International Union of Theoretical and Applied Mechanics

IUTAM Symposium on Optimal Guidance and Control for Autonomous Systems

MARCH 15 - 17, 2023
UNIVERSITY OF HAWAIʻI AT MĀNOA
HONOLULU, HI, USA

UNIVERSITY of HAWAIʻI
MĀNOA

IUTAM Symposium on Optimal Guidance and Control for Autonomous Systems
March 15-17, 2023, Honolulu, Hawaii, USA

Generalization of the boundary uniqueness theorem for $A(z)$-analytic functions

Nasridin M. Jabborov*, Behzod E. Husenov#

*Belarus-Uzbek joint intersectoral institute of applied technical qualifications in Tashkent

Temur Malik str., 4, 100171, Tashkent, Uzbekistan

jabborov61@mail.ru

#Bukhara State University

Muhammad Ikbal str. 11, 705018, Bukhara, Uzbekistan

husenovbehzod@mail.ru

Abstract

We consider $A(z)-$analytic functions in case when $A(z)$ is anti-analytic function. In this paper, the Hardy class for $A(z)-$analytic functions is introduced in the convex domain and the boundary uniqueness theorem for these class of functions is given.

Solutions of the Beltrami equation:

$$\frac{\partial f}{\partial \bar z} - A(z)\frac{\partial f}{\partial z} = 0. \tag{1}$$

It is directly related to quasi-conformal maps. The function $A(z)$ is, in general, assumed to be measurable with $|A(z)| \le c < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. Solutions of equation (1) are often referred to as $A(z)-$*analytic functions* in the literature.

The solutions of equation (1), as well as quasi-conformal homeomorphisms in the complex plane $\mathbb{C}$, have been studied in sufficient details. Here we confine ourselves to giving the references [1-4] and formulating the following three theorems:

Theorem 1 ([3]). For any measurable on the complex plane function $A(z): \|A\|_\infty < 1$ there exists a unique homeomorphic solution $\psi(z)$ of equation (1) with fixes the points $0, 1, \infty$.

Note that if the function $|A(z)| \le c < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A(z) \equiv 0$ outside D, so theorem 1 holds for any domain $D \subset \mathbb{C}$.

Theorem 2 ([1,2]). All generalized solutions of equation (1) have the form $f(z) = F[\psi(z)]$, where $\psi(z)$ is a homeomorhic solution in theorem 1, and $F(\zeta)$ is a holomorphic function from ζ to $\psi(D)$. Moreover, if a generalized solution $f(z)$ has isolated singular points, then the holomorphic function $F = f \circ \psi^{-1}$ also has isolated singularities of the same types.

Theorem 2 implies that an $A(z)-$analytic function $f(z)$ carries out an internal (open) mapping, i. e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain D the maximum of the modulus is reached only on the boundary, i. e. $|f(z)| \le \max_{z \in \partial D} |f(z)|, \ z \in D$. If the function is not zero, then the minimum principle also holds, i. e. $|f(z)| \ge \min_{z \in \partial D} |f(z)|, \ z \in D$.

Theorem 3 ([4]). If a function $A(z)$ belongs to the class $C^m(D)$, then every solution $f(z)$ of equation (1) also belongs, at least, to the same class $C^m(D)$.

Let $A(z)$ be anti-analytic, i. e. $\frac{\partial A}{\partial z} = 0$ in $D \subset \mathbb{C}$, and such that $|A(z)| \le c < 1, \ \forall z \in D$. Then according to (1) the class $f \in O_A(D)$ of $A(z)-$analytic functions in D is characterized by the fact that $\overline{D_A} f = \frac{\partial f}{\partial \bar z} - A(z)\frac{\partial f}{\partial z} = 0$. Since an anti-analytic function is smooth, theorem 3 implies that $O_A(D) \subset C^\infty(D)$.

Now we assume that the domain $D \subset £$, is convex, and $a \in D$ is a fixed point in it. Consider the function

$$K(z;a) = \frac{1}{2\pi i}\frac{1}{z-a+\overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)}d\tau}}, \qquad (2)$$

where $\gamma(a;z)$ is a smooth curve which connects points a and z in D. Since the domain is simply connected and the functions $\overline{A}(z)$ is holomorphic, the integral $I(z) = \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)}d\tau}$ does not depend on a path of integration; it coincides with a primitive, i. e. $I(z) = \overline{A}(z)$.

Theorem 4 ([6]). $K(z;a)$ is an $A(z)-$analytic function outside of the point $z = a$, i. e. $K \in O_A(D)$. Moreover, at $z = a$ the function $K(z;a)$ has a simple pole.

Remark 1 ([6]). If the domain $D \subset £$ is not a convex, but only simply connected, then although the function $\psi(z;a)$ is uniquely defined in the D, but a priori, it might has the other isolated zeros except a: $\psi(z;a)$, $z \in P = \{a_1, a_2, ...\}$. Consequently, $\psi \in O_A(D)$, $\psi(z;a) = 0$ when $z \notin P$ and $K(z;a)$ is analytic function only in $D \setminus P$, it has a poles at the points of P. Due to this fact, we consider the class of $A(z)-$analytic functions only in the convex domain $D \subset £$.

Initially, we introduce the Hardy class for $A(z)-$analytic functions. Let the convex domain $D \subset £$ have a smooth boundary ∂D. Then the Hardy classes H^p by definition consist of $A(z)-$analytic functions $f \in O_A(D)$, and $\forall \varepsilon > 0$, for which

$$\overline{\lim_{\varepsilon \to 0}} \int\limits_{\partial D} |f(\zeta - \varepsilon \nu_\zeta)|^p |d\zeta + A(\zeta)d\overline{\zeta}| < \infty, \qquad (3)$$

where ν_ζ is the vector (unit) of the external normal to ∂D at point ζ and $p > 0$. The Hardy class in the domain of D is $A(z)-$analytic functions is denoted as H_A^p.

Now we give a boundary uniqueness theorem for $A(z)-$analytic functions in a convex domain $D \subset £$.

Theorem 5. Let $f \in O_A(D)$. Suppose that M is the set of positive measure on the boundary ∂D, such that $f(\zeta) = 0$ is for $\zeta \in M$. Then the function $f(z)$ is identically equal to zero.

From this theorem, we obtain the following corollaries:

Corollary 1. Let be given a bounded $f(z)$ function and a set M of a positive measure of a piece of boundary ∂D. If the function $f(\zeta)$ is zero for $\zeta \in M$, then $f(z)$ is identically zero.

Corollary 2. Let $f \in H_A^p$ and subset M of a positive measure on the boundaries ∂D. If at $\zeta \in M$ the function is $f(\zeta) = 0$, then $f \equiv 0$.

References

1. B. Bojarski, Homeomorphic solutions of the Beltrami systems, Dokl. Akad. Nauk USSR, 102, 661-664, 1955. (in Russian).

2. B. Bojarski, Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients, Mat. Sb. (N. S.), 43 (85), №4, 451-503, 1957. (in Russian).

3. L. Alfors, Lectures on quassiconformal mappings, Princeton, N. J., Van Nostrand, 1966.

4. I.N. Vekua, Generalized analytic functions, Nauka, Moscow, 1988. (in Russian).

5. L.A. Aizenberg, Carleman's Formulas in Complex Analysis, Springer Science+Business Media B.V., 1993.

6. A. Sadullaev, N.M. Jabborov. On a class of A-analytic functions. Jour. Sib. Fed. Univ. Math. 9-3 (2016), 374-383.