = 0.1, and at different values of concentration of the Newtonian liquid).

$$s_{2n} = \frac{1 + X\,\mathrm{De}\,a_0^2 - i\sqrt{2\,\mathrm{De}\,a_0^2(2-X) - X^2\,\mathrm{De}^2\,a_0^4 - 1}}{2\,\mathrm{De}} = a - bi;$$

here,

$$a = \frac{1 - X\,\mathrm{De}\,a_0^2}{2\,\mathrm{De}},\quad b = \frac{\sqrt{2\,\mathrm{De}\,a_0^2(2-X) - X^2\,\mathrm{De}^2\,a_0^4 - 1}}{2\,\mathrm{De}},\quad a_0 = \frac{2n+1}{2}\pi,$$

$$\mathrm{De} = \frac{\lambda\nu}{h^2},\quad s_{1n} = a + bi,\quad s_{2n} = a - bi.$$

In this case the solutions (18) and (19) are given by

$$u(y,t) = \frac{h^2}{2\eta}\left(-\frac{dp(x)}{dx}\right)\left[\left(1 - \frac{y^2}{h^2}\right)\right.$$

$$\left. + 32\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\cos\left(\left(\frac{2n+1}{2}\right)\pi\frac{y}{h}\right)2e^{-\frac{\nu}{h^2}a_n t}}{(2n+1)^3\pi^3(M_2^2 + N_2^2)}\left(M_3\cos b_n\frac{\nu}{h^2}t + N_3\sin b_n\frac{\nu}{h^2}t\right)\right],$$

$$\frac{u(0,t)}{u_0} = 1 + 32\sum_{n=0}^{\infty}\frac{2(-1)^{n+1}e^{-\frac{\nu}{h^2}a_n t}}{(2n+1)^3\pi^3(M_2^2 + N_2^2)}\left(M_3\cos b_n\frac{\nu}{h^2}t + N_3\sin b_n\frac{\nu}{h^2}t\right); \tag{25}$$

here,

$$M_1 = 1 - 2a_n\,\mathrm{De} + a_n^2\,\mathrm{De}^2 - b_n^2,\quad N_1 = 2a_n b_n - 2b_n\,\mathrm{De},$$

$$M_2 = 1 - 2a_n\,\mathrm{De} + a_n^2\,\mathrm{De}^2\,X - \mathrm{De}^2\,b_n^2 X,\quad N_2 = 2a_n b_n\,\mathrm{De}^2\,X - 2b_n\,\mathrm{De},$$

$$M_3 = M_1 M_2 + N_1 N_2,\quad N_3 = -(M_1 N_2 - M_2 N_1).$$

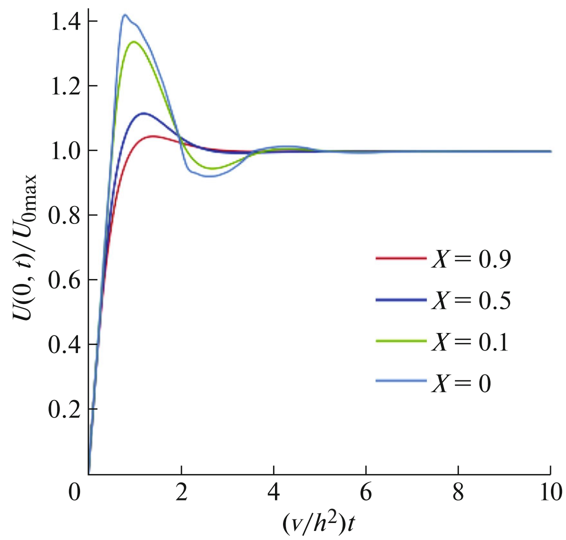

Fig. 3. Time variation in the ratio of the maximum velocity to the maximum velocity of a stationary profile at the nonstationary flow of the viscoelastic liquid (when the Deborah number is De = 0.5, and at different values of concentration of the Newtonian liquid).

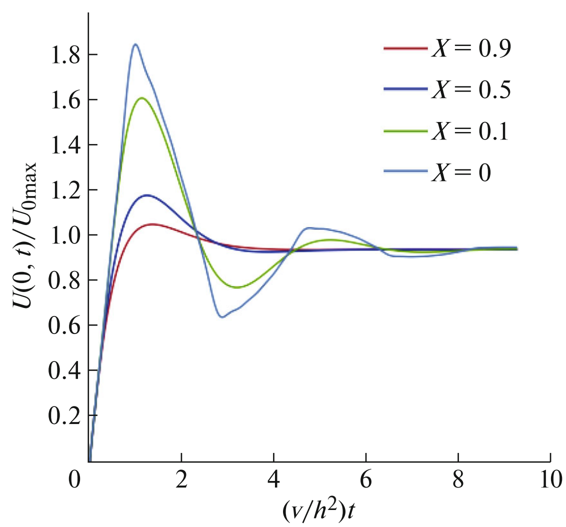

Fig. 4. Time variation in the ratio of the maximum velocity to the maximum velocity of a stationary profile at the nonstationary flow of the viscoelastic liquid (when the Deborah number is De = 1, and at different values of concentration of the Newtonian liquid).

Using formula (25), we analyze the results of numerical calculation of the nonstationary flow of a viscoelastic liquid when the solutions to Eq. (11) consist of complex conjugate roots. In Figs. 3–6 we show the variations with time of the ratio of the maximum velocity to the maximum velocity of the stationary profile at the nonstationary flow of a viscoelastic liquid (when the Deborah number has the values De = 0.5, 1, 3, and 5, respectively, and for different values of the concentration of the Newtonian liquid).

In Figs. 3–6 we see that the process of transition of a viscoelastic liquid from the nonstationary state to the stationary one has a wavelike character, unlike the process of transition of a Newtonian liquid, and the transition time is several times longer than the transition time of a Newtonian liquid. For instance, we can note that the difference is 1.4-fold in Fig. 3, 2-fold in Fig. 4, 3.5-fold in Fig. 5, and 4.5-fold in Fig. 6. In Figs. 5 and 6 we show that an increase in the Deborah number increases the ratio of the maximum velocity to the maximum velocity of the stationary profile of a viscoelastic liquid by 4.5–5 times compared to the

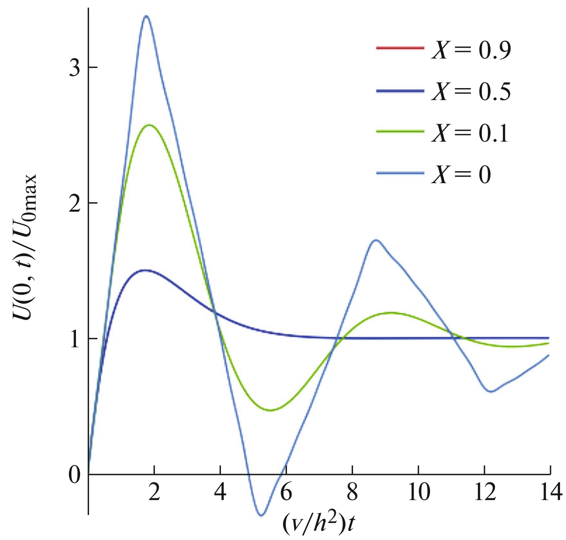

Fig. 5. Time variation in the ratio of the maximum velocity to the maximum velocity of a stationary profile at the nonstationary flow of the viscoelastic liquid (when the Deborah number is De = 3, and at different values of concentration of the Newtonian liquid).

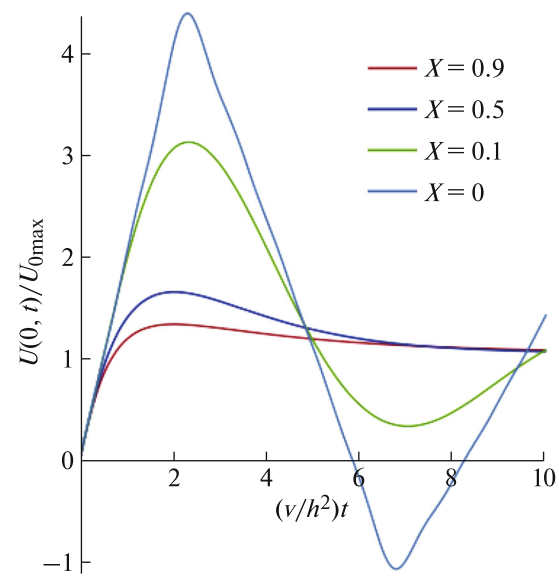

Fig. 6. Time variation in the ratio of the maximum velocity to the maximum velocity of a stationary profile at the nonstationary flow of the viscoelastic liquid (when the Deborah number is De = 5, and at different values of concentration of the Newtonian liquid).

maximum velocity of a Newtonian liquid in the transient process. The main cause of a sharp increase in the hydrodynamic quantities in the course of transition of a viscoelastic liquid can be the inertia force taking into account in the Maxwell model. We can easily see that if remove the action of the inertia force, then it is no different from the Newtonian liquid. Here, the effect of the inertia force is due to the presence of the Deborah number, that is, relaxation. We also revealed that the perturbed processes can arise in the transient process. This perturbation, occurring in the nonstationary flow of a viscoelastic liquid will be stabilized by mixing a Newtonian liquid into it, that is, an instantaneous maximum increase in the velocity of the viscoelastic liquid as a result of increasing the concentration of the Newtonian liquid is normalized. Implementation of this property play an important role in technical and technological processes, in the prevention of technical failures and faults.

CONCLUSIONS

Based on the generalized Maxwell model, we solved the problem about the nonstationary flow of a viscoelastic liquid under the action of a constant pressure gradient in a flat channel. In the course of solution we derived the formulas for determining the profile of velocity and the flow rate in the nonstationary flow of a viscoelastic liquid. On the basis of the established formulas, we analyzed the processes of transition of the characteristics of viscoelastic liquid in a flat channel from the nonstationary state to the stationary one. By the results of our analysis, we demonstrated that the transient processes under the effect of the Deborah number, determining the property of elasticity of liquid in a viscoelastic flow, are principally different from the transient process in a Newtonian liquid. Here, we have detected that the processes of transition of the characteristics of a viscoelastic liquid from the nonstationary state to the stationary one with small values of the Deborah number is almost not different from the processes of transition in a Newtonian liquid. At large values of the Deborah number compared to unity, we established that the process of transition of a viscoelastic liquid from the nonstationary state to the stationary one has a wavelike character, unlike the process of transition of a Newtonian liquid, and the transition time is several times longer than the transition time in a Newtonian liquid.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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