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Boundary uniqueness theorem for bounded $A(z)$–analytic functions

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In this paper, we prove an analogue of Fatou's theorem on radial for $A(z)$–analytic functions. The boundary uniqueness theorem for bounded $A(z)$–analytic functions is also proved.

Keywords: Beltrami equation; radial limits; set of positive measure; $A(z)$–lemniscate.

MSC 2020: 30H05, 31A05

1. Introduction

On a class of $A(z)$–analytic functions

Solutions of the Beltrami equation:

$$\frac{\partial f}{\partial \bar{z}} = A(z)\frac{\partial f}{\partial z}. \tag{1}$$

It is directly related to quasi-conformal mappings. With respect to the $A(z)$ function, it is measurable and

$$|A(z)| \le c < 1$$

almost everywhere in the $D \subset \mathbb{C}$ domain under consideration, where c =const. In the literature, the solution of equation (1) is commonly called $A(z)$–analytic functions.

The work of U. Srebro and E. Yakubov [11] which established a local theorem of the existence and uniqueness of homeomorphic solutions of degenerate Beltrami equations, is written in geometric terms.

One of the fundamental works in the theory of Beltrami equations is a monograph by V. Gutlyanskii, U. Srebro and E. Yakubov [15], which considers a geometric approach to the study of the Beltrami equation.

The solutions of equation (1), as well as quasi-conformal homeomorphisms in the complex plane $\mathbb{C}$, have been studied in sufficient details. Here we confine to do ourselves to giving the references ([3], [4], [10], [14]) and formulating the following three theorems:

Theorem 1.1. (see [14]). *For any measurable on the complex plane function $A(z) : ||A||_\infty < 1$ there exists a unique homeomorphic solution $\chi(z)$ of equation (1) which fixes the points $0, 1, \infty$.*

Note that if the function $|A(z)| \le c < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A(z) \equiv 0$ outside D, so Theorem 1.1 holds for any domain $D \subset \mathbb{C}$.

Theorem 1.2. (see [3], [4]). *All generalized solutions of equation (1) have the form $f(z) = F[\chi(z)]$, where $\chi(z)$ is a homeomorphic solution in Theorem 1.1, and $F(z)$ is a holomorphic function in the domain $\chi(D)$. Moreover, if a generalized solution $f(z)$ has isolated singular points, then the holomorphic function $F = f \circ \chi^{-1}$ also has isolated singularities of the same types.*

Theorem 1.2 implies that an $A(z)$–analytic function f carries out an internal (open) mapping, i.e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain $G \subset\subset D$ the maximum of the modulus is reached only on the boundary, i.e. $|f(z)| \leq \max_{z\in\partial G} |f(z)|, z \in G$. If the function is not zero, then the minimum principle also holds, i.e. $|f(z)| \geq \min_{z\in\partial G} |f(z)|, z \in G$.

Theorem 1.3. (see [10]). *If a function $A(z)$ belongs to the class $C^m(D)$, then every solution f of equation (1) also belongs, at least, to the same class $C^m(D)$.*

Let $A(z)$ be anti-analytic, i.e. $\frac{\partial A}{\partial z} = 0$, in $D \subset \mathbb{C}$, and such that $|A(z)| \leq C < 1, \forall z \in D$. We put

$$D_A = \frac{\partial}{\partial z} - \overline{A(z)}\frac{\partial}{\partial \bar z}, \quad \overline{D}_A = \frac{\partial}{\partial \bar z} - A(z)\frac{\partial}{\partial z}.$$

Then, according to (1), the class $A(z)$–analytic functions in D is characterized by the fact that $\overline{D}_A f = 0$. Since an anti-analytic function is smooth, Theorem 1.3 implies that $O_A(D) \subset C^\infty(D)$. In this case, the following takes place:

Theorem 1.4. (analogue of Cauchy's theorem [13]). *If $f \in O_A(D) \cap C(\overline{D})$, where $D \subset \mathbb{C}$ is a domain with rectifiable boundary ∂D, then*

$$\int_{\partial D} f(z)(dz + A(z)d\bar z) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $\zeta \in D$ is a fixed point in it. Consider the function

$$K(z,\zeta) = \frac{1}{2\pi i}\frac{1}{z - \zeta + \overline{\int_{\gamma(\zeta,z)} \overline{A(\tau)}d\tau}}, \tag{2}$$

where $\gamma(\zeta, z)$ is a smooth curve which points of $\zeta, z \in D$. Since the domain is simply connected and the function $\overline{A}(z)$ is holomorphic, the integral

$$I(z) = \overline{\int_{\gamma(a,z)} \overline{A(\tau)}d\tau}$$

does not depend on a path of integration; it coincides with a primitive, i.e. $I'(z) = \overline{A}(z)$.

Theorem 1.5. (see [16]). *$K(z,\zeta)$ is an $A(z)$–analytic function outside of the point $z = \zeta$, i.e. $K(z,\zeta) \in O_A(D\backslash\{\zeta\})$. Moreover, at $z = \zeta$ the function $K(z,\zeta)$ has a simple pole.*

Remark 1.1. (see [16]). *If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function*

$$\psi(z,\zeta) = z - \zeta + \overline{\int_{\gamma(\zeta,z)} \overline{A(\tau)}d\tau},$$

although well defined in D, may have other isolated zeros except for $\zeta : \psi(z,\zeta) = 0$ for $z \in P\backslash\{\zeta, \zeta_1, \zeta_2, ...\}$. Consequently, $\psi \in O_A(D), \psi(z,\zeta) \neq 0$ when $z \notin P$ and $K(z,\zeta)$ is an $A(z)$–analytic function only in $D\backslash P$, it has poles at the points of P. Due to this fact we consider the class of $A(z)$–analytic functions only in convex domains.

According to Theorem 1.2, the function $\psi(z,a) \in O_A(D)$ carries out an internal mapping. In particular, the set

$$L(a,r) = \left\{z \in D : |\psi(z,a)| = \left|z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)}d\tau}\right| < r\right\}$$

is open in D. For suffiently small $r > 0$ it compactly belongs to D and contains the point a. This set is called an $A(z)$–lemniscate with the center a and denoted by $L(a,r)$. According to the maximum principle the lemniscate $L(a,r)$ is simply connected and to the minimum principle it is connected (see [16]).

First we note that the analog power series for $A(z)$–analytic functions will be

$$\sum_{k=0}^{\infty} c_k \psi^k(z,a), a \in D, c_k = const. \tag{3}$$

The domain of convergence of the series (3) is a lemniscate

$$L(a,r) = \{|\psi(z,a)| < r\},$$

where the radius of convergence is given by the Cauchy-Hadamard formula:

$$\frac{1}{r} = \overline{\lim_{k\to\infty}} \sqrt[k]{|c_k|}.$$

There is true an inverse

Theorem 1.6. (see [16]). *If $f(z) \in O_A(L(a,r)) \cap C(L(a,r))$, where $L(a;r) = \{\zeta \in D : |\psi(\zeta,a)| < r\} \subset\subset D$ is lemniscate, then the function $f(z)$ can be expanded to the Taylor series in $L(a,r)$:*

$$f(z) = \sum_{n=0}^{\infty} c_n \psi^n(z,a), \tag{4}$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{\partial L(a,\rho)} \frac{f(\zeta)}{|\psi(\zeta,a)|^{n+1}} (d\zeta + A(\zeta)d\bar{\zeta}), 0 < \rho < r, n = 0,1,...$$

$A(z)$–harmonic and $A(z)$–subharmonic functions

Let $f = u + iv$.

Theorem 1.7. (see [17]). *The real part of the $A(z)$–analytic functions of $f(z) \in O_A(D)$ satisfies equation* $\Delta_A u = \frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z}\right)\right) +$

$$+\frac{\partial}{\partial \bar{z}}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial z} - 2\bar{A}\frac{\partial u}{\partial \bar{z}}\right)\right) = 0 \tag{5}$$

in the domain of D.

In connection with Theorem 1.7, it is natural to define the $A(z)$–harmonic function as follows.

Definition 1.1. (see [17]). *A double differentiable function $u \in C^2(D)$, $u : D \to R$ is called $A(z)$–harmonic in the D domain if the D domain if it satisfies the differential equation* (5).

The class of $A(z)$–harmonic functions in the domain of D is denoted as $h_A(D)$. Thus, the real part and hence the imaginary part, of the $A(z)$–harmonic function in the domain of D. The inverse theorem is also true for simply connected domains.

Theorem 1.8. (see [17]). *If the function is $u(z) \in h_A(D)$, where D is a simply connected domain, then $f \in O_A(D) : u = Re f$.*

For $A(z)$–analytic and $A(z)$–harmonic functions, the following Dirichlet problem is naturally considered:

Dirichlet problem. (see [17]) *A bounded domain of $G \subset D$ is given and a continuous function of $\omega(\zeta)$ is set at the boundary of ∂G. It is required to find $A(z)$–harmonic in the domain of G, continuous on the closure of G the function of $u(z) \in h_A(G) \cap C(\bar{G}) : u|_{\partial G} = \omega$.*

Theorem 1.9. (see [17]). (an analogue of the Poisson formula for $A(z)$–harmonic functions). *If the $\omega(\zeta)$ function is continuous on the boundary of the lemniscate of $L(a,r) \subset D$, then the function*

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(\zeta,a)|=r} \omega(\zeta)\frac{r^2 - |\psi(z,a)|^2}{|\psi(\zeta,z)|^2}|d\zeta + A(\zeta)d\bar{\zeta}| \tag{6}$$

is the solution of the Dirichlet problem in $L(a,r)$.

The $f(\zeta,z) = \frac{\psi(a,\zeta)+\psi(a,z)}{\psi(z,\zeta)}$ function is an $A(z)$–analytic function for $z \in L(a,r)$, where $\zeta \in \partial L(a,r)$. Then

$$P(\zeta,z) = \frac{1}{2\pi}(f(\zeta,z) + \bar{f}(z,\zeta)) = \frac{1}{2\pi}\left(\frac{\psi(a,\zeta)+\psi(a,z)}{\psi(a,\zeta)-\psi(a,z)} + \frac{\bar{\psi}(a,\zeta)+\bar{\psi}(a,z)}{\bar{\psi}(a,\zeta)-\bar{\psi}(a,z)}\right) =$$

$$= \frac{1}{2\pi}\left(\frac{|\psi(a,\zeta)|^2 - |\psi(a;z)|^2}{|\psi(z,\zeta)|^2}\right) = \frac{1}{2\pi}\left(\frac{r^2 - |\psi(a,z)|^2}{|\psi(z,\zeta)|^2}\right).$$

Formula (6) is called an analogue of the Poisson formula for $A(z)$–harmonic functions.

Definition 1.2. (see [18]) *A function $u : D \to [-\infty, \infty)$ is called $A(z)$–subharmonic in a convex domain $G \subset \mathbb{C}$ if it satisfies the following two conditions:*

1) $u(z)$ is semi-continuous from above, i.e. there is inequality

$$\overline{\lim_{w \to z_0}} u(w) \le u(z_0), \tag{7}$$

(It follows that the function is bounded from above on any compact subset of the domain G);

2) for each point $\forall z_0 \in \mathbb{C}$ there is a number $r(z_0) > 0$ such that inequality

$$u(z_0) \le \frac{1}{2\pi r} \int\limits_{|\psi(\zeta, z_0)|<r} u(\zeta)|d\zeta + A(\zeta)d\bar{\zeta}|. \tag{8}$$

holds for all.

In this case, the function $\psi(\zeta, z_0) = \zeta - z_0 + \overline{\int\limits_{\gamma(\zeta,z_0)} \overline{A(\tau)}d\tau}$ for the convex domain G exists and has a single zero at the point (see Theorem 1.5).

A function $u : D \to [-\infty; \infty)$ is called $A(z)$–subharmonic in an arbitrary domain D if it is $A(z)$–subharmonic in any convex subdomain $G \subset D$.

The class $A(z)$–subharmonic in the domain of D functions is denoted by $sh_A(D)$. In the future, for convenience, we will also include the trivial function $u \equiv -\infty$ in $sh_A(D)$. We give an important property of the maximum principle for $A(z)$–subharmonic functions.

The maximum principle for $A(z)$–subharmonic functions. (see [18]) Let $u \in sh_A(D)$ and at some point $z_0 \in D$ it reaches its maximum, then $u|_D \equiv const$.

2. Main part

Radial limits for $A(z)$–analytic functions

The classical theorem was first proved P. Fatou in 1906. This suggestion about radial limits for $A(z)$–analytic functions consists in the following statement:

Theorem 2.1. (an analogue of Fatou's theorem). *If the function $f \in O_A(L(a, r))$ is bounded at $L(a, r)$, then it exists almost everywhere at $\partial L(a, r)$ it has a radial limit of $\lim\limits_{z \to \zeta \in \partial L(a,r)} f(z)$.*

Proof. Decomposing the function in a series: $f(z) = \sum\limits_{n=0}^{\infty} c_n \psi^n(z, a)$. First we show that for a bounded function $f(z) = \sum\limits_{n=0}^{\infty} c_n \psi^n(z, a), z \in \partial L(a, r)$. Assuming $\psi(z, a) = \rho e^{it}$, we have

$$|f(z)|^2 = f(z)\overline{f(z)} = \sum_{n=0}^{\infty} c_n \rho^n e^{itn} \sum_{n=0}^{\infty} \overline{c_n} \rho^n e^{-itn} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} c_j \overline{c}_{n-j} e^{it(2j-n)} \right) \rho^n,$$

where $\rho < r$.

Series $\sum\limits_{n=0}^{\infty} \left(\sum\limits_{j=0}^{n} c_j \overline{c}_{n-j} e^{it(2j-n)} \right) \rho^n$ converges uniformly at $t \in [0, 2\pi]$ and both rows on the right converge absolutely. After slow integration, we get

$$\exists E > 0, \; E^2 \ge \frac{1}{2\pi\rho} \int\limits_{|\psi(z,a)|=\rho} |f(z)|^2 |dz + A(z)d\bar{z}| = \sum_{n=0}^{\infty} |c_n|^2 \rho^2,$$

and therefore

$$\sum_{j=0}^{\infty} |c_n|^2 \rho^{2n} \le E^2.$$

Since this inequality is true for all $r > \rho$, then $\sum\limits_{n=0}^{\infty} |c_n|^2 r^{2n} \leq E^2$.

According to the Riesz-Fischer theorem [9], it follows from condition $\sum\limits_{n=0}^{\infty} |c_n r^n|^2 < \infty$ that $\sum\limits_{n=0}^{\infty} c_n r^n e^{int} < \infty$, there is a Fourier series and there are functions $g(t) \in L^2[0, 2\pi]$ such that $\int\limits_0^{2\pi} \left| \sum\limits_{n=0}^{\infty} c_n r^n e^{int} - g(t) \right|^2 dt = 0$. So this series is summed by the Cesaro method to $g(t)$ for almost all $t \in [0, 2\pi]$, so it is summed by Abel (see [2], [5]). Whence follows the existence of a limit of $t \in [0, 2\pi]$

$$\lim_{z \to \zeta \in \partial L(a,r)} f(z) = \lim_{\rho \to r-0} \sum_{n=0}^{\infty} c_n \rho^n e^{int}$$

for almost all $t \in [0, 2\pi]$. The proposition is proven. □

Let $u(z) = Re f(z)$.

Corollary 2.1. *If the function $u(z)$ is $A(z)$–harmonic and bounded at $L(a,r)$, then it exists almost everywhere at $\partial L(a,r)$ it has a radial limit of*

$$u(\zeta) = \lim_{z \to \zeta \in \partial L(a,r)} u(z),$$

except, perhaps, some set of measure zero.

For $u(z)$ is the real part of a function $f(z)$ which is $A(z)$–harmonic in lemniscate $L(a,r)$. For a finite real number $u(z) < B$ for all $z \in L(a,r)$ the function $\frac{1}{f(z)-B}$ is $A(z)$–analytic and bounded and therefore has radial limits at almost all points $\zeta \in \partial L(a,r)$. The same is therefore true of $f(z)$ and of $u(z) = Re f(z)$.

Boundary uniqueness theorem for bounded $A(z)$–analytic functions

When studying the boundary properties of functions of a complex variable, the fundamental value has the property of uniqueness of its definition by boundary values. Let us first consider this property for $A(z)$–analytic bounded functions. First, we introduce the measure of the boundary of lemniscate $\partial L(a,r)$ and some of its piece:

Let the measurable set be $M \subset \partial L(a,r)$. The lemniscate boundary $\partial L(a,r)$ consists of the following set:

$$\partial L(a,r) = \{\zeta : |\psi(\zeta, a)| = r\}.$$

The measure of this set is equal to the length of the straightening curve $|\psi(\zeta, a)| = r$:

$$\mu(\partial L(a,r)) = \int\limits_{\partial L(a,r)} |d\zeta + A(\zeta) d\bar{\zeta}| = \int\limits_{|\psi(\zeta,a)|=r} |d\psi(\zeta, a)| =$$

$$= \left[\psi(\zeta, a) = re^{i\varphi}, 0 \leq \varphi < 2\pi, |d\psi(\zeta, a)| = r d\varphi\right] = \int\limits_0^{2\pi} r d\varphi = 2\pi r.$$

Hence, for the measure of the lemniscate boundary $\mu(\partial L(a,r))$ the following equation is executed:

$$\mu(\partial L(a,r)) = \int\limits_{\partial L(a,r)} |d\zeta + A(\zeta) d\bar{\zeta}| = 2\pi r.$$

From the additive measure property [6],

$$\mu(\partial L(a,r)) = \int\limits_{\partial L(a,r)} |d\zeta + A(\zeta) d\bar{\zeta}| = \int\limits_{M \cup (\partial L(a,r) \setminus M)} |d\zeta + A(\zeta) d\bar{\zeta}| =$$

$$= \int\limits_M |d\zeta + A(\zeta) d\bar{\zeta}| + \int\limits_{\partial L(a,r) \setminus M} |d\zeta + A(\zeta) d\bar{\zeta}| = \mu(M) + \mu\left(\partial L(a,r) \setminus M\right).$$

Based on this relation, we can define $\mu(M)$ sets by the measure:

$$\mu(M) = \int_M |d\zeta + A(\zeta)d\bar{\zeta}|.$$

Now let's move on to the main statement:

Theorem 2.2. *If the $f(z) \in O_A(L(a,r))$ bounded in the lemniscate $L(a,r)$, tends along the radius to the value zero on the set of points of the border piece $M \subset \partial L(a;r)$ of a positive measure, then $f(z)$ is identically equal to zero.*

The classical case of this theorem was proved by N.N. Luzin and I.I. Privalov.

Proof. We will consider it bounded $|f(z)| < 1$ and $f(z)$ are not identically equal to zero. We may, without loss of generality, suppose that $f(a) \neq 0$; otherwise we just work with the bounded function $\frac{f(z)}{\psi^m(z,a)}$ instead of $f(z)$, where m is the zero order of function f at point $z = a$. The function $u(z) = ln|f(z)| \in sh_A(L(a,r))$ of negative sign, everywhere in the lemniscate $L(a,r)$, except for the zeros of the function $f(z)$, in which $u(z)$ turns into $-\infty$.

Let's draw the boundaries of the lemniscate $|\psi(\xi,a)| = \rho$ of the radius of the $\rho < r$ so that it does not contain zeros of the $f(z)$ function, where $\xi \in \partial L(a,\rho)$. It is always possible to do this, no matter how close ρ is to r, because the set of zeros of the $f(z)$ function does not have a limit point inside the lemniscate $L(a,\rho)$. At this boundary of the lemniscate $|\psi(\xi,a)| = \rho$, the $u(z)$ function takes negative values of $u(\xi)$, where $z \in L(a,\rho)$. We form the Poisson integral

$$\nu_\rho(z) = \frac{1}{2\pi\rho} \int_{|\psi(\xi,a)|=\rho} u(\xi) \frac{\rho^2 - |\psi(z,a)|^2}{|\psi(\xi,z)|^2} |d\xi + A(\xi)d\bar{\xi}|,$$

which will represent the $A(z)$-harmonic function in $L(a,\rho)$, continuous for $L(a,\rho)$.

We form the difference

$$D_\rho(z) = u(z) - \nu_\rho(z). \tag{9}$$

Obviously, $D_\rho(z)$ is a function, $A(z)$-subharmonic inside the lemniscate of radius ρ, with the exception of a finite number of points at which it is equal to $-\infty$, equal to zero on the boundary of $|\psi(\xi,a)| = \rho$. According to the principle of maximum for $A(z)$-subharmonic functions [18] everywhere at $|\psi(\xi,a)| < \rho$ we have:

$$D_\rho(z) \leq 0.$$

Having noticed this, consider the relation (9), writing it in the form:

$$u(z) = \nu_\rho(z) + D_\rho(z). \tag{10}$$

Considering z to be a constant point other than the zeros of $f(z)$, we show that $\nu_\rho(z)$ tends to $-\infty$ at $\rho \to r$; contradiction from the ratio (10). To evaluate the Poisson integral $\nu_\rho(z)$ from above, we write:

$$\nu_\rho(z) < \frac{\rho + |\psi(z,a)|}{\rho - |\psi(z,a)|} \frac{1}{2\pi\rho} \int_{|\psi(\xi,a)|=\rho} u(\xi)|d\xi + A(\xi)d\bar{\xi}|. \tag{11}$$

Since $\lim\limits_{\xi \to \zeta \in M} u(\xi) = -\infty$, where $\mu(M) > 0$. Also, there is an open set $S \subset M$ on which $\mu\left(\zeta \in S : \lim\limits_{\xi \to \zeta} u(\xi) \neq -\infty\right) = 0$. Then, according to D.F. Egorov's theorem, there exists a perfect set $P := M \setminus S$, $\mu(P) > 0$ for $\forall \varepsilon > 0$ whose measure is $\mu(P) = \mu(M) - \varepsilon$. Therefore, there is such a set of $Q \subset \partial L(a,\rho)$ in all $\xi \in Q$, $\lim\limits_{\xi \to \zeta \in P} u(\xi) = -\infty$, uniformly. That is, a negative function $u(\xi)$ uniformly tends to infinity by radius $\xi \to \zeta$, if for any positive number δ there exists such a number N (depending on δ) that for all $\xi \in Q$ and out of $|\psi(\xi,\zeta)| < \delta$ the inequality $u(\zeta) < -N$ is fulfilled. Since this is $\mathbb{C} \simeq \mathbb{R}^2$ we used this case of Egorov's theorem. The uniform convergence of the integrable function implies the transfer of the limit inside the integral. Noticing this form inequality (11) we have:

$$\nu_\rho(z) < \frac{\rho + |\psi(z,a)|}{\rho - |\psi(z,a)|} \left(\int_Q u(\xi)|d\xi + A(\xi)d\bar{\xi}| + \int_{\partial L(a,\rho) \setminus Q} u(\xi)|d\xi + A(\xi)d\bar{\xi}| \right),$$

from where we conclude that $u_\rho(z)$ at $\rho \to r$ tends to $-\infty$, since

$$\lim_{\xi \to \zeta} \frac{1}{2\pi\rho} \int_{Q} u(\xi)|d\xi + A(\xi)d\bar{\xi}| < -\frac{N}{2\pi r} \int_{P} |d\zeta + A(\zeta)d\bar{\zeta}| = -\frac{N}{2\pi r}\mu(M),$$

and the second integral retains negative values. It means $u \equiv -\infty$, so $f \equiv 0$. Thus, the theorem is fully proved. □

Now we give the following the result of the boundary uniqueness theorem in the general case for function bounded in the lemniscate $L(a, r)$.

Corollary 2.2. *Let $A(z)$-analytic functions $f_1(z)$ and $f_2(z)$ and bounded in lemniscate $L(a, r)$. If these functions tend in radii to a value equal to each other on a positive measure set of points of a piece of the boundary $M \subset \partial L(a, r)$, then these functions are identically equal to:*

$$\lim_{z \to \zeta \in M} f_1(z) = \lim_{z \to \zeta \in M} f_2(z) \Rightarrow f_1 \equiv f_2.$$

Proof. Consider the following function as a subtraction of these functions: $f = f_1 - f_2$. This function $f(z)$ is also $A(z)$-analytic and limited in lemniscate $L(a, r)$. According to Theorem 2.2, if the boundaries of the lemniscate on the piece $M \subset \partial L(a, r)$, where $\mu(M) > 0$, the value of the function $f(z)$ along the radius tends to zero, then the function $f(z)$ is identically zero. The corollary is proved. □

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Chegaralangan $A(z)$– analitik funksiyalar uchun chegaraviy yagonalik teoremasi

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Ushbu maqolada $A(z)$–analitik funksiyalar uchun radial limitlar haqidagi Fatu teoremasining analogi isbotlangan. Shuningdek, chegaralangan $A(z)$–analitik funksiyalar uchun chegaraviy yagonalik teoremasi isbotlangan.

Kalit so'zlar: Beltrami tenglamasi; radial limitlar; musbat o'lchovli to'plam; $A(z)$–lemniskata.

Граничная теорема единственности для ограниченных $A(z)$–аналитических функций

Жабборов Насридин, Хусенов Бехзод

В этой статье, мы докажем аналог теоремы Фату о радиальных для $A(z)$–аналитических функций. Также доказана граничная теорема единственности для ограниченных $A(z)$–аналитических функций.

Ключевые слова: Уравнения Бельтрами; радиальные пределах; множестио положительной меры; $A(z)$–лемниската.

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