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Carleman's Formula for $A(z)$-Analytic Functions

N. M. Zhabborov[1*], **N. Kh. Narzillaev**[2**], and **B. E. Husenov**[3***]

(Submitted by A. M. Elizarov)

[1] *Department of Applied Mathematics of Tashkent State University of Economics, Tashkent, 100066 Uzbekistan*

[2] *Faculty of Mathematics of the National University of Uzbekistan, Tashkent, 100174 Uzbekistan*

[3] *Department of Mathematical Analysis of Bukhara State University, Bukhara, 200118 Uzbekistan*

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Abstract—In this paper, we investigate $A(z)$-analytic functions, where $A(z)$ is a function that is antianalytic. The article demonstrates the existence of an $A(z)$-harmonic measure at most points on the boundary of a lemniscate. The main contribution of this paper is the development of a new quenching function for $A(z)$-analytic functions. This quenching function is used to derive Carleman formula for $A(z)$-analytic functions in the Hardy class.

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1. INTRODUCTION

1.1. $A(z)$-Analytic Functions

One of the major challenges in the classical theory of complex analysis is the integral representation of analytic functions, which allows us to recover a function within a domain from its values along the boundary. Additionally, it is natural to inquire how an analytic function may be reconstructed based on its value at a single point on the boundary of a simply-connected domain. In 1926, T. Carleman achieved a significant breakthrough by solving this issue for certain types of domains. He devised a strategy for constructing a "quenching" function in the context of boundary-value problems. Subsequently, this problem was independently resolved by Cauchy and other researchers. G.M. Goluzin and V.I. Krylov further extended Carleman's findings in 1933, employing a specialized holomorphic function to assist with the process, which relies on a portion of the boundary of the domain [2]. However, this technique was only viable for simple domains. Also, this paper provides an overview and extends some boundary properties of the class of holomorphic functions, such as [9, 10].

Let $A(z)$ be antianalytic function, i.e., $\frac{\partial A}{\partial z} = 0$ in the domain $D \subset \mathbb{C}$ and there is a constant $c < 1$ such that $|A(z)| \leq c$ for all $z \in D$. The function $f(z)$ is said to be $A(z)$-*analytic* in the domain D if for any $z \in D$, the following equality holds

$$\frac{\partial f}{\partial \bar{z}} = A(z)\frac{\partial f}{\partial z}. \tag{1}$$

We denote by $O_A(D)$ the class of all $A(z)$-analytic functions defined in the domain D. Since an antianalytic function is infinitely smooth, then $O_A(D) \subset C^{\infty}(D)$ (see [5]). In this case, the following takes place.

*E-mail: jabborov61@mail.ru

**E-mail: n.narzillaev@nuu.uz

***E-mail: b.e.husenov@buxdu.uz

Theorem 1 (see [3], analogue of Cauchy integral theorem). *If $f \in O_A(D) \cap C\left(\bar{D}\right)$, where $D \subset \mathbb{C}$ is a domain with smooth ∂D, then*

$$\int\limits_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $a \in D$ is a fixed point in it. Since the function $\overline{A}(z)$ is analytic, the integral

$$I(z) = \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)}d\tau}$$

is independent of the path of integration; it coincides with the antiderivative $I'(z) = \overline{A}(z)$. Consider the function

$$K(z,a) = \frac{1}{2\pi i}\frac{1}{z-a+I(z)},$$

where $\gamma(a,z)$ is a smooth curve which connects the points $a, z \in D$ (see [5]).

Theorem 2 (see [5]). *$K(z,a)$ is an $A(z)$-analytic function outside of the point $z = a$, i.e., $K(z,a) \in O_A(D\backslash\{a\})$. Moreover, at $z = a$ the function $K(z,a)$ has a simple pole.*

Remark 1 (see [5]). If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function

$$\psi(a,z) = z - a + I(z),$$

although well defined in D, may have other isolated zeros except $a : \psi(a,z) = 0$ for except $z \in P\backslash\{a, a_1, a_2, ...\}$. Consequently, $\psi \in O_A(D)$, $\psi(a,z) \neq 0$ when $z \notin P$ and $K(z,a)$ is an $A(z)$-analytic function only in $D\backslash P$, it has poles at the points of P. Due to this fact we consider the class of $A(z)$-analytic functions only in convex domains.

According to [5], Theorem 1.2, the function $\psi(a,z)$ is an $A(z)$-analytic function.

The following set is an open subset of D:

$$L(a,r) = \{z \in D : |\psi(a,z)| < r\}.$$

For sufficiently small $r > 0$, this set compactly lies in D (we denote it by $L(a,r) \subset\subset D$) and contains the point a. The set $L(a,r)$ is called an $A(z)$-*lemniscate* centered at the point a. The lemniscate $L(a,r)$ is a simply-connected set (see [5]).

Theorem 3 (see [4], Cauchy's integral formula). *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be an arbitrary subdomain with a smooth or piecewise smooth ∂G. Then, for any function $f(z) \in O_A(G) \bigcap C(\bar{G})$, the following formula holds*

$$f(z) = \int\limits_{\partial G} f(\xi)K(z,\xi)\left(d\xi + A(\xi)d\bar{\xi}\right), \quad z \in G. \tag{2}$$

1.2. A(z)-Harmonic and A(z)-Subharmonic Functions

Let $f(z) = u(z) + iv(z)$.

Theorem 4 (see [6]). *The real part $u(z)$ of the functions $f(z) \in O_A(D)$ satisfies the equation*

$$\Delta_A u := \frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left(\left(1+|A|^2\right)\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{1}{1-|A|^2}\left(\left(1+|A|^2\right)\frac{\partial u}{\partial z} - 2\bar{A}\frac{\partial u}{\partial \bar{z}}\right)\right) = 0 \tag{3}$$

in the domain D.

Conversely, if D is a simply connected domain, and a function $u \in C^2(D)$ satisfies the differential equation (3), then there is $u(z) = Ref(z)$.

In connection with Theorem 4, it is natural to define $A(z)$-harmonic functions as follows.

Definition 1 (see [6]). *A function $u \in C^2(D)$, $u : D \to \mathbb{R}$ is called $A(z)$-harmonic if it satisfies in the domain D the differential equation (3).*

The class of $A(z)$-harmonic functions in the domain D is denoted as $h_A(D)$. Thus, the operator $\triangle_A$ in the theory of $A(z)$-harmonic functions plays the same role as Laplace operator $\triangle$ in the theory of harmonic functions. It follows from Theorem 4 that the real and imaginary parts of $A(z)$-analytic function $f = u + iv$ in the domain D are $A(z)$-harmonic functions. The function v is called the $A(z)$-conjugate harmonic function to u.

Theorem 5 (see [7], on the mean of $A(z)$-harmonic function in lemniscate). *Let D be a convex domain. If $u(z)$ is an $A(z)$-harmonic function in lemniscate $L(a, r) \subset D$, then for any $\rho < r$ we have*

$$u(a) = \frac{1}{2\pi\rho} \int\limits_{|\psi(a,\xi)|=\rho} u(\xi)\left|d\xi + A(\xi)d\bar{\xi}\right|, \tag{4}$$

$$u(a) = \frac{1}{\pi\rho^2} \iint\limits_{|\psi(a,\xi)|\le\rho} u(\xi)\left(1 - |A(\xi)|^2\right) \frac{d\xi \wedge d\bar{\xi}}{2i}. \tag{5}$$

Theorem 6 (see [6], analogue of the Poisson formula for $A(z)$-harmonic functions). *If the function $\omega(\zeta)$ is continuous on the boundary of the lemniscate $L(a, r)$, then the function*

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} \omega(\zeta) \frac{r^2 - |\psi(a, z)|^2}{|\psi(\zeta, z)|^2} \left|d\zeta + A(\zeta)d\bar{\zeta}\right| \tag{6}$$

is the solution of the Dirichlet problem in $L(a, r)$, i.e., $u(z) \in h_A(L(a, r)) \cap C\left(\bar{L}(a, r)\right) : u(z)|_{\partial L(a,r)} = \omega(\zeta)$. Conversely, any function $u(z) \in h_A(L(a, r)) \cap C(\bar{L}(a, r))$ is represented in $L(a, r)$ by the Poisson integral

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} u(\zeta) \frac{r^2 - |\psi(a, z)|^2}{|\psi(\zeta, z)|^2} \left|d\zeta + A(\zeta)d\bar{\zeta}\right|, \quad z \in L(a, r). \tag{7}$$

Formulas (6) and (7) are analogues of the Poisson formula for $A(z)$-harmonic functions and $P(z, \zeta) = \frac{r^2 - |\psi(a,z)|^2}{|\psi(\zeta,z)|^2}$ is the Poisson kernel for $A(z)$-harmonic functions.

Using the averaging operator (4), we can determine $A(z)$-subharmonic functions.

Definition 2 (see [7]). *The function $u(z) : D \to [-\infty, \infty)$ is called $A(z)$-subharmonic in the domain $D \subset \mathbb{C}$ if it is semi-continuous from above, i.e., $\overline{\lim\limits_{w\to z}} u(w) \le u(z)$, $\forall z \in D$ and the inequality of average is valid*

$$u(a) \le \frac{1}{2\pi r} \int\limits_{|\psi(a,\zeta)|=r} u(\zeta)\left|d\zeta + A(\zeta)d\bar{\zeta}\right|$$

for any fixed point $a \in D$ and for any lemniscate $L(a, r) = \{|\psi(a, z)| < r\} \subset\subset D$, where $r > 0$.

The class of $A(z)$-subharmonic in D functions is denoted by $sh_A(D)$.

1.3. Angular Limits and Hardy Classes for $A(z)$-analytic Functions

Let $L(a,r) \subset\subset D$. First, we define the concepts of angular and radial limits of $A(z)$-subharmonic and $A(z)$-analytic functions in lemniscate $L(a,r)$. The radial limits of the function $f(z)$ at some point $\zeta \in \partial L(a,r)$ are denoted as $f^*(\zeta)$, and the angular limits are denoted as $f^*_{\sphericalangle}(\zeta)$ (see [8]).

In the classical case of the disk $U = \{w \in \mathbb{C} : |w| < 1\} \subset \mathbb{C}_w$, the limit by the radius $\tau_\zeta = \{w = t\zeta\}$, $0 \leq t \leq 1$, $\zeta \in \partial U$ of the function $g(w)$,

$$g^*(\zeta) = \lim_{w \to \zeta, w \in \tau_\zeta} g(w)$$

is called the radial limit, and the limit by the angle $\sphericalangle \subset U$, ending at the point $\zeta \in \sphericalangle$, is called the angular limit,

$$g^*_{\sphericalangle}(\zeta) = \lim_{w \to \zeta, w \in \sphericalangle_\zeta} g(w).$$

Since lemniscate $L(a,r)$ is a simply connected domain with a real analytic boundary, then according to Riemann's theorem there exists a conformal map $\chi(z) : U \to L(a,r)$, which is also conformal in some neighborhood of closure $\bar{U}$. Let χ maps the boundary point $\lambda \in \partial U$ to the boundary point $\zeta \in \partial L(a,r)$. Then, the curve $\gamma_\zeta = \chi(\tau_\lambda)$ has the property that it connects points a, ζ and is perpendicular to all lines of level $\partial L(a,\rho) = \{|\psi(a,z)| = \rho\}, 0 < \rho \leq r$. In the theory of $A(z)$-analytic functions, the curve $\gamma_\zeta = \chi(\tau_\lambda)$ plays the role of the radial direction, and the image of the angle $\chi(\sphericalangle)$ plays the role of the angular set at the point $\zeta \in \partial L(a,r)$. We will denote this angle by $\sphericalangle = \sphericalangle_\zeta$. The limit $f^*(\zeta) = \lim\limits_{z \to \zeta, z \in \gamma_\zeta} f(z)$ is called the radial limit, and $f^*_{\sphericalangle}(\zeta) = \lim\limits_{z \to \zeta, z \in \sphericalangle_\zeta} f(z)$ is the angular limit of the function $f(z)$ at the point $\zeta \in \partial L(a,r)$ (see [8]).

Now we will show the smoothness of the boundary of lemniscate $L(a,r)$. For this, we take automorphism $\chi^{-1}(z) : \overline{L}(a,r) \to \overline{U}$ by Riemann's theorem. Let there be some neighborhood $V = \{\psi(a,\zeta) = re^{i\theta}, |\theta| < \varepsilon\}$ for $\forall \varepsilon > 0$. Also has $\chi^{-1}(V) \subset \partial U$ and $\chi^{-1}(\zeta_0) = \lambda_0 \in \partial U$. Further, there is a diffeomorphism $\pi = -i \ln \chi^{-1}(\zeta) : V \to [-1;1]$. This diffeomorphism represents all boundary points of the differentiability of the function $f^*(\zeta)$ and $f^*_{\sphericalangle}(\zeta)$.

Next, we introduce the Hardy class for $A(z)$-analytic functions.

Definition 3 (see [8]). *The Hardy class $H^p, p > 0$, for $A(z)$-analytic functions is the set of all functions $f(z)$ such that its averages*

$$\frac{1}{2\pi\rho} \int\limits_{|\psi(a,z)|=\rho} |f(z)|^p \left|dz + A(z)d\bar{z}\right| \tag{8}$$

are uniformly bounded for $\rho < r$, i.e., $\sup\limits_{\rho<r} \left\{ \frac{1}{2\pi\rho} \int\limits_{|\psi(a,z)|=\rho} |f(z)|^p \left|dz + A(z)d\bar{z}\right| \right\} < \infty.$

The Hardy class for $A(z)$-analytic functions in the domain $L(a,r)$ is denoted as $H^p_A(L(a,r))$. The norms in them are defined by the formula (see [8])

$$\| f \|_{H^p_A} = \sup_{|\psi(a,z)|<r} \left(\frac{1}{2\pi\rho} \int\limits_{|\psi(a,z)|=\rho} |f(z)|^p \left|dz + A(z)d\bar{z}\right| \right)^{\frac{1}{p}} < \infty.$$

Further, from the inequality $b^q < b^p + 1, 0 < q < p, b \geq 0$ we conclude that $f \in H^p_A$ follows $f \in H^q_A$, i.e., $H^p_A \subset H^q_A$ for all p and q. Let us define a class of bounded functions

$$H^\infty_A(L(a,r)) = \left\{ f(z) \in O_A(L(a,r)) : \sup_{|\psi(a,z)|<r} \{|f(z)|\} < \infty \right\}.$$

The norm in $H^\infty_A(L(a,r))$ is defined as $\| f(z) \|_{H^\infty_A} = \sup\limits_{z \in L(a,r)} \{|f(z)|\}$ (see [8]).

1.4. The Fatou's Theorems and Cauchy's Integral Formula for Hardy Class H_A^1

Now, we will consider the Fatou's theorem for the class of functions H_A^1.

Theorem 7 (see [8], the Fatou's theorem for the class of functions H_A^1). *If $f(z) \in H_A^1(L(a,r))$, then the angular limit*

$$f^*_{\triangleleft}(\zeta) = \lim_{z\to\zeta, z\in\triangleleft_\zeta} f(z)$$

exists and is finite for almost all $\zeta \in \partial L(a,r)$, except, perhaps, the points of some set E of measure zero.

The following statements follow from Theorem 7.

Theorem 8 (see [8]). *If $f(z) \in H_A^1(L(a,r))$, then $f^*(\zeta) \in L_A^1(\partial L(a,r))$. As $\rho \to r$*

$$\int\limits_{|\psi(a,z)|=\rho} f(z)\left|dz + A(z)d\bar{z}\right| \;\longrightarrow\; \int\limits_{|\psi(a,\zeta)|=r} f^*(\zeta)\left|d\zeta + A(\zeta)d\bar{\zeta}\right| \tag{9}$$

and

$$\int\limits_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|\left|dz + A(z)d\bar{z}\right| \longrightarrow 0. \tag{10}$$

According to Cauchy integral formula (2), for lemniscates $L(a,r)$

$$f(z) = \frac{1}{2\pi i}\int\limits_{|\psi(a,\xi)|=\rho} f(\xi)K(\xi,z)\left(d\xi + A(\xi)d\bar{\xi}\right),$$

we conclude that

$$f(z) = \frac{1}{2\pi i}\int\limits_{|\psi(a,\zeta)|=r} f^*(\zeta)K(\zeta,z)\left(d\zeta + A(\zeta)d\bar{\zeta}\right). \tag{11}$$

This is the Cauchy integral formula for functions of H_A^1.

We show a boundary uniqueness theorem for the Hardy class H_A^1.

Theorem 9 (see [8]). *Let $f(z) \in H_A^1(L(a,r))$. Suppose that for some set $M \subset \partial L(a,r)$ of positive measure $f^*(\zeta) = 0\ \forall\zeta \in M$. Then, $f(z) \equiv 0$.*

2. CARLEMAN'S FORMULA FOR $A(z)$-ANALYTIC FUNCTIONS

2.1. $A(z)$-Harmonic Measure of a Boundary Set

For a boundary measurable subset of a lemniscate $L(a,r)$, the $A(z)$-harmonic measure $\omega(z, M, L(a,r))$ is defined very simply, according to the Poisson formula. If

$$\aleph_M(\zeta) = \begin{cases} -1, & \zeta \in M, \\ 0, & \zeta \in \partial L(a,r)\backslash M \end{cases}$$

is a characteristic function of the set $M \subset \partial L(a,r)$, then the $A(z)$-harmonic measure is

$$\omega(z, M, L(a,r)) = \frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r} P(z,\zeta)\aleph_M(\zeta)\left|d\zeta + A(\zeta)d\bar{\zeta}\right|. \tag{12}$$

Note that the $A(z)$-harmonic measure $\omega(z, M, L(a,r))$ is a $A(z)$-harmonic function inside the lemniscate $L(a,r)$ and

$$-1 \le \omega(z, M, L(a,r)) \le 0.$$

Theorem 10. *The function $\omega(z,M,L(a,r))$ either does not vanish anywhere, $\omega(z,M,L(a,r))<0$, or is identically zero, $\omega(z,M,L(a,r))\equiv 0$. Moreover, $\omega(z,M,L(a,r))\equiv 0$ if and only if the bounded set $M\subset\partial L(a,r)$ has measure zero.*

Proof. If $\omega(z^0,M,L(a,r))=0$ at some inner point $z^0\in L(a,r)$, then according to the maximum principle for $A(z)$—harmonic functions $\omega(z,M,L(a,r))\equiv 0$. Now, if mes$M=0$, then by definition

$$\omega(z,M,L(a,r))=\frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r}P(z,\zeta)\aleph_M(\zeta)\big|d\zeta+A(\zeta)d\bar{\zeta}\big|=-\frac{1}{2\pi r}\int\limits_{M}P(z,\zeta)\big|d\zeta+A(\zeta)d\bar{\zeta}\big|=0.$$

If mes$M>0$, then $\omega(a,M,L(a,r))<0$, i.e., $\omega(z,M,L(a,r))\neq 0$ in $L(a,r)$. The theorem is proved. □

The following theorem is very important in qualitative estimates of $A(z)$-analytic functions.

Theorem 11. *Let $M\subset\partial L(a,r)$ be a measurable boundary set of positive measure. Then, for almost all points $\zeta^0\in M$ there exist radial (angular) limits $\omega^*(\zeta^0,M,L(a,r))=-1$.*

Proof. According to (12),

$$\omega(z,M,L(a,r))=\frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r}P(z,\zeta)\aleph_M(\zeta)\big|d\zeta+A(\zeta)d\bar{\zeta}\big|,$$

where

$$\aleph_M(\zeta)=\begin{cases}-1, & \zeta\in M,\\ 0, & \zeta\in\partial L(a,r)\backslash M\end{cases}$$

is the characteristic function of the measurable set $M\subset\partial L(a,r)$. We put

$$d\mu(\zeta)=\aleph(\zeta)\Big|d\zeta+A(\zeta)d\bar{\zeta}\Big|.$$

Then, the measure μ will be finite on $\partial L(a,r)$, the measure of the arc $l_t=\{\psi(a,\zeta)=re^{i\tau},0\leq\tau\leq t\}$ is

$$\mu(t)=\int\limits_{l_t}\big|d\mu(\tau)\big|.$$

Using this formula, we parameterize $\zeta\in\partial L(a,r)$ through $-\pi<t\leq\pi$, such that $\psi(a,\zeta)=re^{it}$, and denote $\mu(\zeta)$ through $\mu(t)$. According to the Radon–Nicodim theorem (see [1]), there is a derivative $\mu'(\zeta)=\mu'(t)$ at almost all points $\zeta\in\partial L(a,r)$, $\psi(a,\zeta)=re^{it}$. We fix a point $-\pi<t_0\leq\pi$, $\psi(a,\zeta_0)=re^{it_0}$ such that there is a finite derivative $\mu'(\zeta_0)=\mu'(t_0)$ and show that

$$\omega(z,M,L(a,r))=\frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r}P(z,\zeta)d\mu(\zeta)=\frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r}\frac{r^2-|\psi(a,z)|^2}{|\psi(\zeta,z)|^2}d\mu(\zeta)$$

tends to $\mu'(\zeta_0)$ with z tending to ζ_0, inside any angle $\sphericalangle_{\zeta_0}$,

$$\lim_{z\to\zeta_0,z\in\sphericalangle_{\zeta_0}}\omega(z,M,L(a,r))=\mu'(\zeta_0).$$

Without lost of generality, for simplicity, assume that $\psi(a,\zeta_0)=re^{it_0}=r$, $t_0=0$. We need to show that

$$\frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r}P(z,\zeta)d\mu(\zeta)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t)\to\mu'(0)=-1,$$

at $\psi(a,z)=\rho e^{i\theta}$, $\rho e^{i\theta}\in\sphericalangle_{\zeta_0}$, $\rho\to r$.

Take $d\nu(t) = d\mu(t) - \frac{\mu'(0)}{r}\left|d\zeta + A(\zeta)d\bar{\zeta}\right|$. Then, $\nu'(0) = 0$ and for any $\varepsilon > 0$ there exists $\delta > 0$, such that $|\nu(t)| \leq \varepsilon|t|$ for the arc $l_\delta = \{\psi(a,\zeta) = re^{it}, -\delta \leq t \leq \delta\}$. We have

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t)$$

$$= \frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t) + \frac{1}{2\pi}\int\limits_{-\pi+\delta}^{\pi-\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t).$$

At $\rho \to r$ the last integral here tends to zero, because $r^2 - \rho^2 \to 0$ and

$$|r^2 - 2\rho r\cos(t-\theta) + \rho^2| \geq c_\delta > 0,$$

so, for ρ sufficiently close to r,

$$\left|\frac{1}{2\pi}\int\limits_{-\pi+\delta}^{\pi-\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t)\right| < \varepsilon$$

is fulfilled.

Next,

$$\frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t) = \frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\nu(t)$$

$$+ \frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}\left[\mu'(0)\frac{\left|d\zeta + A(\zeta)d\bar{\zeta}\right|}{r}\right]$$

$$= \frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\nu(t) + \mu'(0),$$

where

$$\left|\frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\nu(t)\right| \leq \frac{\varepsilon}{\pi}\int\limits_{0}^{\delta}\left|\frac{1}{2\pi}\int\limits_{-\delta}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}tdt\right|.$$

Direct calculation shows that at $\rho e^{i\theta} \in \sphericalangle_{\zeta_0}$ the next integral is bounded,

$$\int\limits_{0}^{\delta}\left|\frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}\right|tdt \leq \tilde{c} = \text{const}.$$

Hence,

$$\left|\frac{1}{2\pi}\int\limits_{0}^{\delta} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t) - \mu'(0)\right| \leq \varepsilon + \tilde{c}\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, then at $\psi(a,z) = \rho e^{i\theta}$, $\rho e^{i\theta} \in \sphericalangle_{\zeta_0}, \rho \to r$,

$$\frac{1}{2\pi r}\int\limits_{|\psi(a,\zeta)|=r} P(z,\zeta)d\mu(\zeta) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{r^2-\rho^2}{r^2-2\rho r\cos(t-\theta)+\rho^2}d\mu(t) \to \mu'(0) = -1.$$

The theorem is proved. □

2.2. Construction of a Quenching Function

Let $D \subset \mathbb{C}$ be a convex domain, $L(a, r) \subset\subset D$ be some lemniscate, on the boundary of which the set $M \subset \partial L(a, r)$ of positive measure is given. The task is to restore the function $f(z) \in H^1_A(L(a, r))$ to $L(a, r)$ by its boundary values given not over the entire boundary $\partial L(a, r)$, as in (11), but only on M. Applying Carleman's simple idea, we will construct a *"quenching" function* that will allow us to get rid of (11) by integrating over $\partial L(a, r)\backslash M$. For this purpose, it is necessary to construct an auxiliary function $\varphi(z) \in H^\infty_A(L(a, r))$ satisfying two conditions:

1. $|\varphi^*(\zeta)| = 1$ almost everywhere on $\partial L(a, r)\backslash M$.
2. $|\varphi(z)| > 1$ at $L(a, r)$.

This can be done by constructing the $A(z)$-harmonic measure $\omega(z, M, L(a, r))$ of the boundary set $M \subset \partial L(a, r)$. According to Theorem 10, $\omega(z, M, L(a, r)) \in h_A(L(a, r))$, $-1 \leq \omega(z, M, L(a, r)) < 0$ and

$$\omega^*(\zeta, M, L(a, r)) = \lim_{z\to\zeta, z\in\sphericalangle} \omega(z, M, L(a, r)) = -1$$

almost everywhere at M and

$$\omega^*(\zeta, \partial L(a, r)\backslash M, L(a, r)) = \lim_{z\to\zeta, z\in\sphericalangle} \omega(z, \partial L(a, r)\backslash M, L(a, r)) = 0$$

almost everywhere at $\partial L(a, r)\backslash M$.

Since $L(a, r) \subset\subset D$ is simply connected, there is an $A(z)$-harmonic function $v(z)$, conjugated to $\omega(z, M, L(a, r))$. Then, $\omega(z, M, L(a, r)) + iv(z) = w(z) \in O_A(L(a, r))$. Consider function $\varphi(z) = e^{-w(z)} \in O_A(L(a, r))$. It satisfies the above conditions

$$|\varphi(z)| = e^{-\omega(z,M,L(a,r))} \leq e$$

everywhere in $L(a, r)$, i.e.,

$$\varphi(z) \in H^\infty_A(L(a, r)), \quad |\varphi^*(\zeta)| = e^{-\omega^*(\zeta,M,L(a,r))} = e^0 = 1$$

almost everywhere on $\partial L(a, r)\backslash M$ and

$$|\varphi(z)| = e^{-\omega(z,M,L(a,r))} > 1 \quad \forall z \in L(a, r).$$

This function is called the quenching function with respect to the set M.

2.3. The Carleman Formula in Class H^1_A

Now we prove an the Carleman formula.

Theorem 12. *If $f \in H^1_A(L(a, r))$ and $M \subset \partial L(a, r)$ is the set of positive measure, then the formula*

$$f(z) = \frac{1}{2\pi i} \lim_{m\to\infty} \int\limits_M f^*(\zeta) \left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^m K(\zeta, z) \left(d\zeta + A(\zeta)d\bar{\zeta}\right), \tag{13}$$

will be true for any point $z \in L(a, r)$. Moreover, the convergence in (13) will be uniform on compacts from $L(a, r)$.

Proof. Note that for any fixed $m \in \mathbb{N}$, the function $f\varphi^m$ belongs to the Hardy class $H^1_A(L(a, r))$. Applying the Cauchy formula (11), we get

$$f(z)\varphi^m(z) = \frac{1}{2\pi i} \lim_{m\to\infty} \int\limits_{\partial L(a,r)} f^*(\zeta) \left[\varphi^*(\zeta)\right]^m K(\zeta, z) \left(d\zeta + A(\zeta)d\bar{\zeta}\right), \tag{14}$$

further

$$f(z) = \frac{1}{2\pi i} \lim_{m\to\infty} \int\limits_{\partial L(a,r)\backslash M} f^*(\zeta) \left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^m K(\zeta, z) \left(d\zeta + A(\zeta)d\bar{\zeta}\right)$$

$$+\frac{1}{2\pi i}\lim_{m\to\infty}\int\limits_M f^*(\zeta)\left[\frac{\varphi^*(\zeta)}{\varphi(z)}\right]^m K(\zeta,z)\left(d\zeta+A(\zeta)d\bar{\zeta}\right), \tag{15}$$

where $z\in L(a,r)$. As $m\to\infty$, the first integral in (15) tends to zero, because $|\varphi^*(\zeta)|=1$ almost everywhere at $\partial L(a,r)\backslash M$, and

$$|\varphi(z)|=e^{-\omega(z,M,L(a,r))}>1,\quad \forall z\in L(a,r).$$

Thus, the formula (13) has. The theorem is proved. □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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