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GOLUSIN-KRYLOV THEOREM FOR $A(z)-$ANALYTIC FUNCTION

Behzod Husenov

Department of Mathematics,
Bukhara State University, Bukhara, Uzbekistan
husenovbehzod@mail.ru

Abstract: The article explores $A(z)-$analytic functions. The Hardy class is given for $A(z)-$analytic functions is proved an analog of theorem Golusin-Krylov and Carleman's formula is given for $A(z)-$analytic functions.

INTRODUCTION

The integral representation of holomorphic functions plays an important role in the classical theory of functions of one complex variable and in multidimensional complex analysis (in the latter case, along with integration over the entire ∂D boundary of the D domain, integration over the Shilov $S=S(D)$ boundary is often encountered). He solves the classical problem. Which are set as follows: to restore a holomorphic function in a D from its values on a certain set $M\subset\partial D$ that does not contain S. Of course, an M must be a set of uniqueness for the considered class of holomorphic functions (for example, for those that are continuous on D o enter the Hardy $H^p(D),\ p\geq 1$ class).

The first result in the direction of solving such a problem was obtained by T. Carleman in 1926 [3] for the $D\subset C^1$ field for one special kind. His idea of introducing a "quenching" function into the Cauchy integral formula was developed by G. M. Golusin and V. I. Krylov in 1933 [4], adherently to simple connected flat domains. Their method provided for the construction of some auxiliary holomorphic function depending on the set of M, which was possible for simply connected domains of the $D\subset C^1$, but generally speaking, it is already possible for multiply connected domains in the C or for the domains of $C^n,\ n>1$. Another metod based on the approximation of the kernel of the integral representation was proposed by M. M. Lavrentiev in 1956 [5]. It turned out that this method works successfully in the cases noted when the Golusin-Krylov approach is not applicable.

Academician A. Sadullayev and his apprentices published an article in 2016 on a class $A(z)-$analytic functions.

ON A CLASS OF $A(z)-$ANALYTIC FUNCTIONS

Let the convex domain of $D\subset C$. It is known if $z=x+iy$, then $\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$. Let $D_A=\frac{\partial}{\partial z}-\overline{A(z)}\frac{\partial}{\partial \bar z},\ \overline{D}_A=\frac{\partial}{\partial \bar z}-A(z)\frac{\partial}{\partial z}$ for function $|A(z)|\leq c<1,\ c=const$.

Definition 1 [7]. If the differentiable function $f:D\to C,\ f\in C^1(D)$ satisfies, the following equality:

$\overline{D}_A f(z)=\frac{\partial f}{\partial \bar z}-A(z)\frac{\partial f}{\partial z}=0$, (1) then the function $f(z)$ is called the $A(z)-$**analytic functions** and denote the $f\in O_A(D)$, where $|A(z)|\leq c<1,\ c=const$ ($A(z)-$antianalytic functions $\frac{\partial A}{\partial \bar z}=0$). Equality (1) is called the Beltrami equation. If in the domain of $D_A f(z)=\frac{\partial f}{\partial z}-\overline{A(z)}\frac{\partial f}{\partial \bar z}=0$, then the $f(z)$ is called the $A(z)-$**antianalytic functions**.

Definition 2 [7]. The following form of the set in domain D: $L(a,r)=\{|\psi(z,a)|=|z-a+\int_{\gamma(\zeta,z)}\overline{A(\tau)}d\tau|<r\}$ is called lemniscate, where $a\in D,\ r>0$.

Theorem 1 (analogue of Cauchy's theorem [6]). If $f(z)\in O_A(D)\cap C(\overline{D})$, where $D\subset C$ is a domain with restifiable boundary ∂D, then

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

In the study of $A(z)-$analytic functions, when the function is $A(z)-$analytic, the

$$K(z;\zeta) = \frac{1}{2\pi i}\frac{1}{z-\zeta+\int_{\gamma(\zeta;z)} \overline{A(\tau)d\tau}} \qquad (2),$$

kernel plays a large role, where $\gamma(\zeta;z)$ is a smooth curve connecting the points of the $\zeta, z \in D.$

Theorem 2 (Cauchy formula [6]). Let $D \subset C$ is an arbitrary convex domain and $G \subset D$ is a subdomain, with piecewise smooth boundary $\partial G.$ Then for any function $f(z) \in O_A(D) \cap C(\bar{G})$ we have a formula

$$f(z) = \int_{\partial G} K(\zeta;z) f(z)(d\zeta + A(\zeta)d\bar{\zeta}),\ z \in G. \qquad (3)$$

Now we briefly dwell on power series in the class of $A(z)-$analytic functions, when $A(z)-$antianalytic is some convex domain of $D \subset C.$ In this case, the function $\psi(z;\zeta)$ has the form of

$$\psi(z;\zeta) = z - \zeta + \int_{\gamma(\zeta;z)} \overline{A(\tau)d\tau} \in O_A(D),$$

In particular, the set $L(a,r) = \{|\psi(z,a)| = |z - a + \int_{\gamma(\zeta;z)} \overline{A(\tau)d\tau}| < r\}$ is open in $D.$ This set is called $A-$lemniscate with center ζ and denoted by $L(\zeta;r).$ It is a simply connected domain. [8]

Definition 3 [7]. The function $f(z)$ is called belonging to the Hardy class $H_A^p(L(a,r)), p \geq 1,$ if the function $f(z)$ is regular and bounded in the lemniscate $L(a,r)$:

$$\exists T > 0, \int_{\partial L(a;r)} |f(z)|^p |dz + A(z)d\bar{z}| \leq T,\ 0 < p < \infty;$$

Hardy H^p space with $0 < p < \infty -$ this is the class of functions in the lemniscate of $L(a,r)$:

$$|f|_{H_A^p} = \sup_{0<R<r}\left(\frac{1}{2\pi R}\int_{\partial L(a;r)} |f(z)|^p |dz + A(z)d\bar{z}|\right)^{\frac{1}{p}} < \infty.$$

We denote $f \in H_A^p(L(a,r)) - f \in O_A(L(a,r))$ and $f \in H^p(L(a,r)), p \geq 1.$

The classic Hardy class case is defined in books [2].

MAIN RESULTS

Let the lemniscate $L(a,r)$ be given. For the $f \in H_A(L(a,r)),\ p = 1$ function, the Cauchy

$$f(z) = \frac{1}{2\pi i}\int_{\partial L(a;r)} \frac{f(z)(d\zeta + A(\zeta)d\bar{\zeta})}{\zeta - z + \int_{\gamma(z;\zeta)} \overline{A(\tau)d\tau}},\ z \in L(a,r). \qquad (4)$$

integral formula holds. Let M be the set of positive Lebesgue measure on the $\partial L(a,r)$ lemniscate. The task is to restore the $f(z)$ of $A(z)-$analytic functions in $L(a,r)$ from boundary values not on the entire boundary of the $\partial L(a,r),$ as in (4), but only on the $M \subset \partial L(a,r).$ For this purpose, it is necessary to construct an auxiliary function of the $\varphi(z) \in H_A^{\infty} L(a,r)$ that satisfies two conditions:

1) $|\varphi(\zeta)| = 1$ almost everywhere on $\partial L(a,r) \setminus M,$
2) $|\varphi(\zeta)| > 1$ in $L(a,r).$

We show an analog of the Golusin-Krylov theorem for an $A(z)-$analytic function.

If $A(z)$ is antianalytic in $L(a,r),$ then the following statement holds.

Theorem 3. If the $f \in H_A(L(a,r))$ and the set $M \subset \partial L(a,r)$ are of positive Lebesgue measure, then for any point of $z \in L(a,r)$ the Carleman formula of

$$f(z) = \frac{1}{2\pi i}\lim_{m\to\infty}\int_M f(\zeta)\left[\frac{\varphi(\zeta)}{\varphi(z)}\right]^m \frac{d\zeta + Ad\bar{\zeta}}{\zeta - z + \int_{\gamma(\zeta;z)} \overline{A(\tau)d\tau}}, \qquad (5)$$

is convergent uniformly on compact sets in $L(a,r).$

Proof. The $f\varphi^m$ function belongs to the Hardy class for the $H_A(L(a,r)),$ here is an $m-$ positive integer. By the Cauchy formula (4)

$$f(z)=\frac{1}{2\pi i}\int_{\partial L(a;r)}\frac{f(z)(d\zeta+A(\zeta)d\overline{\zeta})}{\zeta-z+\int_{\gamma(z;\zeta)}\overline{A(\tau)}d\overline{\tau}},\quad z\in L(a;r),$$

then

$$f(z)=\frac{1}{2\pi i}\int_{M}\frac{f(z)(d\zeta+A(\zeta)d\overline{\zeta})}{\zeta-z+\int_{\gamma(z;\zeta)}\overline{A(\tau)}d\overline{\tau}}+\frac{1}{2\pi i}\int_{\partial L(a;r)\setminus M}\frac{f(z)(d\zeta+A(\zeta)d\overline{\zeta})}{\zeta-z+\int_{\gamma(z;\zeta)}\overline{A(\tau)}d\overline{\tau}}. \qquad (6)$$

It remains to note that the second integral (6) uniformly tends to zero for $m\to\infty$ on every compact set $K\subset L(a,r)$.
The theorem is proved.

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SHORT BIODATA OF THE AUTHOR

Behzod Husenov
He is graduated from the National University of Uzbekistan. His scientific interests are connected with classical theory of functions of one complex variable and on a class of $A(z)-$analytic functions, in particular, Carleman's formula is given for $A(z)-$analytic functions. At the present B. E. Husenov is a teacher of Department of Mathematics at the Bukhara State University.

