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Analogue of the Carleman's Formula for $A(z)$-analytic Functions

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Abstract. In this paper, an analogue of the Carleman formula is proved for $A(z)$-analytic functions from the Hardy class. The idea of obtaining the Carleman formulas and the concept of the Carleman function for $A(z)$-analytic functions from the Hardy class belong to M.M. Lavrentiev. In the proof of Carleman's formula, $A(z)$-harmonic functions and the Poisson formula in lemniscates $L(a, r)$, compactly belonging to the domain under consideration $D \subset \mathbb{C}$, are used substantially.

Keywords: $A(z)$-analytic function, Hardy class, $A(z)$-lemniscate, multiple Cauchy integral formula for $A(z)$-analytic functions.

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One of the major challenges in the classical theory of complex analysis is the integral representation of analytic functions, which allows us to recover a function within a domain from its values along the boundary. Additionally, it is natural to inquire how an analytic function may be reconstructed based on its value at a single point on the boundary of a simply-connected domain. In 1926, T. Carleman achieved a significant breakthrough by solving this issue for certain types of domains. He devised a strategy for constructing a "quenching" function in the context of boundary-value problems. G. M. Goluzin and V. I. Krylov further extended Carleman's findings in 1933, employing a specialized holomorphic function to assist with the process, which relies on a portion of the boundary of the domain. Another method based on the approximation of the kernel of the integral representation was proposed by M. M. Lavrentiev in 1956. It turned out that this method works successfully in the noted cases when the Goluzin-Krylov approach is not applicable [2].

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1. Introduction and prelimiraries

1.1. $A(z)$-analytic functions

Let $A(z)$ be an antianalytic function, i.e. $\dfrac{\partial A}{\partial z} = 0$ in the domain $D \subset \mathbb{C}$ and there is a constant $c < 1$ such that $|A(z)| \leqslant c$ for all $z \in D$. The function $f(z)$ is said to be $A(z)$-*analytic* in the domain D if for any $z \in D$, the following equality holds:

$$\frac{\partial f}{\partial \bar{z}} = A(z) \frac{\partial f}{\partial z}. \tag{1}$$

We denote by $O_A(D)$ the class of all $A(z)$-analytic functions defined in the domain D. Since an antianalytic function is smooth, $O_A(D) \subset C^{\infty}(D)$ (see [1]). In this case, the following takes place:

Theorem 1.1 (see [3], analogue of the Cauchy integral theorem)**.** *If $f \in O_A(D) \cap C\left(\bar{D}\right)$, where $D \subset \mathbb{C}$ is a domain with smooth ∂D, then*

$$\int\limits_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex and $\xi \in D$ is a fixed point in it. Since the function $\overline{A}(z)$ is analytic, the integral

$$I(z) = \overline{\int\limits_{\gamma(\xi,z)} \overline{A(\tau)} d\tau}$$

is independent of the path of integration; it coincides with the antiderivative $I'(z) = \overline{A}(z)$. Consider the function

$$K(z,\xi) = \frac{1}{2\pi i} \frac{1}{z - \xi + I(z)},$$

where $\gamma(\xi, z)$ is a smooth curve which connects the points $\xi, z \in D$ (see [5]).

Theorem 1.2 (see [5])**.** *$K(z,\xi)$ is an $A(z)$-analytic function outside of the point $z = \xi$, i.e. $K(z,\xi) \in O_A(D\backslash\{\xi\})$. Moreover, at $z = \xi$ the function $K(z,\xi)$ has a simple pole.*

Remark 1.1 (see [5])**.** *If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function*

$$\psi(\xi, z) = z - \xi + I(z),$$

although well defined in D, may have other isolated zeros except $\xi : \psi(\xi, z) = 0$ for $z \in P\backslash\{\xi, \xi_1, \xi_2, ...\}$. Consequently, $\psi \in O_A(D)$, $\psi(\xi, z) \neq 0$ when $z \notin P$ and $K(z,\xi)$ is an $A(z)$-analytic function only in $D\backslash P$, it has poles at the points of P. Due to this fact we consider the class of $A(z)$-analytic functions only in convex domains.

According to [5], Theorem 1.2, the function $\psi(\xi, z)$ is an $A(z)$-analytic function.

The following set is an open subset of D:

$$L(a,r) = \{z \in D : |\psi(a,z)| < r\}.$$

For suffiently small $r > 0$, this set compactly lies in D (we denote it by $L(a,r) \subset\subset D$) and contains the point a. The set $L(a,r)$ is called an $A(z)$-*lemniscate* centered at the point a. The lemniscate $L(a,r)$ is a simply-connected set (see [5]).

Theorem 1.3 (see [4], Cauchy's integral formula)**.** *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be an arbitrary subdomain with a smooth or piecewise smooth ∂G. Then for any function $f(z) \in O_A(G) \bigcap C(\bar{G})$, the following formula holds:*

$$f(z) = \int\limits_{\partial G} f(\xi) K(z,\xi) \left(d\xi + A(\xi) d\bar{\xi}\right), \quad z \in G\cdot \tag{2}$$

Note that from formula (2) it follows that if $f(z) \in O_A(L(a,r)) \bigcap C\left(\bar{L}(a,r)\right)$, where $L(a,r) \subset\subset D$ is a fixed $A(z)$-lemniscate, then in $L(a,r)$ the function $f(z)$ is expanded in a Taylor series:

$$f(z) = \sum_{k=0}^{\infty} c_k \psi^k(a,z), \tag{3}$$

where $c_k = \dfrac{1}{2\pi i} \displaystyle\int\limits_{|\psi(a,\xi)|=\rho} \dfrac{f(\xi)}{(\psi(a,\xi))^{k+1}} \left(d\xi + A(\xi) d\bar{\xi}\right), 0 < \rho < r,\ k = 0, 1, 2, \dots .$

1.2. $A(z)$-harmonic functions

Theorem 1.4 (see [6])**.** *The real part $u(z)$ of the functions $f(z) \in O_A(D)$ satisfies the equation*

$$\begin{aligned}\Delta_A u := &\frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left(\left(1+|A|^2\right)\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z}\right)\right) + \\ &+ \frac{\partial}{\partial \bar{z}}\left(\frac{1}{1-|A|^2}\left(\left(1+|A|^2\right)\frac{\partial u}{\partial z} - 2\bar{A}\frac{\partial u}{\partial \bar{z}}\right)\right) = 0\end{aligned} \tag{4}$$

in the domain D.

Conversely, if D is a simply connected domain, and a function $u \in C^2(D)$ satisfies the differential equation (4), then there is $u(z) = Ref(z)$.

In connection with Theorem 1.4, it is natural to define $A(z)$-harmonic functions as follows.

Definition 1.1 (see [6])**.** *A function $u \in C^2(D)$, $u : D \to \mathbb{R}$ is called $A(z)$-harmonic if it satisfies in the domain D the differential equation (4).*

The class of $A(z)$-harmonic functions in the domain D is denoted as $h_A(D)$. Thus, the operator $\triangle_A$ in the theory of $A(z)$-harmonic functions plays the same role as Laplace operator $\triangle$ in the theory of harmonic functions. It follows from Theorem 1.4 that the real and imaginary parts of $A(z)$-analytic function $f = u + iv$ in the domain D are $A(z)$-harmonic functions. The function v is called the $A(z)$-conjugate harmonic function to u.

Theorem 1.5 (see [6], analogue of the Poisson formula for $A(z)$-harmonic functions)**.** *If the function $\omega(\zeta)$ is continuous on the boundary of the lemniscate $L(a,r)$, then the function*

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} \omega(\zeta) \frac{r^2 - |\psi(a,z)|^2}{|\psi(\zeta,z)|^2} \left|d\zeta + A(\zeta) d\bar{\zeta}\right| \tag{5}$$

is the solution of the Dirichlet problem in $L(a,r)$, i.e. $u(z) \in h_A(L(a,r)) \cap C\left(\bar{L}(a,r)\right)$: $u(z)|_{\partial L(a,r)} = \omega(\zeta)$. Conversely, any function $u(z) \in h_A(L(a,r)) \cap C(\bar{L}(a,r))$ is represented in $L(a,r)$ by the Poisson integral:

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} u(\zeta) \frac{r^2 - |\psi(a,z)|^2}{|\psi(\zeta,z)|^2} \left|d\zeta + A(\zeta) d\bar{\zeta}\right|, \quad z \in L(a,r)\cdot \tag{6}$$

Formulas (5) and (6) are analogues of the Poisson formula for $A(z)$-harmonic functions. Here, $P(z,\zeta) = \dfrac{r^2 - |\psi(a,z)|^2}{|\psi(\zeta,z)|^2}$ is the Poisson kernel.

1.3. Angular limits and Hardy classes for $A(z)$-analytic functions

Let $L(a,r) \subset\subset D$ and $f(z) \in O_A(L(a,r))$. We define the concepts of angular and radial limits of $A(z)$-subharmonic and $A(z)$-analytic functions in lemniscate $L(a,r)$. The radial limits of the function $f(z)$ at some point $\zeta \in \partial L(a,r)$ are denoted as $f^*(\zeta)$ and the angular limits are denoted as $f^*_{\sphericalangle}(\zeta)$ (see [8]).

In the classical case of the disk $U = \{w \in \mathbb{C} : |w| < 1\} \subset \mathbb{C}_w$, the limit by the radius $\tau_\zeta = \{w = t\zeta\},\ 0 \leqslant t \leqslant 1,\ \zeta \in \partial U$ of the function $g(w)$,

$$g^*(\zeta) = \lim_{w\to\zeta, w\in\tau_\zeta} g(w)$$

is called the radial limit, and the limit by the angle $\sphericalangle \subset U$, ending at the point $\zeta \in \sphericalangle$, is called the angular limit,

$$g^*_{\sphericalangle}(\zeta) = \lim_{w\to\zeta, w\in\sphericalangle_\zeta} g(w)\cdot$$

Since lemniscate $L(a,r)$ is a simply connected domain with a real analytic boundary, according to Riemann's theorem there exists a conformal map $\chi(z) : U \to L(a,r)$, which is also conformal in some neighborhood of closure $\bar{U}$. Let χ maps the boundary point $\lambda \in \partial U$ to the boundary point $\zeta \in \partial L(a,r)$. Then the curve $\gamma_\zeta = \chi(\tau_\lambda)$ has the property that it connects points a, ζ and is perpendicular to all lines of level $\partial L(a,\rho) = \{|\psi(a,z)| = \rho\}$, $0 < \rho \leqslant r$. In the theory of $A(z)$-analytic functions, the curve $\gamma_\zeta = \chi(\tau_\lambda)$ plays the role of the radial direction, and the image of the angle $\chi(\sphericalangle)$ plays the role of the angular set at the point $\zeta \in \partial L(a,r)$. We will denote this angle by $\sphericalangle = \sphericalangle_\zeta$. The limit $f^*(\zeta) = \lim\limits_{z\to\zeta, z\in\gamma_\zeta} f(z)$ is called the radial limit, and $f^*_{\sphericalangle}(\zeta) = \lim\limits_{z\to\zeta, z\in\sphericalangle_\zeta} f(z)$ is the angular limit of the function $f(z)$ at the point $\zeta \in \partial L(a,r)$ (see [8]).

Now we will show the smoothness of the boundary of lemniscate $L(a,r)$. For this, we take the automorphism $\chi^{-1}(z) : \overline{L}(a,r) \to \overline{U}$ by Riemann's theorem. Let there be some neighborhood $V = \{\psi(a,\zeta) = re^{i\theta}, |\theta| < \varepsilon\}$ for $\forall\varepsilon > 0$. Also has $\chi^{-1}(V) \subset \partial U$ and $\chi^{-1}(\zeta_0) = \lambda_0 \in \partial U$. Further, there is a diffeomorphism $\pi = -i\ln\chi^{-1}(\zeta) : V \to [-1;1]$. This diffeomorphism represents all boundary points of differentiability of the function $f^*(\zeta)$ and $f^*_{\sphericalangle}(\zeta)$ (see [8]).

Next we introduce the Hardy class for $A(z)$-analytic functions:

Definition 1.2 (see [8]). *The Hardy class H^p, $p > 0$, for $A(z)$-analytic functions is the set of all functions $f(z)$ such that its averages*

$$\frac{1}{2\pi\rho} \int\limits_{|\psi(a,z)|=\rho} |f(z)|^p \big|dz + A(z)d\bar{z}\big| \tag{7}$$

are uniformly bounded for $\rho < r$, i.e. $\sup\limits_{\rho<r}\left\{\dfrac{1}{2\pi\rho}\int\limits_{|\psi(a,z)|=\rho}|f(z)|^p\big|dz + A(z)d\bar{z}\big|\right\} < \infty\cdot$

The Hardy class for $A(z)$-analytic functions in the domain $L(a,r)$ is denoted as $H^p_A(L(a,r))$. The norms in them are defined by the formula (see [8]):

$$\| f \|_{H^p_A} = \sup_{|\psi(a,z)|<r}\left(\frac{1}{2\pi\rho}\int\limits_{|\psi(a,z)|=\rho}|f(z)|^p\big|dz + A(z)d\bar{z}\big|\right)^{\frac{1}{p}} < \infty\cdot$$

Further, from the inequality $b^q < b^p + 1,\ \ 0 < q < p,\ \ b \geqslant 0$ we conclude that $f \in H_A^p$ follows $f \in H_A^q$, i.e. $H_A^p \subset H_A^q$ for all p and q. Let us define a class of bounded functions

$$H_A^\infty(L(a,r)) = \left\{ f(z) \in O_A(L(a,r)) : \sup_{|\psi(a,z)|<r} \{|f(z)|\} < \infty \right\}.$$

The norm in $H_A^\infty(L(a,r))$ is defined as $\| f(z) \|_{H_A^\infty} = \sup_{z \in L(a,r)} \{|f(z)|\}$ (see [8]).

1.4. The Fatou theorems and Cauchy's integral formula for the Hardy class H_A^1

Now, we will consider the Fatou theorem for the class of functions H_A^1 :

Theorem 1.6 (See [8], the Fatou theorem for the class of functions H_A^1)**.** *If $f(z) \in H_A^1(L(a,r))$, then the angular limit*

$$f_{\sphericalangle}^*(\zeta) = \lim_{z \to \zeta, z \in \sphericalangle_\zeta} f(z)$$

exists and is finite for almost all $\zeta \in \partial L(a,r)$, except, perhaps, the points of some set E of measure zero.

The following statements follow from Theorem 1.6:

Theorem 1.7 (see [8])**.** *If $f(z) \in H_A^1(L(a,r))$, then $f^*(\zeta) \in L_A^1(\partial L(a,r))$. As $\rho \to r$*

$$\int\limits_{|\psi(a,z)|=\rho} f(z)\left|dz + A(z)d\bar{z}\right| \longrightarrow \int\limits_{|\psi(a,\zeta)|=r} f^*(\zeta)\left|d\zeta + A(\zeta)d\bar{\zeta}\right| \tag{8}$$

and

$$\int\limits_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|\left|dz + A(z)d\bar{z}\right| \longrightarrow 0. \tag{9}$$

According to Cauchy integral formula (2) for lemniscates $L(a,r)$

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\psi(a,\xi)|=\rho} f(\xi)K(\xi,z)\left(d\xi + A(\xi)d\bar{\xi}\right),$$

we conclude that

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\psi(a,\zeta)|=r} f^*(\zeta)K(\zeta,z)\left(d\zeta + A(\zeta)d\bar{\zeta}\right). \tag{10}$$

This is the Cauchy integral formula for functions of H_A^1.

We show a boundary uniqueness theorem for the Hardy class H_A^1:

Theorem 1.8 (see [8])**.** *Let $f(z) \in H_A^1(L(a,r))$. Suppose that for some set $M \subset \partial L(a,r)$ of positive measure $f^*(\zeta) = 0,\ \forall \zeta \in M$. Then $f(z) \equiv 0$.*

2. Carleman's formula for $A(z)$-analytic functions

2.1. $A(z)$-harmonic measure of a boundary set

For a measurable boundary subset of a lemniscate $L(a,r)$, the $A(z)$-harmonic measure $\omega(z, M, L(a,r))$ is defined very simply, according to the Poisson formula. If

$$\aleph_M(\zeta) = \begin{cases} -1, & \zeta \in M, \\ 0, & \zeta \in \partial L(a,r) \backslash M \end{cases}$$

is a characteristic function of the set $M \subset \partial L(a,r)$, then the $A(z)$-harmonic measure is

$$\omega(z, M, L(a,r)) = \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} P(z,\zeta)\aleph_M(\zeta)\big|d\zeta + A(\zeta)d\bar{\zeta}\big|\cdot \quad (11)$$

Note that the $A(z)$-harmonic measure $\omega(z, M, L(a,r))$ is a $A(z)$-harmonic function inside the lemniscate $L(a,r)$ and

$$-1 \leqslant \omega(z, M, L(a,r)) \leqslant 0\cdot$$

Theorem 2.1 (see [9])**.** *The function $\omega(z, M, L(a,r))$ either does not vanish anywhere, $\omega(z, M, L(a,r)) < 0$, or is identically zero, $\omega(z, M, L(a,r)) \equiv 0$. Moreover, $\omega(z, M, L(a,r)) \equiv 0$ if and only if the bounded set $M \subset \partial L(a,r)$ has measure zero.*

The following theorem is very important in qualitative estimates of $A(z)$-analytic functions.

Theorem 2.2 (see [9])**.** *Let $M \subset \partial L(a,r)$ be a measurable boundary set of positive measure. Then for almost all points $\zeta^0 \in M$ there exist radial (angular) limits $\omega^*(\zeta^0, M, L(a,r)) = -1$.*

2.2. Construction of a quenching function and the Carleman formula in class H_A^1.

Let $D \subset \mathbb{C}$ be a convex domain, $L(a,r) \subset\subset D$ be some lemniscate, on the boundary of which the set $M \subset \partial L(a,r)$ of positive measure is given. The task is to restore the function $f(z) \in H_A^1(L(a,r))$ to $L(a,r)$ by its boundary values given not over the entire boundary $\partial L(a,r)$, as in (10), but only on M. Applying Carleman's simple idea, we will construct a *"quenching" function* that will allow us to get rid of (10) by integrating over $\partial L(a,r)\backslash M$. For this purpose, it is necessary to construct an auxiliary function $\varphi(z) \in H_A^\infty(L(a,r))$ satisfying two conditions (see [9]):

1. $|\varphi^*(\zeta)| = 1$ almost everywhere on $\partial L(a,r)\backslash M$.
2. $|\varphi(z)| > 1$ at $L(a,r)$.

This can be done by constructing the $A(z)$-harmonic measure $\omega(z, M, L(a,r))$ of the boundary set $M \subset \partial L(a,r)$. According to Theorem 2.2, $\omega(z, M, L(a,r)) \in h_A(L(a,r))$, $-1 \leqslant \omega(z, M, L(a,r)) < 0$ and

$$\omega^*(\zeta, M, L(a,r)) = \lim_{z\to\zeta, z\in\sphericalangle} \omega(z, M, L(a,r)) = -1$$

almost everywhere at M and

$$\omega^*(\zeta, \partial L(a,r)\backslash M, L(a,r)) = \lim_{z\to\zeta, z\in\sphericalangle} \omega(z, \partial L(a,r)\backslash M, L(a,r)) = 0$$

almost everywhere at $\partial L(a,r)\backslash M$ (see [9]).

Since $L(a,r) \subset\subset D$ is simply connected, there is an $A(z)$-harmonic function $v(z)$, conjugated to $\omega(z, M, L(a,r))$. Then $\omega(z, M, L(a,r)) + iv(z) = w(z) \in O_A(L(a,r))$. Consider function $\varphi(z) = e^{-w(z)} \in O_A(L(a,r))$. It satisfies the above conditions:

$$|\varphi(z)| = e^{-\omega(z,M,L(a,r))} \leqslant e$$

everywhere in $L(a,r)$, i.e.

$$\varphi(z) \in H_A^\infty(L(a,r)), \quad |\varphi^*(\zeta)| = e^{-\omega^*(\zeta,M,L(a,r))} = e^0 = 1$$

almost everywhere on $\partial L(a,r)\backslash M$ and

$$|\varphi(z)| = e^{-\omega(z,M,L(a,r))} > 1, \quad \forall z \in L(a,r).$$

This function is called the quenching function with respect to the set M (see [9]).

Now we look at the important formula:

Theorem 2.3 (see [9]). *If $f \in H_A^1(L(a,r))$ and $M \subset \partial L(a,r)$ is the set of positive measure, then the formula*

$$f(z) = \frac{1}{2\pi i} \lim_{m\to\infty} \int\limits_M f^*(\zeta) \left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^m K(\zeta, z) \left(d\zeta + A(\zeta)d\bar{\zeta}\right), \tag{12}$$

will be true for any point $z \in L(a,r)$. Moreover, the convergence in (12) will be uniform on compacts from $L(a,r)$.

3. M. M. Lavrentiev's method Carleman's formula for $A(z)$-analytic functions

Let in the set $\partial L(a,r)\backslash M$ with the Cauchy kernel $K(z,\zeta)$ (here $z \in L(a,r)$ is fixed) be approximated by $A(z)$-analytic functions $g_{z,m}(\zeta) \in H_A^\infty(L(a,r))$. These functions are orthogonal to the considered $A(z)$-analytic functions $f \in H_A^1(L(a,r))$ and integration over $\partial L(a,r)$. In addition,

$$\lim_{m\to\infty} \int\limits_{\partial L(a,r)\backslash M} f^*(\zeta)\left(K(z,\zeta) - g_{z,m}(\zeta)\right)\left(d\zeta + A(\zeta)d\bar{\zeta}\right) = 0\cdot \tag{13}$$

We arrive at the following formula:

$$f(z) = \lim_{m\to\infty} \int\limits_M f^*(\zeta)\left(K(z,\zeta) - g_{z,m}(\zeta)\right)\left(d\zeta + A(\zeta)d\bar{\zeta}\right)\cdot \tag{14}$$

In formula (14), z is included as a parameter under the integral sign.

Now we construct the Carleman function for $A(z)$-analytic functions. This idea of obtaining the Carleman formula (12) with kernel approximation can be described using the Carleman concept introduced by M. M. Lavrentyev for $A(z)$-analytic functions. A function of two complex ζ, z and a positive variable α, which we denote by $G(z,\zeta,\alpha)$, is called the Carleman function $A(z)$-analytic set M in the domain D if:

1) $G(z,\zeta,\alpha) = \dfrac{1}{\psi(z,\zeta)} + \tilde{G}(z,\zeta,\alpha)$, where $\tilde{G}$ to ζ is a function of class $H_A^\infty(L(a,r))$;

2) $\dfrac{1}{2\pi} \int\limits_{\partial L(a,r)\backslash M} |G(z,\zeta,\alpha)|\left|d\zeta + A(\zeta)d\bar{\zeta}\right| \leqslant |C(z)|\,\alpha$, where the constant $C(z)$ depends on z.

An example of a generalized Carleman function is the kernel in formula (12):

$$G(z,\zeta,\alpha) = \left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^{\frac{1}{\alpha}} \frac{1}{\psi(z,\zeta)}.$$

And in general, if the Carleman function is $A(z)$-analytic G, then the Carleman formula for $A(z)$-analytic functions is also true:

$$f(z) = \frac{1}{2\pi i} \lim_{\alpha\to 0} \int\limits_M f^*(\zeta) G(z,\zeta,\alpha)\left(d\zeta + A(\zeta)d\bar{\zeta}\right), \tag{15}$$

It is obvious that generalized formulas (14) and (15) are equivalent.

Let us prove the theorem of M. M. Lavrentiev for the $A(z)$-analytic function:

Theorem 3.1. *Let $L(a,r) \subset\subset D$ be a set whose boundary consists of a finite number of closed piecewise smooth Jordan disjoint curves, and let M be an open subset of $\partial L(a,r)$. Then there is a Carleman formula (14) for function $f(z)$ from class $H^1_A(L(a,r))$:*

$$f(z) = \lim_{m\to\infty} \int\limits_M f^*(\zeta)\left(K(z,\zeta) - g_{z,m}(\zeta)\right)\left(d\zeta + A(\zeta)d\bar{\zeta}\right)$$

which we construct using a chain of integrals and expansions in series.

We first check the limit (13) for uniform convergence by representing $g_{z,m}(\zeta)$, the approximating Cauchy kernel $K(z,\zeta)$, as a series. $\forall z \in L(a,r)$, $\zeta \in \partial L(a,r)$ of

$$2\pi i K(z,\zeta) = \frac{1}{\psi(\zeta,z)} = \frac{1}{\zeta - z + \overline{\int\limits_{\gamma(z,\zeta)} \overline{A(\tau)}d\tau}} = \frac{1}{(\zeta-a) + \overline{\int\limits_{\gamma(a,\zeta)} \overline{A(\tau)}d\tau} - (z-a) - \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)}d\tau}} =$$

$$= \frac{1}{\psi(a,\zeta)} \frac{1}{1 - \frac{\psi(a,z)}{\psi(a,\zeta)}} = \sum_{k=0}^{\infty} \frac{(\psi(a,z))^k}{(\psi(a,\zeta))^{k+1}}.$$

The Cauchy kernel $K(z,\zeta)$ is extended to $L(a,r)$ lemniscates as indicated above. Now let us check $g_{z,m}(\zeta)$ functions. From the end of subsection 1 we select a function in the form of a finite series:

$$g_{z,m}(\zeta) = \frac{1}{2\pi i} \sum_{k=0}^{m} \frac{(\psi(a,z))^k}{(\psi(a,\zeta))^{k+1}}.$$

Now we move on to limit

$$\lim_{m\to\infty} \int\limits_M f^*(\zeta)\left(K(z,\zeta) - g_{z,m}(\zeta)\right)\left(d\zeta + A(\zeta)d\bar{\zeta}\right) =$$

$$= \lim_{m\to\infty} \int\limits_M f^*(\zeta)\left(\sum_{k=0}^{\infty} \frac{(\psi(a,z))^k}{(\psi(a,\zeta))^{k+1}} - \sum_{k=0}^{m} \frac{(\psi(a,z))^k}{(\psi(a,\zeta))^{k+1}}\right)\left(d\zeta + A(\zeta)d\bar{\zeta}\right).$$

Now let us prove Theorem 3.1:

Proof. From the generalized Runge theorem (see [10], p. 20) it follows that the compact set $\partial L(a,r)\backslash M$ is, for small $\varepsilon > 0$, a compact set $O_A(L(a,r)_\varepsilon)$ — convex. Consider a sequence of compacts $\mathcal{K}_m, m \in \mathbb{N}, \mathcal{K}_m \subset \mathcal{K}_{m+1}$, $\bigcup\limits_{m=1}^{\infty} \mathcal{K}_m = L(a,r)$ each $\mathcal{K}_m$ is $O_A(L(a,r)_\varepsilon)$ — convex. For example, we choose this compact set in this form:

$$\mathcal{K}_m = L\left(a, \left(1 - \frac{1}{m+1}\right) r\right) = L\left(a, \frac{mr}{m+1}\right).$$

Function $K(z,\zeta)$ is $A(z)$-analytic on $(\partial L(a,r)\backslash M) \times \mathcal{K}_m$, and each function from $O_A\left((\partial L(a,r)\backslash M) \times \mathcal{K}_m\right)$ is uniformly approximated on this compact by functions from $O_A\left(L(a,r)_\varepsilon \times L(a,r)_\varepsilon\right)$. We showed uniform convergence above by representing the $g_{z,m}(\zeta)$

functions as a series. Indeed, from $O_A\left(L(a,r)_\varepsilon\right)$ — the convexity of $(\partial L(a,r)\backslash M)$ and $\mathcal{K}_m$ follows $O_A\left(L(a,r)_\varepsilon \times L(a,r)_\varepsilon\right)$ — the convexity of $(\partial L(a,r)\backslash M)\times\mathcal{K}_m$, which is easy to prove using the double integral Cauchy formula (in the special case of space $\mathbb{C}^2$, formula (4) from [7]):

$$f(z) = \frac{1}{(2\pi i)^2} \int\limits_{(\partial L(a,r)\backslash M)\times\mathcal{K}_m} \frac{f_{m,n}(z,\zeta)\left(d\zeta + A(\zeta)d\bar{\zeta}\right)\wedge(dz + A(z)d\bar{z})}{\psi(z,\zeta)\times\psi(a,z)},$$

applied to the neighborhood of $(\partial L(a,r)\backslash M)\times\mathcal{K}_m$, and the fact that the integral is the limit of integral sums. So, evenly on

$$K(z,\zeta) = \lim_{n\to\infty} f_{m,m(n)}(\zeta,z),$$

where

$$f_{m,m(n)}(\zeta,z) \in O_A\left(L(a,r)_\varepsilon \times L(a,r)_\varepsilon\right).$$

Function $g_{z,m}(\zeta)$ is $A(z)$-analytic on $L(a,r)_\varepsilon \times L(a,r)_\varepsilon$ and Carleman's formula (14) is valid. □

In this proof we actually did not use the openness of M, but only the fact that M contains a neighborhood of some point, so we have

Corollary 3.1. *The statement of Theorem 3.1 holds if M contains at least one point from $\partial L(a,r)$ together with its neighborhood on $\partial L(a,r)$.*

Finally, we note that the $A(z)$-analytic kernel $K(z,\zeta)$ in the Carleman formula for $A(z)$-analytic functions from Theorem 3.1 or Corollary 3.1 is constructed as constructively as one can construct an approximation in Runge's theorem (see [10]), i.e. using a chain of integrals and expansions in series.

M. M. Lavrentiev's method has found important applications in the theory of ill-posed problems of analysis and mathematical physics in [11]. And the method of A. M. Kytmanov appeared in relation to homogeneous domains in $\mathbb{C}^n$ (see [12]).

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Аналог формулы Карлемана для $A(z)$-аналитических функций

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Аннотация. В работе для $A(z)$-аналитических функций из класса Харди доказывается аналог формулы Карлемана. Идея получения формулы Карлемана и понятие функции Карлемана для $A(z)$-аналитических функций из класса Харди принадлежат М. М. Лаврентьеву. В доказательстве формулы Карлемана существенно используются $A(z)$-гармонические функции и формула Пуассона в лемнискатах $L(a, r)$, компактно принадлежащих в рассматриваемой области $D \subset \mathbb{C}$.

Ключевые слова: $A(z)$-аналитическая функция, класс Харди, $A(z)$-лемниската, кратная интегральная формула Коши для $A(z)$-аналитических функций.