# Generalization of the Boundary Uniqueness Theorem for $A(z)$–Analytic Functions

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**Abstract.** We consider $A(z)$–analytic functions in case when $A(z)$ is antianalytic function. In this paper, the Nevanlinna class for $A(z)$–analytic functions is are introduced and for these classes, the boundary values of the function are investigated. For the Nevanlinna class of functions, an analogue of Fatou's theorem was proved as a proposition to show that the function has a value on the boundary of the domain. Also, the Privalov's ice-cream cone consruction is introduced for $A(z)$–analytic functions and Egoroff's theorems are applied for them. As the main result, the analog generalized boundary uniqueness theorem for $A(z)$–analytic functions is proven and the boundary uniqueness theorem for Nevanlinna classes of functions are given as a corollary.

**Keywords:** $A(z)$–analytic functions, $A(z)$–lemniscate, the angular limit for $A(z)$–analytic functions, the Nevanlinna class for $A(z)$–analytic functions, the analog generalized boundary uniqueness theorem for $A(z)$–analytic functions.

## 1 Introduction

Let $A(z)$ be antianalytic function, i.e. $\frac{\partial A}{\partial z} = 0$ in the domain $D \subset \mathbb{C}$; moreover, let $|A(z)| \leq c < 1$ for all $\forall z \in D$, where $c$ =const. The function $f(z)$ is said to be $A(z)$–analytic in the domain $D$ if for any $z \in D$, the following equality holds:

$$\frac{\partial f}{\partial \bar{z}} = \overline{A}(z)\frac{\partial f}{\partial z}. \tag{1}$$

We denote by $O_A(D)$ the class of all $A(z)$–analytic functions defined in the domain $D$. Since an antianalytic function is infitely smooth, $O_A(D) \subset C^\infty(D)$ (see [7]). In this case, the following takes place:

**Theorem 1.** *(analogue of Cauchy's theorem [5]). If $f \in O_A(D) \cap C(\bar{D})$, where $D \subset \mathbb{C}$ is a domain with rectifiable boundary $\partial D$, then*

$$\int\limits_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $\xi \in D$ is a fixed point in it. Consider the function

$$K(z,\xi) = \frac{1}{2\pi i} \frac{1}{z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \overline{A(\tau)} d\tau}},$$

where $\gamma(\xi, z)$ is a smooth curve which points of $\xi, z \in D$. Since the function $\overline{A}(z)$ is analytic and the domain $D$ is simply-connected, the integral

$$I(z) = \overline{\int\limits_{\gamma(\xi,z)} \overline{A(\tau)} d\tau}$$

is independent of the path of integration; it coincides with the antiderivative $I'(z) = \overline{A}(z)$ (see [7]).

**Theorem 2.** *(see [7]). $K(z,\xi)$ is an $A(z)$–analytic function outside of the point $z = \xi$, i.e. $K(z,\xi) \in O_A(D\backslash\{\xi\})$. Moreover, at $z = \xi$ the function $K(z,\xi)$ has a simple pole.*

**Remark 1.** (see [7]). *If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function*

$$\psi(z,\xi) = z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \overline{A(\tau)} d\tau},$$

*although well defined in $D$, may have other isolated zeros except for $\zeta : \psi(z,\xi) = 0$ for $z \in P\backslash\{\xi, \xi_1, \xi_2, ...\}$. Consequently, $\psi \in O_A(D)$, $\psi(z,\xi) \neq 0$ when $z \notin P$ and $K(z,\xi)$ is an $A(z)-$analytic function only in $D\backslash P$, it has poles at the points of $P$. Due to this fact we consider the class of $A(z)$–analytic functions only in convex domains.*

According to [7, Theorem 2], the function

$$\psi(z,a) = z - a + \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)} d\tau}$$

is an $A(z)$–analytic function.

The following set is an open subset of $D$:

$$L(a,r) = \left\{ z \in D : |\psi(z,a)| = |z - a + \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)} d\tau}| < r \right\}$$

For suffiently small $r > 0$, this set compactly lies in $D$ (we denote this fact by $L(a,r) \subset\subset D$) and contains the point $a$. The set $L(a,r)$ is called an $A(z)$–lemniscate with the centered at the point $a$. The lemniscate $L(a,r)$ is a simply-connected set (see [7]).

**Theorem 3.** *(see [6] Cauchy formula). Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be an arbitrary subdomain with a smooth or piecewise smooth boundary $\partial G$ that compactly lies in $D$. Then for any function $f(z) \in O_A(G) \bigcap C(\bar{G})$, the following formula holds:*

$$f(z) = \int_{\partial G} f(\xi) K(z, \xi) \left( d\xi + A(\xi) d\bar{\xi} \right), \quad z \in G \cdot \tag{2}$$

## 2 Some classes and concepts for $A(z)$–analytic functions

### 2.1 Angular limit for $A(z)$–analytic functions.

Initially, we introduce an angular limit for $A(z)$–analytic functions. Let $L(a,r) \subset\subset D$ and $f(z) \in O_A(L(a,r))$.

**Definition 1.** *Let's put $\psi(z,a) = \rho e^{i\theta}$ and $\psi(\zeta_0, a) = re^{i\phi_0}$, where $0 < \rho < r$, $0 < \theta < 2\pi$ and $\phi_0 \in (0, 2\pi)$ – fixed angle. Then there is an "angular" limit of*

$$\lim_{z \to \zeta_{0\angle}} f(z)$$

*with $\rho \to r$ on $\zeta_0 \in \partial L(a,r)$ and we denote them $f^*(\zeta_0)$, where $|\theta - \phi_0| < d(r - \rho)$, $d =$const.*

**Remark 2.** *Thus, it is prescribed that point $z$ tends to $\zeta_0$, remaining inside the sector of solution$< \pi$ with a vertex at point $\zeta_0$, symmetrical with respect to the radius leading from $a$ to $\zeta_0$. In this case, they say that $f(z) \to f^*(\zeta_0)$ with $z$, tending to $\zeta_0$ in non-tangential directions. We will write it down like this:*

$$f(z) \to f^*(\zeta_0) \;\; by \;\; z \to \zeta_{0\angle}.$$

### 2.2 The Nevanlinna classes for $A(z)$–analytic functions.

Now, we introduce the Nevanlinna class for $A(z)$–analytic functions. $A(z)$–analytic function $f(z)$ is not identically equal to 0 in lemniscate $L(a,r)$, belongs to class $N$ if integral

$$\int_{|\psi(z,a)|=\rho} \ln |f(z)|^p |dz + A(z) d\bar{z}|$$

was bounded at $z \in L(a, \rho)$.

This class in the domain of $D$ is $A(z)$–analytic functions is denoted as $N_A(D)$. We will also look at the following properties of the function class $N_A$:

The function $f(z)$ is represented in the lemniscate $L(a,r)$ as the ratio of two bounded functions:

$$f(z)=\frac{g(z)}{h(z)} \tag{3}$$

functions $g(z)$ and $h(z)$ can always be considered bounded in lemniscate $L(a,r)$ modulo one. Class $N_A$ can be characterized differently based on the following statement by Nevanlinna.

**Statement 1.** *In order for the function $f(z)\not\equiv 0$ to belong to class $N_A$, it is necessary and sufficient that the integral*

$$\int\limits_{|\psi(z,a)|=\rho} \ln^+|f(z)||dz+A(z)d\bar{z}| \tag{4}$$

*is bounded at $0<\rho<r$ by some finite number $E$, independent of $\rho$, where if $\tilde{d}\geq 1$, then $\ln^+\tilde{d}=\ln\tilde{d}$.*

*Proof.* If function $f(z)\not\equiv 0$ belongs to class $N_A$, i.e. it is representable as a relation (4) with $|g(z)|\leq 1,\ \ |h(z)|\leq 1$ in $|\psi(z,a)|<r$, then since $|f(z)|\leq\frac{1}{|h(z)|}$ in $|\psi(z,a)|<r$:

$$\int\limits_{|\psi(z,a)|=\rho} \ln^+|f(z)||dz+A(z)d\bar{z}|\leq-\int\limits_{|\psi(z,a)|=\rho} \ln|h(z)||dz+A(z)d\bar{z}|. \tag{5}$$

Now, if $h(z)=\sum\limits_{k=m}^{\infty}c_k\psi^k(z,a),\quad m\geq 0$, then according to Jensen's formula applied to function $\frac{h(z)}{\psi^m(z,a)}$, we have:

$$\frac{1}{2\pi\rho}\int\limits_{|\psi(z,a)|=\rho} \ln|h(z)||dz+A(z)d\bar{z}|$$

$$=\ln|c_m|+\sum_{0<|\psi_k(z,a)|\leq\rho}\ln\frac{\rho}{|\psi_k(z,a)|}+m\ln\rho; \tag{6}$$

here the zeros of function $\psi_k(z,a)$ in $h(z)$ are denoted by $0<|\psi(z,a)|<r$, and the sum is taken for all zeros of function $h(z)$ lying in $0<|\psi(z,a)|<r$. Since the right part in (6) is a non-decreasing function from $\rho$ to $0<\rho<r$, the right part in (5) will be a non-increasing function from $\rho$ and, therefore, will be bounded from above at $\rho$. This proves that the integral (4) will also be bounded in $0<\rho<r$.

Let now, inversely, the function $f(z)\not\equiv 0$ be such that the integral (4) is bounded at $\rho$. Then by the Jensen-Schwartz formula in $|\psi(z,a)|<r$ and $\xi\in\partial L(a,\rho)$ we have:

$$\ln f(z)=\sum_{|\psi_k(z,a)|<\rho}\ln\frac{\rho(\psi(z,a)-\psi_k(z,a))}{\rho^2-\bar{\psi}_k(z,a)\psi(z,a)}$$

$$+\frac{1}{2\pi\rho}\int\limits_{|\psi(\xi,a)|=\rho} \ln|f(z)|\frac{\psi(\xi,a)+\psi(z,a)}{\psi(\xi,a)-\psi(z,a)}|d\xi+A(\xi)d\bar{\xi}|+ic_1,$$

where $c_1$ is a real constant. This can be rewritten as:

$$f(z) = \frac{g_\rho(z)}{h_\rho(z)}, \tag{7}$$

where

$$g_\rho(z) = \prod_{|\psi_k(z,a)|<\rho} \frac{\rho(\psi(z,a) - \psi_k(z,a))}{\rho^2 - \bar{\psi}_k(z,a)\psi(z,a)} e^{-\frac{1}{2\pi\rho}\int\limits_{|\psi(\xi,a)|=\rho} \ln^+ \frac{1}{|f(z)|}\frac{\psi(\xi,a)+\psi(z,a)}{\psi(\xi,a)-\psi(z,a)}|d\xi + A(\xi)d\bar{\xi}| + ic_2}$$

and

$$h_\rho(z) = e^{-\frac{1}{2\pi\rho}\int\limits_{|\psi(\xi,a)|=\rho} \ln^+ |f(z)|\frac{\psi(\xi,a)+\psi(z,a)}{\psi(\xi,a)-\psi(z,a)}|d\xi + A(\xi)d\bar{\xi}|}$$

is put, where $c_2$ is also real constant.

Functions $g_\rho(z), h_\rho(z) \in O_A(L(a,r))$ and $|g_\rho(z)| \leq 1$, $|h_\rho(z)| \leq 1$ in $L(a,r)$. Taking the sequence of numbers $\rho_k \to r$, according to the principle of condensation, a subsequence $h_{\rho'_k}(z)$ can be distinguished from the sequence of function $h_{\rho_k}(z)$, which converges uniformly inside the lemniscate $L(a,r)$ to the $A(z)-$analytic function $h(z)$, and $|h(z)| < 1$ to $|\psi(z,a)| < r$, where $k \in \mathbb{N}$. Since the values of $|h_{\rho_k}(a)|$ are bounded from below by a positive value independent of $k$, then $h(z) \not\equiv 0$. From (7) it follows that function $g_{\rho'_k}$ converges in $\{|\psi(z,a)| < r\}$ to some function $g(z)$, $A(z)$–analytical in $\{|\psi(z,a)| < r\}$, and $|g(z)| \leq 1$ in $\{|\psi(z,a)| \leq r\}$.

The statement is proved. □

Since from the representation (4) for the function $f(z) \not\equiv 0$ of class $N_A$ we have:

$$\begin{aligned}\int\limits_{|\psi(z,a)|=\rho} |\ln|f(z)|||dz + A(z)d\bar{z}| \leq & \int\limits_{|\psi(z,a)|=\rho} |\ln|g(z)|||dz + A(z)d\bar{z}| \\ & + \int\limits_{|\psi(z,a)|=\rho} |\ln|h(z)|||dz + A(z)d\bar{z}| \\ & = -\int\limits_{|\psi(z,a)|=\rho} \ln|g(z)||dz + A(z)d\bar{z}| \\ & - \int\limits_{|\psi(z,a)|=\rho} \ln|h(z)||dz + A(z)d\bar{z}|,\end{aligned}$$

and according to the Statement 1 proved in the first part of the proof, the last two integrals do not decrease by $0 < \rho < r$, then not only the integral (4), but also the integral

$$\int\limits_{|\psi(z,a)|=\rho} |\ln|f(z)|||dz + A(z)d\bar{z}| \tag{8}$$

at $0 < \rho < r$ will be bounded to a horse value independent of $\rho$. This property of class $N_A$ functions will be used now. That is, in order to show the existence of limit values for the class of functions $N_A$, we will prove Fatou's theorems as a proposition.

**Proposition 1.** *If the function $f(z) \not\equiv 0$ in $L(a,r)$ and belongs to class $N_A(L(a,r))$, then it has almost everywhere on the boundary of the lemniscate $\partial L(a,r)$ certain limit values of $f^*(\zeta)$ along all non-tangential paths (angular limit), and $|\ln|f^*(\zeta)||$ is summed by $|\psi(\zeta,a)| = r$.*

*Proof.* If the function $f(z) \in N_A$, $f(z) \not\equiv 0$ and bounded in $L(a,r)$ it has already been noted that it has almost everywhere on $|\psi(\zeta,a)| = r$ certain limit values of $f^*(\zeta)$ along all non-tangential paths, and in particular along radial paths. We denote by $\psi(z,a) = \rho e^{i\theta}$ we have $f(\rho) := |f(z)|$, where $0 \le \rho \le r$. According to the maximum modulus principle for $A(z)$–analytic functions, it follows that the monotonicity of functions $f^*(r) = \max\limits_{0\le\rho\le r} f(\rho)$. Hence, by Fatou's lemma we have:

$$\int\limits_{|\psi(z,a)|=\rho} |\ln|f^*(\zeta)|||d\zeta + A(\zeta)d\bar{\zeta}| \le \lim_{\rho\to r} \int\limits_{|\psi(z,a)|=\rho} |\ln|f(z)|||dz + A(z)d\bar{z}|, \quad (9)$$

moreover, according to what has been said about the integral (9), the right part is bounded here. In this case, in the "radial" convergence, taking $\psi(z,a) = \rho e^{i\theta}$, $\psi(\zeta,a) = re^{i\theta}$ will be $\rho \to r$.

Therefore, $|\ln|f^*(\zeta)||$ is summable by $\partial L(a,r)$. But then the values of $|\ln|f^*(\zeta)||$ are almost everywhere on $\partial L(a,r)$ are finite, i.e. the values of $f^*(\zeta)$ are almost everywhere on $\partial L(a,r)$ are different from zero. But then the values of $|\ln|f^*(\zeta)||$ are almost everywhere finite by $|\psi(\zeta,a)| = r$, i.e. the values of $f^*(\zeta)$ are almost everywhere $|\psi(\zeta,a)| = r$ different from zero.

If now $f(z) \not\equiv 0$ is any function of class $N_A$, then in its representation (3) functions $g(z)$ and $h(z)$ have almost everywhere on $|\psi(\zeta,a)| = r$ certain limit values along non-rotational paths and these limit values are almost everywhere non-zero. But then $|\ln|f^*(\zeta)||$ almost everywhere on $\partial L(a,r)$ has certain limiting values of $f^*(\zeta)$; by applying again Fatou's lemma to the integral (8), we conclude that $|\ln|f^*(\zeta)||$ is summable by $\partial L(a,r)$.

The proposition are proven. □

The finite angular limit values of the function $f \in N_A(L(a,r))$ that exist almost everywhere on $|\psi(\zeta,a)| = r$ along non-tangential paths are now called its boundary values.

# 3 Boundary uniqueness theorem for $A(z)$–analytic functions.

## 3.1 Privalov's ice-cream cone construction for $A(z)$–analytic functions

**Definition 2.** *For $|\psi(\zeta,a)| = r$, let's put such an domain*

$$S_\zeta := \left\{ z : |\psi(z,a)| > \frac{r}{\sqrt{2}}, \quad |\arg\psi(\zeta,z)| < \frac{\pi}{4} \right\}.$$

Let's make a number of obvious remarks:

(a) $\bigcup\limits_{|\psi(\zeta,a)|=r} S_\zeta$ – is all of $\{r/\sqrt{2} < |\psi(z,a)| < r\}$.

(b) If $r/\sqrt{2} < |\psi(z,a)| < r$ for some $z$, $\{\zeta : |\psi(\zeta,a)| = r, \quad \zeta \in S_\zeta\}$ is the (open) arc $\widetilde{\zeta_1,\zeta_2}$ of the boundary of lemniscate $\partial L(a,r)$.

(c) If $J = \widetilde{\zeta_1,\zeta_2}$ is the arc of the lemniscate boundary $\partial L(a,r)$, contracting the angle no more than $\pi/2$, then the set of points $z$, $r/\sqrt{2} < |\psi(z,a)| < r$, such that $z$ is contained only in those sets $S_\zeta$, which have $\zeta \in J$, forms a closed curved triangle $T$.

Now we can describe the Luzin-Privalov construction for $A(z)$–analytic functions. For a given closed set $M$ on the lemniscate boundary $\partial L(a,r)$, let $\{J_k\}$ be the set (no more than countable) of arcs on the boundary $\partial L(a,r)$ adjacent (additional) to $M$. Using each arc $J_k$ as a base, we will build a triangle or trapezoid $T_k$ on it in accordance with the procedure described in (c). Let's take the closed domain

$$\bar{D} = \{|\psi(z,a)| \le r\} \setminus \bigcup_{k=1}^{\infty} \breve{T}_k \setminus \bigcup_{k=1}^{\infty} J_k$$

(the icon ˇ denotes the interior, and the dash on top denotes the closure).

Our domain $\bar{D}$ has the following important property:

**Proposition 2.** *Each point $z \in \bar{D}$, modulo a large $r/\sqrt{2}$, belongs to $\bar{S}_\zeta$ for some $\zeta \in M$.*

This follows directly from the observations (a) and (c) made above.

Note that $\partial\bar{D}$ is the Jordan curve. Indeed, if $\zeta_\theta$ denotes the only point at which the ray going from $a$ to $\psi(\zeta,a) = re^{i\theta}$, intersects $\partial\bar{D}$, then $\zeta_\theta$ will be a continuous one-to-one mapping of the lemniscate $L(a,r)$ to $\partial\bar{D}$. The boundary $\bar{D}$ will even be a rectified Jordan curve, since for each $k$ the perimeter $T_k$ does not exceed $\tilde{c}|J_k|$, where $\tilde{c}$ is a constant that can be calculated for geometric reasons.

## 3.2 Use of Egoroff's theorem for $A(z)$–analytic functions.

**Proposition 3.** *Let be $A(z)$–analytic functions in $L(a,r)$, and put, for $|\psi(\zeta,a)| = r$,*

$$G_f(\zeta) = \sup_{z \in S_\zeta} |f(z)|.$$

*Then $G_f(\zeta)$ is measurable.*

*Proof.* For $n \geq 3$, take $r_n = r(1 - 1/n)$ and put, for $|\psi(\zeta, a)| = r$,

$$G_n(\zeta) = \sup\{|f(z)| : r/\sqrt{2} \leq |\psi(\zeta, a)| \leq r_n, \quad |\arg\psi((r_n/r)\zeta, z)| \leq \pi/4\}.$$

Because $f(z)$ is continuous for $|\psi(z, a)| \leq r_n$, $G_n(z)$ is continuous. Since clearly $G_f(\zeta) = \lim_{n\to\infty} G_n(\zeta)$ pointwise in $\zeta \in \partial L(a, r)$, $G_f(\zeta)$ is measurable.

The proposition is proved. □

**Proposition 4.** *Let $f(z)$ be $A(z)$–analytic in $L(a, r)$. Suppose there is a set $M$ of positive measure on the boundary of lemniscate $\partial L(a, r)$ such that*

$$\lim_{z\to\zeta_\angle} f(z) = 0$$

*for each $\zeta \in M$. Then there is a closed set $M$, $\mu(M) > 0$, such that $|f(z)| \to 0$ uniformly for $|\psi(z, a)| \to r$ and $z$ in the union of the $S_\zeta$ with $\zeta \in S_\zeta$.*

*Proof.* For $n \geq 3$ and $|\psi(\zeta, a)| = r$ put

$$P_n(\zeta) = \sup\Big\{|f(z)| : \quad z \in S_\zeta, \quad |\psi(z, a)| \geq \Big(r - \frac{r}{n}\Big)\Big\}.$$

The argument used in the proof of the previous proposition shows that each $P_n(\zeta)$ is measurable; so, then, is the set $M^*$ of $\zeta$ for which $\lim_{n\to\infty} P_n(\zeta) = 0$.

By hypothesis, $\lim_{n\to\infty} P_n(\zeta) = 0$ for $\zeta \in M$, and $\mu(M) > 0$. Therefore $\mu(M^*) > 0$, and Egoroff's theorem gives us a measurable $M_0 \subset M^*$, $\mu(M_0) > 0$, with $\lim_{n\to\infty} P_n(\zeta) = 0$ uniformly for $\zeta \in M_0$. Also, on an open set $M^* \backslash M_0$ has $\mu\Big(\zeta \in M^* \backslash M_0 : \lim_{n\to\infty} P_n(\zeta)\Big) = 0$. Taking a closed $M \subseteq M_0$ with $\mu(M) > 0$, we have the proposition.

The proposition is proved. □

### 3.3 Generalization of the boundary uniqueness theorem for $A(z)$–analytic functions.

Now we will prove the analog generalized boundary uniqueness theorem for $A(z)$–analytic functions.

**Theorem 4.** *Let $f \in O_A(L(a, r))$. Suppose that $M$ is the set of positive measure on the boundary $\partial L(a, r)$, such that $\lim_{z\to\zeta_\angle} f(z) = 0$ is for $\zeta \in M$. Then the function $f(z)$ is identically equal to zero.*

*Proof.* This theorem for analytic functions was proved by Luzin N.N. and Privalov I.I. in 1924. By Proposition 4, we can find a closed set $M$, $\mu(M) > 0$, on the lemniscate boundary $L(a, r)$, such that $S_\zeta$, $\zeta \in M$ is uniform at $|\psi(z, a)| \to r$, if $z$ belongs to the union of sets $S_\zeta$, $\zeta \in M$. This means that if we carry out the Luzin-Privalov construction described in Subsection 3.1, starting from the set $M$, we get a domain $G \subset L(a, r)$ on which $f(z) \to 0$ is uniformly at

$|\psi(z,a)| \to r, \quad z \in G$. In the construction of the Luzin-Privalov construction for $A(z)$–analytic functions, we see that $\partial D$ consists of segments in $\{|\psi(z,a)| < r\}$, going to the points of the set $M$ on the boundary $\partial L(a,r)$, and, in addition, from the set $M$ itself. Therefore, if we define function $f(z)$ as zero by $M$, we get a continuous function by $\bar{D}$ and an $A(z)$–analytic function in $D$.

In accordance with what is stated in Subsection 3.1, $\partial D$ is a Jordan straighten curve. We take the homeomorphic map $\varphi$ of lemniscate $\{|\psi(w,b)| < r\}$ to $D$ and for $\{|\psi(w,b)| < r\}$ we put $f(w) = F(\varphi(w))$, where point $b$ is the center of this lemniscate (see [1], [2]). According to the Carateodori theorem, $\varphi$ actually continues up to $\{\psi(\nu,b) = r\}$ and displays this boundary one-to-one and continuously at $\partial D$, where $\zeta \in \partial L(b,r)$. This means that $f(w)$ continues continuously up to $\{|\psi(\nu,b)| = r\}$, since $f(z)$ continues continuously up to $\partial D$. Let $S = \psi^{-1}(M)$. Then $f(\nu) = 0$ for $\nu \in S$. The subset $M$ of the rectified curve $\partial D$ has a positive measure: $\mu(S) = \int\limits_S |d\nu + A(\nu)d\bar{\nu}|$. We take $d\mu = d\nu + A(\nu)d\bar{\nu}$, then $\mu$ will be a measure of $\{|\psi(\nu,b)| = r\}$. Measure $\mu$ is absolutely continuous. That is, for every $\varepsilon > 0$ there is such a $\delta > 0$ that

$$\left| \int\limits_S |d\mu| \right| < \varepsilon$$

for every measurable $S \subset \partial L(b,r)$ such that $\mu(S) < \delta$. Denote the (pairwise disjoint) open intervals additional to $S$ - the so-called adjacent intervals - by $(\alpha_k, \beta_k), \quad 1 \le k \le n$; there are no more than a countable set of them. In other words, the arcs $\{\psi(\zeta,b) = re^{it} : \quad \alpha_k < t < \beta_k\}$ do not intersect in pairs and together with $\{t : \nu \in S\}$, fill exactly the entire boundary of the lemniscate $\{|\psi(\nu,b)| = r\}$. From $\mu(S) < \delta$ it follows that

$$\lim_{n\to\infty} \left| \int\limits_{\partial L(b,r)} |d\mu| - \sum_{k=1}^{n} (\beta_k - \alpha_k) \right| = \left| \int\limits_S |d\mu| \right| = \mu(S) < \varepsilon = \delta.$$

Therefore, it follows from this that $\mu(S) > 0$. Since the function $f(w)$ $A(w)$–analytic in $L(b,r)$, is continuous on a closed lemniscate $\bar{L}(b,r)$ and is zero on $S$, then $f \equiv 0$. That is, the Cauchy formula for $A(z)$–analytic functions (2) on the piece boundary $S \subset \partial L(b,r)$ follows

$$f(w) = \int\limits_S f(\nu)K(w,\nu)(dw + A(w)d\bar{w}) = 0, \quad w \in L(b,r)\cdot$$

From where $f(w) \equiv 0$. If we perform

$$z = r^2 \frac{\psi(w,b) - \psi(a,b)}{r^2 - \bar{\psi}(a,b)\psi(w,b)}$$

isomorphism, here $z = z(w) : L(b,r) \to L(a,r)$ (see [?]), we get $f(z) \equiv 0$. □

From this theorem, we obtain the following corollary:

**Corollary 1.** *Let be given $f \in N_A$ function and a set $M$ of a positive measure of a piece of boundary $\partial L(a, r)$. If the $\lim_{z \to \zeta_\angle} f(z) = 0$ is for $\zeta \in M$, then $f(z)$ is identically zero.*

*Proof.* Obviously, $f \in N_A$ and if $f \not\equiv 0$ is in $|\psi(z, a)| < r$, then, according to Proposition 1, $\ln |f^*(\zeta)|$ is almost everywhere on $|\psi(\zeta, a)| = r$ is finite, i.e. $f^*(\zeta)$ is almost everywhere on $|\psi(\zeta, a)| = r$ is finite and different from zero; this also contradicts the condition that $f^*(\zeta) = 0$ is almost everywhere on $M$ (the points at which $f(z) \equiv 0$ are excluded). □

If we consider these statement as $f = f_1 - f_2$ we get $f_1 \equiv f_2$. This relation, on the other hand, means that the $A(z)$–analytic function belonging to all classes is uniquely defined in our $L(a, r)$ lemniscates, which we will consider.

## References

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