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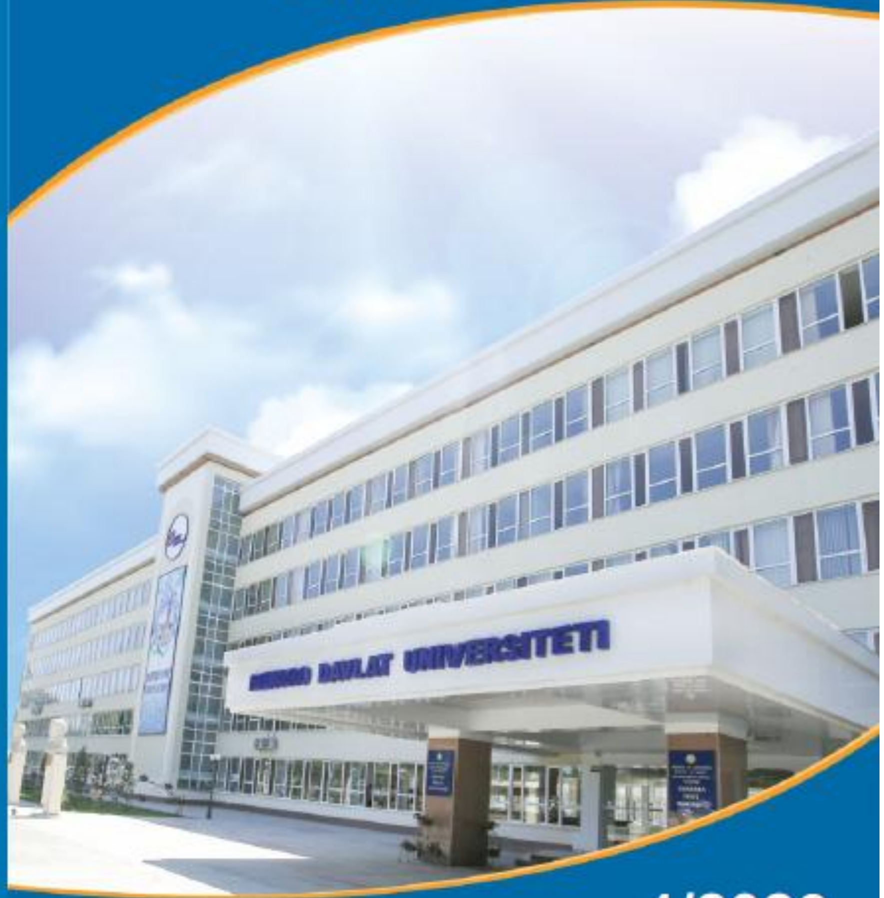

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ANIQ VA TABIIY FANLAR *** EXACT AND NATURAL SCIENCES *** ТОЧНЫЕ И ЕСТЕСТВЕННЫЕ НАУКИ

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POISSON REPRESENTATION FOR $A(z)$ − HARMONIC FUNCTIONS BELONGING TO SOME CLASS

Husenov Behzod Erkin ugli,
Basic doctoral (PhD) student, Bukhara State University, Uzbekistan
Muhammad Ikbal str. 11, 200117, Bukhara, Uzbekistan
b.e.husenov@buxdu.uz

Annotatsiya. *Ushbu maqolada ba'zi sinflarga tegishli bo'lgan $A(z)$ − garmonik funksiyalar uchun Puasson yadrosining xossalari keltirilgan. Jumladan, qatorga yoyiluvchi $A(z)$ − garmonik funksiyalarning xossalari hamda $A(z)$ − garmonik funksiyalar uchun Puasson yadrosining chegaraviy nuqtalardagi xossalari o'rganilgan.*

Kalit so'zlar: *Beltrami tenglamasi, $A(z)$ − lemniskata, $A(z)$ − garmonik funksiyalar uchun Puasson yadrosi, ω_A^* − yaqinlashish.*

Аннотация. *В этой статье приведены свойства ядра Пуассона для $A(z)$ − гармонических функций, принадлежащих некоторым классам. В частности, были исследованы свойства $A(z)$ − гармонических функций, разлагающихся на ряд, и свойства граничного поведения ядра Пуассона для $A(z)$ − гармонических функций.*

Ключевые слова: *Уравнения Бельтрами, $A(z)$ − лемниската, Ядро Пуассона для $A(z)$ − гармонических функций, ω_A^* − сходимость.*

Abstract. *This article presents properties of the Poisson kernel for $A(z)$ − harmonic functions belonging to some classes. In particular, the properties of $A(z)$ − harmonic functions decomposing into a series and the properties of the boundary behavior of the Poisson kernel for $A(z)$ − harmonic functions were investigated.*

Key words: *Beltrami equation, $A(z)$ − lemniscate, Poisson kernel for $A(z)$ − harmonic functions, ω_A^* −* convergence.

1. Introduction.

Solutions of the Beltrami equation:

$$\frac{\partial f}{\partial \bar{z}} - A(z)\frac{\partial f}{\partial z} = 0. \tag{1}$$

It is directly related to quasi-conformal maps. The function $A(z)$ is, in general, assumed to be measurable with $|A(z)| \le c < 1$ almost everywhere in the domain $D \subset \mathbb{C}$, where $c = const$. Solutions of equation (1) are often referred to as $A(z)$ − *analytic functions* in the literature.

The solutions of equation (1), as well as quasi-conformal homeomorphisms in the complex plane $\mathbb{C}$, have been studied in sufficient details. Here we confine ourselves to giving the references [1-3,6] and formulating the following three theorems:

Theorem 1 (see [3]). *For any measurable on the complex plane function $A(z): \|A\|_\infty < 1$ there exists a unique homeomorphic solution $\psi(z)$ of equation (1) with fixes the points $0, 1, \infty$.* Note that if the function $|A(z)| \le c < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A(z) \equiv 0$ outside D, so theorem 1 holds for any domain $D \subset \mathbb{C}$.

Theorem 2 (see [1,2]). *All generalized solutions of equation (1) have the form $f(z) = \Phi[\psi(z)]$, where $\psi(z)$ is a homeomorhic solution in theorem 1, and $\Phi(\zeta)$ is a holomorphic function from ζ to $\psi(D)$. Moreover, if a generalized solution $f(z)$ has isolated singular points, then the holomorphic function $\Phi = f \circ \psi^{-1}$ also has isolated singularities of the same types.*

Theorem 2 implies that an $A(z)-$analytic function $f(z)$ carries out an internal (open) mapping, i. e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain D the maximum of the modulus is reached only on the boundary, i. e. $|f(z)|\le \max_{z\in\partial D}|f(z)|$, $z\in D$. If the function is not zero, then the minimum principle also holds, i. e. $|f(z)|\ge \min_{z\in\partial D}|f(z)|$, $z\in D$. (see [7]). Theorem 3 (see [4]). *If a function $A(z)$ belongs to the class $C^{m}(D)$, then every solution $f(z)$ of equation (1) also belongs, at least, to the same class $C^{m}(D)$.*

Let $A(z)$ be anti-analytic, i. e. $\frac{\partial A}{\partial z}=0$ in $D\subset \mathbb{C}$, and such that $|A(z)|\le c<1$, $\forall z\in D$. Then according to (1) the class $f\in O_A(D)$ of $A(z)-$analytic functions in D is characterized by the fact that $\overline{D_A}f=\frac{\partial f}{\partial \bar z}-A(z)\frac{\partial f}{\partial z}=0$. Since an anti-analytic function is smooth, Theorem 3 implies that $O_A(D)\subset C^{\infty}(D)$. In this case , the following takes place:

Theorem 4 (analogue of Cauchy's theorem (see [6])). *If $f\in O_A(D)\cap C(\overline{D})$, where $D\subset \mathbb{C}$ is a domain with rectifiable boundary ∂D, then*

$$\int_{\partial D} f(z)\left(dz+A(z)d\bar z\right)=0.$$

Now we assume that the domain $D\subset \mathbb{C}$, is convex, and $a\in D$ is a fixed point in it. Consider the function

$$K(z;a)=\frac{1}{2\pi i}\frac{1}{z-a+\overline{\int_{\gamma(a,z)} \overline{A(\tau)}d\tau}}, \qquad (2)$$

where $\gamma(a,z)$ is a smooth curve which connects points a and z in D. Since the domain is simply connected and the functions $\overline{A}(z)$ is holomorphic, the integral $I(z)=\overline{\int_{\gamma(a,z)}\overline{A(\tau)}d\tau}$ does not depend on a path of integration; it coincides with a primitive, i. e. $I(z)=\overline{A}(z)$. (see [7]).

Theorem 5 (see [7]). *$K(z;a)$ is an $A(z)-$analytic function outside of the point $z=a$, i. e. $K\in O_A(D)$. Moreover, at $z=a$ the function $K(z;a)$ has a simple pole.*

Remark 1 (see [7]). *If the domain $D\subset \mathbb{C}$ is not a convex, but only simply connected, then although the function $\psi(z;a)$ is uniquely defined in the D, but a priori, it might has the other isolated zeros except a: $\psi(z;a)$, $z\in P=\{a_1,a_2,...\}$. Consequently, $\psi\in O_A(D)$, $\psi(z;a)\neq 0$ when $z\notin P$ and $K(z;a)$ is analytic function only in $D\setminus P$, it has a poles at the points of P. Due to this fact, we consider the class of $A(z)-$analytic functions only in the convex domain $D\subset \mathbb{C}$.*

According to Theorem 2, the function $\psi(z;a)\in O_A(D)$ carries out an internal mapping. In particular, the set

$$L(a;r)=\left\{z\in D:\left|\psi(z;a)\right|<\left|z-a+\overline{\int_{\gamma(a,z)}\overline{A(\tau)}d\tau}\right|<r\right\}$$

is open in D. For suffiently small $r>0$ it compactly belongs to D and contains the point a. This set is called an $A(z)-$lemniscate with the center a and denoted by $L(a;r)$. According to the maximum principle the lemniscate $L(a;r)$ is simply connected and to the minimum principle it is connected. (see [7]).

Here we note that the analog power series for $A(z)-$analytic functions will be

$$\sum_{k=-\infty}^{\infty} c_k\psi^k(z;a), \qquad (3)$$

where $c_k=\mathrm{const}$. The domain of convergence of the series (3) is a lemniscate $L(a;r)$, where the radius of convergence is given by the Cauchy-Hadamard formula:

$$\frac{1}{r}=\overline{\lim_{k\to\infty}}\sqrt[k]{|c_k|}.$$

(see [7]). There is true an inverse

Theorem 6 (see [7]). If $f(z)\in O_A\left(L(a;r)\right)\cap C\left(\overline{L}(a;r)\right)$, where $L(a;r)=\{z\in D:|\psi(z;a)|<r\}\subset\subset D$ is a lemniscate, then the function $f(z)$ can be expanded to the Taylor series in $L(a;r)$:

$$f(z)=\sum_{k=0}^{\infty}c_k\psi^k(z;a), \tag{4}$$

where $c_k=\frac{1}{2\pi i}\int_{\partial L(a,\rho)}\frac{f(\zeta)}{(\psi(\zeta;a))^{k+1}}(d\zeta+A(\zeta)d\overline{\zeta}),\ 0<\rho<r,\ k=0,1,2,\dots$.

Theorem 7 (Laurent expansion (see [7])). *Let* $f(z)$ *be* $A(z)-$*analytic in a ring of lemniscates:* $f\in O_A\left(L(a;R)\setminus L(a;r)\right)$, $R>r$. *Then* $f(z)$ *will be expanded to the Loran series in this ring:*

$$f(z)=\sum_{k=-\infty}^{\infty}c_k\psi^k(z;a), \tag{5}$$

(5) where a coefficients of Taylor-Laurent series which determines by the formula

$$c_k=\frac{1}{2\pi i}\int_{\partial L(a,\rho)}\frac{f(\zeta)}{(\psi(\zeta;a))^{k+1}}(d\zeta+A(\zeta)d\overline{\zeta}),\ r<\rho<R,\ k=0,\pm1,\pm2,\dots.$$

The series (5) converges uniformly inside of the ring

$$L(a;R)\setminus L(a;r)=\{z\in D: r<|\psi(z;a)|<R\}.$$

Let $f=u+iv$.

Theorem 8 (see [8]). *The real part of the* $A(z)-$*analytic functions of* $f\in O_A(D)$ *satisfies equation*

$$\Delta_A u=\frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial\overline{z}}-2A\frac{\partial u}{\partial z}\right)\right)+\frac{\partial}{\partial\overline{z}}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial z}-2\overline{A}\frac{\partial u}{\partial\overline{z}}\right)\right)=0 \tag{6}$$

in the domain of D.

In connection with Theorem 8, it is natural to define the $A(z)-$harmonic function as follows.

Definition 1 (see [8]). *A double differentiable function* $u\in C^2(D)$, $u:D\to\mathbb{R}$ *is called* $A(z)-$*harmonic in the* D *domain if the* D *domain if it satisfies the differential equation (6).*

The class of $A(z)-$harmonic functions in the domain of D is denoted as $h_A(D)$. Thus, the real part and hence the imaginary part, of the $A(z)-$harmonic function in the domain of D. The inverse theorem is also true for simply connected domains.

Theorem 9 (see [8]). *If the function is* $u(z)\in h_A(D)$, *where* D *is a simply connected domain, then* $f\in O_A(D)$: $u=\mathrm{Re}\,f$. For $A(z)-$analytic and $A(z)-$harmonic functions, the following Dirichlet problem is naturally considered:

Dirichlet problem (see [8]). *A bounded domain of* $G\subset D$ *is given and a continuous function of* $\omega(\zeta)$ *is set at the boundary of* ∂G. *It is required to find* $A(z)-$*harmonic in the domain of G, continuous on the closure of* G *the function of* $u(z)\in h_A(G)\cap C(\overline{G})$: $u|_{\partial G}=\omega$.

Theorem 10 (Poisson formula for $A(z)-$harmonic functions (see [8])). *If the* $\omega(\zeta)$ *function is continuous on the boundary of the lemniscate of* $L(a,r)\subset D$, *then the function*

$$u(z)=\frac{1}{2\pi r}\oint_{|\psi(a,\zeta)|=r}\omega(\zeta)\frac{r^2-|\psi(a,z)|^2}{|\psi(\zeta;z)|^2}\left|d\zeta+A(\zeta)d\overline{\zeta}\right| \tag{7}$$

is the solution of the Dirichlet problem in $L(a,r)$.

The $f(\zeta;z)=\frac{\psi(a;\zeta)+\psi(a;z)}{\psi(\zeta;z)}$ function is an $A(z)-$analytic function for $z\in L(a,r)$, where $\zeta\in L(a,r)$. Then $\Pi(\zeta;z)=\mathrm{Re}\,f(\zeta;z)\in h_A(L(a,r))$ and besides

$$\Pi(\zeta;z)=\frac{1}{2\pi}\left(f(\zeta;z)+\overline{f}(\zeta;z)\right)=\frac{1}{2\pi}\left(\frac{\psi(a;\zeta)+\psi(a;z)}{\psi(a;\zeta)-\psi(a;z)}+\frac{\overline{\psi}(a;\zeta)+\overline{\psi}(a;z)}{\overline{\psi}(a;\zeta)-\overline{\psi}(a;z)}\right)=$$

$$=\frac{1}{2\pi}\left(\frac{|\psi(a;\zeta)|^2-|\psi(a;z)|^2}{|\psi(\zeta;z)|^2}\right)=\frac{1}{2\pi}\left(\frac{r^2-|\psi(a;z)|^2}{|\psi(\zeta;z)|^2}\right). \tag{8}$$

Formula (8) is called an analogue of the Poisson formula for $A(z)-$harmonic functions. (see [8]).

2. Poisson representation for $A(z)-$harmonic functions belonging to some class.

First we will introduce L^p, $p\geq 1$ classes for $A(z)-$analytic functions. Among the functional spaces is the space L^p, consisting of all functions summable with $p-$th degree in the lemniscate boundary $\partial L(a,\rho)$ for $A(z)-$analytic functions, i. e. such $A(z)-$analytic functions $f(z)$ that

$$\frac{1}{2\pi\rho}\int_{|\psi(z,a)|=\rho}|f(z)|^p|dz+A(z)d\bar{z}|<\infty,$$

where $o<\rho<r$. Of this space, the function is $A(z)-$analytic functions in the domain of D, then which we denote $L_A^p(D)$. Using the Minkowski integral inequality, we make sure that L_A^p becomes a normalized space if we put

$$\|f(z)\|_{L_A^p}=\left(\frac{1}{2\pi\rho}\int_{|\psi(z,a)|=\rho}|f(z)|^p|dz+A(z)d\bar{z}|\right)^{\frac{1}{p}}.$$

Space L_A^∞ measurable $A(z)-$analytic functions, bounded almost everywhere in the lemniscate $L(a,r)$, by the identification of functions that differ only on the set of measure zero, and, assuming by definition:

$$\|f(z)\|_{L_A^\infty}=ess\sup_{|\psi(z,a)|=r}|f(z)|.$$

This representation is the norm of space L_A^∞, where $ess\sup$ is the essential supremum of the function.

Convergence in L_A^p is convergence on average with the index p. It is $f_n(z)\to f(z)$ that means

$$\frac{1}{2\pi\rho}\int_{|\psi(z,a)|=\rho}|f_n(z)-f(z)|^p|dz+A(z)d\bar{z}|\to 0.$$

Let $u(z)\in h_A(L(a,r))$. It is remarkable that it can often be represented in this lemniscate $L(a,r)$ by the Poisson formula (8).

Theorem 11. *Assume that the average*

$$\int_{|\psi(z,a)|=\rho}|u(z)|^p|dz+A(z)d\bar{z}|$$

is bounded at $\rho<r$. Then there is a function $f\in L_A^p(L(a,r))$, such that

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r}P(\zeta;z)f(\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|$$

is for $z\in L(a,r)$.

Proof. At $p>1$, the space L_A^p is conjugate to L_A^q, where $\frac{1}{p}+\frac{1}{q}=1$. For functions $u_n(\zeta)=u\left(\left(1-\frac{1}{n}\right)\zeta\right)\in h_A(L(a,r))$ (instead of $1-\frac{1}{n}$, any sequence ρ_n tending to r from below is suitable), we have $\|u_n\|_{L_A^p}\leq d$, $d=const$, ($\|\ \|_{L_A^p}$ here, of course, is taken along the boundary $\partial L(a,r)$), so that by a Cantor diagonal process we can isolate from them a subsequence $u_{n_j}\in h_A(L(a,r))$, such that for all functions g running through some countable everywhere dense subset of space L_A^q, there exists the which is

$$L_A g=\lim_{j\to\infty}\int_{|\psi(\zeta,a)|=r}g(\zeta)u_{n_j}(\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

Since $\|u_n\|_{L_A^p}\leq d$, then this limit of $L_A g$ actually exists for all $g\in L_A^q$ and $L_A g$ is a bounded linear functional on L_A^q. Therefore, since the space L_A^p is conjugate to L_A^q, there exists such a function $f\in L_A^p$ that

$$L_A g=\int_{|\psi(\zeta,a)|=r}f(\zeta)g(\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|$$

for all $g\in L_A^q$.

We will consider the Poisson formula (8) in the form

$$P(\zeta;z)=\frac{r^2-|\psi(a;z)|^2}{|\psi(\zeta;z)|^2}. \qquad (9)$$

Now, for each $n\in\Gamma$, the function $u_n(z)=u\left(\left(1-\frac{1}{n}\right)z\right)$ $A(z)-$harmonious in $L\left(a;\frac{nr}{n-1}\right)$, so if $z\in L(a;r)$, then

$$u_{n_j}(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r}P(\zeta;z)u_{n_j}(\zeta)\,|\,d\zeta+A(\zeta)d\overline{\zeta}\,|.$$

Let's fix an arbitrary $z\in L(a;r)$ and take $g(\zeta)=P(\zeta;z)$, $g\in L_A^q$. Then

$$\lim_{j\to\infty}\int_{|\psi(\zeta;a)|=r}P(\zeta;z)u_{n_j}(\zeta)\,|\,d\zeta+A(\zeta)d\overline{\zeta}\,|=L_A^1 g=\int_{|\psi(\zeta;a)|=r}g(\zeta)f(\zeta)\,|\,d\zeta+A(\zeta)d\overline{\zeta}\,|=$$

$$=\int_{|\psi(\zeta;a)|=r}P(\zeta;z)f(\zeta)\,|\,d\zeta+A(\zeta)d\overline{\zeta}\,|.$$

In this equality, there is

$$\lim_{j\to\infty}2\pi\rho u_{n_j}(z)=2\pi\rho u(z)$$

on the left. So

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r}P(\zeta;z)f(\zeta)\,|\,d\zeta+A(\zeta)d\overline{\zeta}\,|,\ \text{where } f\in L_A^p.$$

Remark 2. *The same result holds with the same proof and at* $p=\infty$, *if we slightly change the wording of the proposition:*

Proposition 1. *If* $u(z)$ *is a bounded* $A(z)-$*harmonic function in lemniscate* $L(a;r)$, *then there exists a function* $f\in L_A^\infty(L(a;r))$, *such that*

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r}P(\zeta;z)f(\zeta)\,|\,d\zeta+A(\zeta)d\overline{\zeta}\,|.$$

But what about the case of $p=1$? Space $L_A^1(L(a;r))$, unfortunately, is not conjugate to any other. But M - the space of finite real measures μ by $\partial L(a;r)$ with a norm $\|\mu\|$, equal to the total variation of measure μ - is conjugate to $C(\partial L(a;r))$ - the space of continuous $A(z)-$analytic functions by $\partial L(a;r)$. If $g^*\in L_A^1(L(a;r))$, then we can associate measure μ_{g^*} with g, putting

$$\int_{|\psi(\zeta;a)|=r}g(\zeta)d\mu g^*(\zeta)=\int_{|\psi(\zeta;a)|=r}g(\zeta)_{g^*}(\zeta);$$

at the same time $\|\mu_{g^*}\|=\|g^*\|_{L_A^1}$.

Now the reasoning carried out in the proof of the ninth theorem shows that the following is true

Proposition 2. *If* $u(z)\in h_A(L(a;r))$ *and the average*

$$\int_{|\psi(z;a)|=\rho}|u(z)|\,d\mu(z)$$

are bounded at $z\in L(a;r)$, *such that*

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r}P(\zeta;z)d\mu(\zeta).$$

Corollary 1. *Let* $u(z)$ *be a function,* $A(z)-$*harmonic and positive in lemniscate* $L(a;r)$. *Then there is a finite positive measure* μ *by* $\partial L(a;r)$, *such that*

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r}P(\zeta;z)d\mu(\zeta).$$

Proof. For $z \in L(a,r)$ (using, for example, the decomposition $u(z)=\sum_{-\infty}^{\infty} c_n \psi^{|n|}(z;a)\psi^n(\zeta;z)$, which takes place in lemniscate $L(a,r)$), we get

$$2\pi\rho u(a)=\int_{|\psi(z;a)|=\rho} u(z)\,|dz+A(z)d\bar{z}|=\int_{|\psi(z;a)|=\rho} u(z)\,|dz+A(z)d\bar{z}|,$$

since $u \geq 0$. And now we apply the Theorem 11. Measure μ is positive, because in this case (see again the proof of the ninth theorem) it turns out that the integral $\int_{|\psi(\zeta;a)|=r} g(\zeta)d\mu(\zeta)$ is positive for any positive function $g \in O_A(L(a,r)) \cap C(\partial L(a,r))$ as the limit of positive numbers.

3. **Summability properties of $A(z)-$harmonic functions given by the Poisson formula.**

First we get some rough results sufficient for many considerations. The Poisson kernel in the form (9)

$$P(\zeta;z)=\frac{r^2-|\psi(z;a)|^2}{|\psi(\zeta;z)|^2}=\sum_{m=-\infty}^{\infty}|\psi(z;a)|^{|m|}\psi^m(\zeta;a)$$

has the following properties for any $z \in L(a,r)$:

1) $P(\zeta;z)>0$;

2) $\int_{|\psi(\zeta;a)|=r} \frac{r^2-|\psi(a;z)|^2}{|\psi(\zeta;z)|^2}\,|d\zeta+A(\zeta)d\bar{\zeta}|=2\pi r;$

The second property is proved in [9].

Proposition 3. *If $f \in L_A^p(L(a,r))$, and $u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r} P(\zeta;z)f(\zeta)\,|d\zeta+A(\zeta)d\bar{\zeta}|$, then the function $u(z)\in h_A(L(a,r))$ and $\int_{|\psi(z;a)|=\rho} |u(z)|^p\,|dz+A(z)d\bar{z}|\leq d$, $d=const$.*

Proof. Let $\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r} \psi^n(\zeta;a)f(\zeta)\,|d\zeta+A(\zeta)d\bar{\zeta}|=c_n$. Then for $z \in L(a,r)$ we have

$$u(z)=\sum_{m=-\infty}^{\infty} c_n\,|\psi(z;a)|^n\psi^n(z;a).$$

It is directly verified that the function $u(z)\in h_A(L(a,r))$, since the series converges uniformly in the interior of the lemniscate $L(a,r)$. (what does uniform convergence mean on compact subsets - such is the language of complex analysis!).

Take now $g \in L_A^q(L(a,r))$, $\|g\|_{L_A^q}=1$, so that (for any fixed $z \in L(a,r)$; the function g, of course, will depend on z)

$$\left(\int_{|\psi(z;a)|=\rho} |u(z)|^p\,|dz+A(z)d\bar{z}|\right)^{\frac{1}{p}}=\int_{|\psi(z;a)|=\rho} u(z)g(z)\,|dz+A(z)d\bar{z}|.$$

According to Fubini's theorem, the integral on the right is

$$\frac{1}{2\pi r}\iint_{|\psi(z;a)|\leq r} P(\zeta;z)f(z)g(z)\left(1-|A(z)|^2\right)\frac{dz\wedge d\bar{z}}{-2i},$$

which modulo does not exceed

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho} P(\zeta;z)\|f(z)\|_{L_A^p}\|g(z)\|_{L_A^q}\,|dz+A(z)d\bar{z}|=\|f(z)\|_{L_A^p}$$

(due to the choice of g and its property 2)).

Finally,

$$\int_{|\psi(z;a)|=\rho} |u(z)|^p\,|dz+A(z)d\bar{z}|\leq\|f\|_{L_A^p}^p.$$

That's all.

Proposition 4. *Let* μ *be a finite measure on* $\partial L(a;r)$. *Then the function* $u(z)$ $A(z)-$*harmonious in lemniscata* $L(a;r)$ *and*

$$\int_{|\psi(z;a)|=\rho} |u(z)||dz+A(z)d\bar{z}|\leq d,$$

where $z\in L(a;r)$.

Proof. $A(z)-$harmony is established in the same way as above. Let $z\in L(a;r)$ be given, and let the function $g\in L_x^\infty(L(a;r))$, $\|g\|_{L_x^\infty}=1$, be such that

$$\int_{|\psi(z;a)|=\rho} |u(z)||dz+A(z)d\bar{z}|=\int_{|\psi(z;a)|=\rho} |g(z)u(z)||dz+A(z)d\bar{z}|.$$

According to Fubini's theorem, the integral on the right side is equal to

$$\frac{1}{2\pi r}\iint_{|\psi(\zeta;a)|<r} P(\zeta;z)g(\zeta)\left(1-|A(\zeta)|^2\right)\frac{d\mu(\zeta)\wedge d\bar{\mu}(\zeta)}{-2i},$$

and by virtue of 1) - 2) modulo does not exceed

$$\frac{1}{2\pi r}\iint_{|\psi(\zeta;a)|<r} P(\zeta;z)\|g\|_{L_x^\infty}d\mu(\zeta)\wedge d\bar{\mu}(\zeta)=\|g\|_{L_x^\infty}\int_{|\psi(\zeta;a)|=r}|d\mu(\zeta)|=\int_{|\psi(\zeta;a)|=r}|d\mu(\zeta)|.$$

That's all.

4. Initial study of the boundary behavior of the Poisson integral for $A(z)-$harmonic functions.

The Poisson kernel for $A(z)-$harmonic functions $P(\zeta;z)$ also has a third property:

3) For $\delta>0$ evenly for $l_\delta=\left\{z:\psi(z;a)=\rho e^{i\theta},\delta\leq|\theta|\leq\pi\right\}$ at a radius of $z\to\zeta$ (or $\rho\to r$).

This immediately follows from the formula for $P(\zeta;z)$.

Proposition 5. *Let the function* f *be continuous on lemniscates* $L(a;r)$ *and*

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r} P(\zeta;z)f(\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

Then $u(z)\to f(\zeta)$, *when the radius is* $z\to\zeta$, *and the convergence is uniformly along* ζ.

Proof. Let's write down

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r} P(\zeta;z)f(\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

For a given arbitrary ζ, we have by property 2) and if we take $\psi(\zeta;a)=re^{i\varphi}$,

$$f(\varphi)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r} P(\zeta;z)f(\varphi)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

Hence,

$$u(z)-f(\varphi)=\frac{1}{2\pi r}\int_{|\psi(\zeta;a)|=r} (f(\zeta)-f(\varphi))P(\zeta;z)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

Let $\delta<\frac{\pi}{2}$ be such that $|f(x)-f(\varphi)|<\varepsilon$ is at $|x-\varphi|<2\delta$; the number δ here depends only on ε, and not on φ, because of the (uniform!) continuity of the function f.

Let's write the integral in the right part as the sum of two:

$$|u(z)-f(\varphi)|=\frac{1}{2\pi r}\int_{\{|\psi(\zeta;a)|=r\}\setminus l_\delta} |f(\zeta)-f(\varphi)|P(\zeta;z)|d\zeta+A(\zeta)d\bar{\zeta}|+$$

$$+\frac{1}{2\pi r}\int_{l_\delta}|f(\zeta)-f(\varphi)|P(\zeta;z)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

If $|\theta-\varphi|<\delta$, then the first integral on the right does not exceed

$$\frac{\varepsilon}{2\pi r}\int_{l_2}|P(\zeta;z)|d\zeta + A(\zeta)d\overline{\zeta}|<\varepsilon$$

Let M be the upper face of value $|f(\zeta)|$. Then the second integral does not exceed

$$\frac{M}{2\pi r}\int_{l_2}|P(\zeta;z)|d\zeta + A(\zeta)d\overline{\zeta}|,$$

which is less than ε, if z is close sufficiently in radius to ζ, by virtue of property 3).

Thus, $|u(z)-f(\varphi)|<2\varepsilon$, if $|\theta-\varphi|<\delta$, and z is close sufficiently in radius to ζ.

Remark 3. *Properties 1), 2) and 3) taken together show that* $\frac{P(\zeta;z)}{2\pi r}$ *represents the so-called approximate unit. The proven proposition takes place due to these properties: not only for the analog of the Poisson kernel* $P(\zeta;z)$, *but also for other kernels that are approximate units, similar results are valid.*

Statement 1. *Let* $f\in L^1_A(L(a,r))$, *and let the function* f *be continuous at point* ζ_0. *Then*

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r} P(\zeta;z)f(\zeta)|d\zeta + A(\zeta)d\overline{\zeta}|.$$

tends to $f(\zeta_0)$ *while striving along the radius of* z *tends to* ζ_0.

The proof is the same as the previous proposition.

Theorem 12. *Let's say* $f\in L^p_A(L(a,r))$, $1\le p<\infty$ *and*

$$u(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r} P(\zeta;z)f(\zeta)|d\zeta + A(\zeta)d\overline{\zeta}|.$$

Then $\int_{|\psi(z,a)|=\rho}|u(z)-f(z)||dz+A(z)d\overline{z}|\to 0$ *at radius* $z\to\zeta$, *i. e.* $u(z)$ *tends to* $f(z)$ *at* L^p_A *the norm at radius* $z\to\zeta$.

Proof. Let's put $f_\rho(z)=u(z)$. Then

$$f_\rho(z)-f(z)=\frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r}\left(f(\zeta)-f(z)\right)P(\zeta;z)|d\zeta + A(\zeta)d\overline{\zeta}|.$$

Using properties 1) and 2) (we consider $f_\rho(z)-f(z)$ as the limit of convex combinations of functions $f(\zeta)-f(z)$, considering ζ as a parameter, and z as a variable), we have, by an obvious generalization of the triangle inequality

$$\left(\int_{|\psi(\zeta,a)|=r}\left|f_\rho(z)-f(z)\right|^p|d\zeta + A(\zeta)d\overline{\zeta}|\right)^{\frac{1}{p}}\le$$

$$\le\frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r}\left(\int_{|\psi(z,a)|=\rho}\left|f(\zeta)-f(z)\right|^p|dz + A(z)d\overline{z}|\right)^{\frac{1}{p}}P(\zeta;z)|d\zeta + A(\zeta)d\overline{\zeta}|.$$

Assuming

$$F(\zeta)=\left(\int_{|\psi(z,a)|=\rho}\left|f(\zeta)-f(z)\right|^p|dz + A(z)d\overline{z}|\right)^{\frac{1}{p}},$$

we get

$$\left\|f_\rho-f\right\|\le\frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r}F(\zeta)|d\zeta + A(\zeta)d\overline{\zeta}|.$$

But $F(\zeta)\to 0$ by radius at $z\to\zeta$! This is so because the shift is continuous in the L^p_A – norm for $1\le p<\infty$. This follows in turn from the elementary facts of the theory of functions. Indeed, let $f\in L^p_A(L(a,r))$ and $\varepsilon>0$ be given. Find a continuous function g, such that $\|f-g\|_{L^p_A}<\varepsilon$. Then obviously

$\int\limits_{|\psi(z;a)|=\rho} |g(\zeta)-g(z)|^{p}\,|dz+A(z)d\bar{z}|<\varepsilon^{p}$ for $\{|\psi(\zeta;a)|=r\}\setminus l_{\delta}$ for sufficiently small δ due to uniform continuity; hence $\|f(\zeta)-f(z)\|_{l_{\delta}^{c}}<3\varepsilon$ for $\{|\psi(\zeta;a)|=r\}\setminus l_{\delta}$.

In any case, the function F is continuous in ζ where it is zero. Therefore, according to Propositions 5, $\|f_{\rho}-f\|\to 0$ by radius at $z\to\zeta$.

At $p=\infty$ all we have is ω_{A}^{*} −convergence:

Proposition 6. *If* $f\in L_{A}^{\infty}(L(a;r))$ *and* $u(z)=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta;a)|=r} P(\zeta;z)f(\zeta)\,|d\zeta+A(\zeta)d\bar{\zeta}|$,

then $u(z)\xrightarrow[\omega_{A}^{*}]{} f(z)$ *by radius at* $z\to\zeta$.

Proof. Let's take an arbitrary function $g\in L_{A}^{1}(L(a;r))$. We need to prove that

$$\int\limits_{|\psi(z;a)|=\rho} u(z)g(z)\,|dz+A(z)d\bar{z}|\to \int\limits_{|\psi(z;a)|=\rho} f(z)g(z)\,|dz+A(z)d\bar{z}|$$

by radius at $z\to\zeta$. But this is so, because (we use parity $P(\zeta;z)$!)

$$\int\limits_{|\psi(z;a)|=\rho} g(z)P(\zeta;z)\,|dz+A(z)d\bar{z}|=\int\limits_{|\psi(z;a)|=\rho} P(z;\zeta)g(z)\,|dz+A(z)d\bar{z}|$$

tends to $g(\zeta)$ at by radius at $z\to\zeta$ according to the previous theorem. It remains only to apply Fubini's theorem.

Similarly fair.

Corollary 2. *Let* $u(z)=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta;a)|=r} P(\zeta;z)\,|d\mu(\zeta)|$, *where* μ *the final measure is on* $\partial L(a;r)$. *Then* $u(z)\xrightarrow[\omega_{A}^{*}]{} d\mu(\zeta)$ *by radius at* $z\to\zeta$, *i. e. for any functions* $g\in O_{A}(L(a;r))\cap C(\overline{L(a;r)})$

$$\int\limits_{|\psi(z;a)|=\rho} u(z)g(z)\,|dz+A(z)d\bar{z}|\to \int\limits_{|\psi(z;a)|=\rho} g(z)\,|d\mu(z)|,$$

when by radius at $z\to\zeta$.

Proof. This corollary is proved by applying Fubini's theorem together with the Proposition 5 of this section.

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