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PROPERTIES OF THE $A(z)$–HARMONIC MAJORANT

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We consider $A(z)$–analytic functions in the case when $A(z)$ is an antianalytic function. Here we will introduce $A(z)$–harmonic majorants and we will show their properties.

Keywords: $A(z)$–analytic functions, $A(z)$–harmonic functions, $A(z)$–subharmonic functions, $A(z)$–lemniscata.

Let $A(z)$ be an antianalytic function, i.e. $\frac{\partial A}{\partial z} = 0$ in the convex domain $D \subset \mathbb{C}$; moreover, let $|A(z)| \le c < 1$ for all $z \in D$. The function $f(z)$ is said to be $A(z)$–*analytic* in the domain D if for any $z \in D$, the following equality holds:

$$\frac{\partial f}{\partial \bar{z}} = A(z)\frac{\partial f}{\partial z} \tag{1}$$

We denote by $O_A(D)$ the class of all $A(z)$–analytic functions defined in the domain D.

According to, the function $\psi\left(a,z\right) = z - a + \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)}d\tau}$ is an $A(z)$–analytic function.

The following set is an open subset of arbitrary convex domain D:

$$L\left(a,r\right) = \left\{ \left|\psi\left(a,z\right)\right| = \left| z - a + \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)}d\tau} \right| < r \right\}.$$

For sufficiently small $r > 0$, this set compactly lies in D (we denote this fact by $L(a,r) \subset\subset D$) and contains the point a. This set $L(a,r)$ is called the $A(z)$–lemniscate centered at the point a. The lemniscate $L(a,r)$ is a simply - connected set (see [1]).

Let $f = u + iv$.

Theorem 1. (see [3]). *The real part of the $A(z)$–analytic functions of $f(z) \in O_A(D)$ satisfies equation*

$$\Delta_A u = \frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z}\right)\right) +$$
$$+\frac{\partial}{\partial \bar{z}}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial z} - 2\bar{A}\frac{\partial u}{\partial \bar{z}}\right)\right) = 0 \tag{2}$$

in the domain of D.

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In connection with Theorem 1, it is natural to define the $A(z)$–harmonic function as follows.

Definition 1 (see [3]). *A double differentiable function $u \in C^2(D)$, $u : D \to \mathbb{R}$ is called $A(z)-$harmonic in the D domain if the D domain if it satisfies the differential equation (2).*

The class of $A(z)$–harmonic functions in the domain of D is denoted as $h_A(D)$. Thus, the real part and hence the imaginary part, of the $A(z)$–harmonic function in the domain of D.

Theorem 2 (see [2], on the mean of $A(z)$–harmonic function in lemniscate). *Let D be a convex domain. Then if $u(z)$ is an $A(z)$–harmonic function in lemniscate $L(a,r) \subset D$, then for any $\rho < r$ take place*

$$u(a) = \frac{1}{2\pi\rho} \int\limits_{|\psi(a,\xi)|=\rho} u(\xi)\left|d\xi + A(\xi)d\bar{\xi}\right|, \tag{3}$$

Using the averaging operator (3), we can determine $A(z)$–subharmonic functions.

Definition 2 (see [2]). *The function $u(z) : D \to [-\infty, \infty)$ is called $A(z)$–subharmonic in the domain $D \subset \mathbb{C}$ if it is semi-continuous from above, $\overline{\lim\limits_{w \to z}}\, u(w) \le u(z)$, $\forall z \in D$ and the inequality of average is valid:*

$$u(a) \le \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} u(\zeta)\left|d\zeta + A(\zeta)d\bar{\zeta}\right|,$$

for any fixed point $a \in D$ and for any lemniscate $L(a,r) \subset\subset D$, where $r > 0$.

The class $A(z)$–subharmonic in the domain of D functions is denoted by $sh_A(D)$.

To do this, we need the concepts of harmonic majorants of subharmonic functions.

Definition 3. *Let $v(z) \in sh_A(L(a,r))$. Then $A(z)$–harmonic in lemniscate $L(a,r)$ function $u(z) \in h_A(L(a,r))$ is called the $A(z)$–harmonic majorant of v, if $v(z) \le u(z)$, $\forall z \in L(a,r)$. Note that if $v(z) \in sh_A(D)$ and $L(a,r) \subset\subset D$, then the solution of the Dirichlet problem*

$$\omega(z) \in h_A(L(a,r)), \quad \omega(z)|_{\partial L(a,r)} = v(z)|_{\partial L(a,r)}$$

is the $A(z)$–harmonic majorant of the function $v(z)$, $\omega(z) \ge v(z)$, $\forall z \in L(a,r)$.

Note that if $v(z) \in sh_A(D)$ and $L(a,r) \subset\subset D$, then the solution of the Dirichlet problem $\omega(z) \in h_A(L(a,r))$, $\omega(z)|_{\partial L(a,r)} = v(z)|_{\partial L(a,r)}$ is the $A(z)$–harmonic majorant of the function $v(z)$, $\omega(z) \ge v(z)$, $\forall z \in L(a,r)$.

Theorem 3. *In order to the $A(z)$–subharmonic function $v(z) \in sh_A(L(a,r))$ in lemniscate $L(a,r)$ has a harmonic majorant $u(z) \in h_A(L(a,r))$, $u(z) \geq v(z)$, it is necessary and sufficient that the family of integrals $\int\limits_{\partial L(a,\rho)} |v(z)| \left|dz + A(z)d\bar{z}\right|$ is uniformly bounded, i.e.*

$$\sup_{\rho<r} \left\{ \int\limits_{\partial L(a,\rho)} |v(z)| \left|dz + A(z)d\bar{z}\right| \right\} < \infty \cdot$$

Theorem 4. *Let $u(z) \in h_A(L(a,r))$ be such that $\int\limits_{\partial L(a,\rho)} |u(z)| \left|dz + A(z)d\bar{z}\right|$ is uniformly bounded, $\sup\limits_{\rho<r} \left\{ \int\limits_{\partial L(a,\rho)} |u(z)| \left|dz + A(z)d\bar{z}\right| \right\} < \infty \cdot$ Then $u(z)$ is represented as the difference of two positive $A(z)$–harmonic functions, $u(z) = u_1(z) - u_2(z)$, where $u_1(z) > 0$, $u_2(z) > 0$.*

References

1. *Sadullayev, A., Zhabborov, N.M.* On a class of A–analytic functions // J. Sib. Fed. Univ. Math. Phys., **9**:3 (2016), 374-383

2. *Khursanov, Sh.Y.* Some properties of $A(z)$–subharmonic functions // Bulleten of NUUz, **4**:3 (2020) 474-484.

3. *Zhabborov N.M., Otaboev T.U., Khursanov Sh.Ya.* The Schwartz inequality and the Schwartz formula for A–analytic functions // Journal of Mathematical Sciences, **264**:6 (2022) 703-714.