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Non-tangential boundary values for $A(z)$−analytic functions

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Non-tangential Boundary Values for $A(z)-$Analytic Functions

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Abstract. We consider $A(z)-$analytic functions in case when $A(z)$ is anti-analytic function. This article introduces L^p classes of $A(z)-$analytics functions by $p \geq 1$. In this paper, we introduce for a non-tangential boundary value of the $A(z)-$analytic function. Thus, this paper proves Fatou's theorem on non-tangential values for $A(z)-$analytic functions.

INTRODUCTION

Solutions of the Beltrami equation:

$$\frac{\partial f}{\partial \bar{z}} = A(z)\frac{\partial f}{\partial z}. \tag{1}$$

It is directly related to quasi-conformal mappings. With respect to the $A(z)$ function, it is measurable and. With respect to the $A(z)$ function, it is measurable and

$$|A(z)| \leq c < 1$$

almost everywhere in the $D \subset \mathbb{C}$ domain under consideration, where $c =$const. In the literature, the solution of equation (1) is commonly called $A(z)-$analytic functions.

The solutions of equation (1), as well as quasi-conformal homeomorphisms in the complex plane $\mathbb{C}$, have been studied in sufficient details. Here we confine to do ourselves to giving the references ([1,3,6]) and formulating the following three theorems:

Theorem 1. (see [6]). For any measurable ont the complex plane function $A(z) : ||A||_\infty < 1$ there exists a unique homeomorphic solution $\psi(z)$ of equation (1) which fixes the points $0, 1, \infty$.

Note that if the function $|A(z)| \leq c < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A(z) \equiv 0$ outside D, so Theorem 1 holds for any domain $D \subset \mathbb{C}$.

Theorem 2. (see [1]). All generalized solutions of equation (1) have the form $f(z) = F[\psi(z)]$, where $\psi(z)$ is a homeomorphic solution in Theorem 1, and $F(z)$ is a holomorphic function in the domain $\psi(D)$. Moreover, if a generalized solution $f(z)$ has isolated singular points, then the holomorphic function $F = f \circ \psi^{-1}$ also has isolated singularities of the same types.

Theorem 2 implies that an $A(z)-$analytic function f carries out an internal (open) mapping, i. e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain $G \subset\subset D$ the maximum of the modulus is reached only on the boundary, i. e. $|f(z)| \leq \max_{z\in\partial G} |f(z)|, z \in G$. If the function is not zero, then the minimum principle also holds, i. e. $|f(z)| \geq \min_{z\in\partial G} |f(z)|, z \in G$.

Theorem 3. (see [3]). If a function $A(z)$ belongs to the class $C^m(D)$, then every solution f of equation (1) also belongs, at least, to the same class $C^m(D)$.

Let $A(z)$ be anti-analytic, i. e. $\frac{\partial A}{\partial z} = 0$, in $D \subset \mathbb{C}$, and such that $|A(z)| \leq c < 1, \forall z \in D$. We put

$$D_A = \frac{\partial}{\partial z} - \overline{A(z)}\frac{\partial}{\partial \bar{z}}, \ \ \overline{D}_A = \frac{\partial}{\partial \bar{z}} - A(z)\frac{\partial}{\partial z}.$$

Then, according to (1), the class $A(z)-$analytic functions in D is characterized by the fact that $\overline{D}_A f = 0$. Since an anti-analytic function is smooth, Theorem 3 implies that $O_A(D) \subset C^\infty(D)$. In this case , the following takes place:

Theorem 4. (analogue of Cauchy's theorem (see [5])). If $f \in O_A(D) \cap C(\bar{D})$, where $D \subset \mathbb{C}$ is a domain with rectifiable boundary ∂D, then

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

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Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $\zeta \in D$ is a fixed point in it. Consider the function

$$K(z,\zeta) = \frac{1}{2\pi i} \frac{1}{z - \zeta + \overline{\int\limits_{\gamma(\zeta,z)} \overline{A(\tau)} d\tau}}, \tag{2}$$

where $\gamma(\zeta,z)$ is a smooth curve which points of $\zeta, z \in D$. Since the domain is simply connected and the function $\overline{A}(z)$ is holomorphic, the integral

$$I(z) = \overline{\int\limits_{\gamma(a,z)} \overline{A(\tau)} d\tau}$$

does not depend on a path of integration; it coincides with a primitive, i.e. $I'(z) = \overline{A}(z)$. (see [7]).

Theorem 5. (see [7]). $K(z,\zeta)$ is an $A(z)-$analytic function outside of the point $z = \zeta$, i. e. $K(z,\zeta) \in O_A(D \backslash \{\zeta\})$. Moreover, at $z = \zeta$ the function $K(z,\zeta)$ has a simple pole.

Remark 1. (see [7]). If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function

$$\psi(z,\zeta) = z - \zeta + \overline{\int\limits_{\gamma(\zeta,z)} \overline{A(\tau)} d\tau},$$

although well defined in D, may have other isolated zeros except for $\zeta : \psi(z,\zeta) = 0$ for $z \in P \backslash \{\zeta, \zeta_1, \zeta_2, ...\}$. Consequently, $\psi \in O_A(D)$, $\psi(z,\zeta) \neq 0$ when $z \notin P$ and $K(z,\zeta)$ is an $A(z)-$analytic function only in $D \backslash P$, it has poles at the points of P. Due to this fact we consider the class of $A(z)-$analytic functions only in convex domains.

According to Theorem 2, the function $\psi(z;a) \in O_A(D)$ carries out an internal mapping. In particular, the set

$$L(a;r) = \{z \in D : |\psi(z;a)| = |z - a + \overline{\int\limits_{\gamma(a;z)} \overline{A(\tau)} d\tau}| < r\}$$

is open in D. For suffiently small $r > 0$ it compactly belongs to D and contains the point a. This set is called an $A(z)-$lemniscate with the center a and denoted by $L(a;r)$. According to the maximum principle the lemniscate $L(a;r)$ is simply connected and to the minimum principle it is connected. (see [7]).

Let $f = u + iv$.

Theorem 6. (see [8]). The real part of the $A(z)-$analytic functions of $f(z) \in O_A(D)$ satisfies equation

$$\Delta_A u = \frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{1}{1-|A|^2}\left((1+|A|^2)\frac{\partial u}{\partial z} - 2\bar{A}\frac{\partial u}{\partial \bar{z}}\right)\right) = 0 \tag{3}$$

in the domain of D.

In connection with Theorem 6, it is natural to define the $A(z)-$harmonic function as follows.

Definition 1. (see [8]). A double differentiable function $u \in C^2(D)$, $u : D \to R$ is called $A(z)-$harmonic in the D domain if the D domain if it satisfies the differential equation (3).

The class of $A(z)-$harmonic functions in the domain of D is denoted as $h_A(D)$. Thus, the real part and hence the imaginary part, of the $A(z)-$harmonic function in the domain of D. The inverse theorem is also true for simply connected domains.

Theorem 7. (see [8]). If the function is $u(z) \in h_A(D)$, where D is a simply connected domain, then $f \in O_A(D)$: $u = \mathrm{Re}\, f$.

For $A(z)-$analytic and $A(z)-$harmonic functions, the following Dirichlet problem is naturally considered:

Dirichlet problem. (see [8]). A bounded domain of $G \subset D$ is given and a continuous function of $\omega(\zeta)$ is set at the boundary of ∂G. It is required to find $A(z)-$harmonic in the domain of G, continuous on the closure of $\bar{G}$ the function of $u(z) \in h_A(G) \cap C(\bar{G}) : u|_{\partial G} = \omega$.

Theorem 8. (Poisson formula for $A(z)-$harmonic functions (see [8])). If the $\omega(\zeta)$ function is continuous on the boundary of the lemniscate of $L(a;r) \subset D$, then the function

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(\zeta;a)|=r} \omega(\zeta) \frac{r^2 - |\psi(z;a)|^2}{|\psi(\zeta;z)|^2} |d\zeta + A(\zeta) d\bar{\zeta}| \tag{4}$$

is the solution of the Dirichlet problem in $L(a;r)$.

The $f(\zeta;z)=\frac{\psi(a;\zeta)+\psi(a;z)}{\psi(z;\zeta)}$ function is an $A(z)-$analytic function for $z\in L(a;r)$, where $\zeta\in\partial L(a;r)$. Then $\mathscr{P}(\zeta;z)=Ref(\zeta;z)$ and besides

$$\mathscr{P}(\zeta;z)=\frac{1}{2\pi}(f(\zeta;z)+\bar{f}(z;\zeta))=\frac{1}{2\pi}\left(\frac{\psi(a;\zeta)+\psi(a;z)}{\psi(a;\zeta)-\psi(a;z)}+\frac{\bar{\psi}(a;\zeta)+\bar{\psi}(a;z)}{\bar{\psi}(a;\zeta)-\bar{\psi}(a;z)}\right)=\frac{1}{2\pi}\left(\frac{|\psi(a;\zeta)|^2-|\psi(a;z)|^2}{|\psi(z;\zeta)|^2}\right)$$

$$=\frac{1}{2\pi}\left(\frac{r^2-|\psi(z;a)|^2}{|\psi(z;\zeta)|^2}\right). \qquad (5)$$

Formula (5) is called an analogue of the Poisson formula for $A(z)-$harmonic functions. (see [8]).

L_A^p CLASS OF FUNCTIONS

First we will introduce L^p classes for $A(z)-$analytic functions by $p\geq 1$.

Definition 2. It is said that an $A(z)$-analytic function $f(z)$ belongs to the class L^p, if its mean

$$\frac{1}{2\pi\rho}\int\limits_{|\psi(z;a)|=\rho}|f(z)|^p|dz+A(z)d\bar{z}| \qquad (6)$$

is bounded in the lemniscate $L(a;r)$, where $0<\rho<r$.

Of this class, the function is $A(z)-$analytic functions in the domain of D, then which we denote $L_A^p(D)$.

We can also consider L_A^p as a space of integrable functions with degree $p, p\geq 1$. In this space, the norm is introduced as follows:

$$\| f(z) \|=\left(\frac{1}{2\pi\rho}\int\limits_{|\psi(z;a)|=\rho}|f(z)|^p|dz+A(z)d\bar{z}|\right)^{\frac{1}{p}}.$$

Non-negativity and uniformly follow directly from the properties of this integral and Minkowski's inequality is the triangle inequality for this norm.

Now we give space L^∞ for $A(z)-$analytic functions:

Definition 3. Space L^∞ for $A(z)-$analytic functions measurable functions, bounded almost everywhere in the lemniscate $L(a;r)$, by the identification of functions that differ only on the set of measure zero, and, assuming by definition:

$$\| f(z) \|=ess\sup_{|\psi(z;a)|<r}|f(z)|.$$

This representation is the norm of space L_A^∞, where $ess\sup$ is the essential supremum of the function.

Essential supremum $ess\sup$ for $A(z)-$analytic functions f is the infimum of the lemniscate $L(a;r)$ of such number b, what

$$|f(z)|\leq b,$$

almost all at $z\in L(a;r)$. In other words,

$$ess\sup f=\inf\{b\in\mathbb{R}:\mu^*(\{z:|f(z)|>b\})=0\},$$

where μ^*- measure on the lemniscate $L(a;r)$. In this lemniscate, the measure is introduced through the mean value theorem in [8] as follows:

$$\mu^*(L(a;r))=\iint\limits_{|\psi(\zeta;a)|\leq r}d\mu^*(\zeta)=\iint\limits_{|\psi(\zeta;a)|\leq r}\left(1-|A(\zeta)|^2\right)\frac{d\zeta\wedge d\bar{\zeta}}{2i}.$$

Note that the functions included in space L_A^m are included in every space L_A^k, if only $k<m$. That is $L_A^m \subset L_A^k$. At the same time

$$\| f(z) \|_{L_A^j} \leq \| f(z) \|_{L_A^m} .$$

This follows directly from the integral Helder inequality. Indeed, assuming $p=\frac{m}{k}, q=\frac{m}{m-k}$ $\left(p>0, q>0; \frac{1}{p}+\frac{1}{q}=1\right)$, we get

$$\| f(z) \|_{L_A^k} = \left(\frac{1}{2\pi\rho} \int\limits_{|\psi(z;a)|=\rho} |f(z)|^k |dz + A(z)d\bar{z}| \right)^{\frac{1}{k}} \leq$$

$$\leq \left(\frac{1}{2\pi\rho} \int\limits_{|\psi(z;a)|=\rho} |f(z)|^{pk} |dz + A(z)d\bar{z}| \right)^{\frac{1}{pk}} \left(\frac{1}{2\pi\rho} \int\limits_{|\psi(z;a)|=\rho} 1^{qk} |dz + A(z)d\bar{z}| \right)^{\frac{1}{qk}}$$

$$= \left(\frac{1}{2\pi\rho} \int\limits_{|\psi(z;a)|=\rho} |f(z)|^{m} |dz + A(z)d\bar{z}| \right)^{\frac{1}{m}} = \| f(z) \|_{L_A^m} .$$

NON-TANGENTIAL BOUNDARY VALUES AND FATOU'S THEOREM FOR $A(z)-$ANALYTIC FUNCTIONS

13 March 2024 12:48:00

We will consider the Poisson formula (5) in the form $P(\zeta;z)=\frac{r^2-|\psi(z;a)|^2}{|\psi(z;\zeta)|^2}$. If the function $u(z)\in h_A(L(a;r))$, admits one of the representations

$$u(z)=\frac{1}{2\pi r} \int\limits_{|\psi(\zeta;a)|=r} P(z;\zeta) f(\zeta) |d\zeta + A(\zeta)d\bar{\zeta}|,$$

where $f\in L_A^p(L(a;r))$,

$$u(z)=\frac{1}{2\pi r} \int\limits_{|\psi(\zeta;a)|=r} P(z;\zeta) |d\mu|, \tag{7}$$

then it is still necessary to investigate the pointwise behavior $u(z)$ at z, tending to the point ζ on the boundary of lemniscate $L(a;r)$.

Both the representations recorded above for $u(z)$ are contained in the second, since if $f\in L_A^p(L(a;r))$ and we take $d\mu = f(z)(dz+A(z)d\bar{z})$, then μ will be a measure of $|\psi(z;a)|=\rho$, where $0<\rho<r$. Dealing with such a measure, it is convenient to conduct an $A(z)-$analytic function $\mu(z)$ bounded variation on the boundary of the lemniscate $\partial L(a;\rho)$, given by formula

$$\mu(z)=\int\limits_{l_\rho} |d\mu(\tau)|,$$

where $l_\rho = \{Re\psi(z;a)>0, |\psi(z;a)|=\rho\}$ (with the usual integral at $Re\psi(z;a)<0$). Then we have the following result:

Theorem 9. (analogue of Fatou 's theorem). Let $\zeta_0 \in \partial L(a;r)$ and the derivative $\mu'(z_0)$ exist and be finite. $\psi(z;a)=\rho e^{i\nu}, \psi(\zeta_0;a)=re^{it_0}$. Then (7) tends to $\mu'(\zeta_0)$ with z tending to ζ_0 inside any sector of the lemniscate $L(a;r)$ of the form $|\nu - t_0| \leq d(r-\rho)$, where $d=$const.

Remark 2. Thus, it is prescribed that point $z \in \partial L(a;\rho)$ tends to ζ_0, remaining inside the sector of solution$< \pi$ with a vertex at point ζ_0, symmetrical with respect to the radius leading from a to ζ_0. In this case, they say that $u(z) \to \mu'(\zeta_0)$ with z, tending to ζ_0 in non-negative directions. We'll write it down like this: $u(z) \to \mu'(\zeta_0)$ by $z \underset{\triangleleft}{\to} \zeta_0$.

Remark 3. A similar result holds for $t_0 = \pm 180^0$ provided that there exists a properly defined derivative $\mu'(\zeta_0)$.

Proof. (Proofs analog of Fatou's theorem). To simplify the recording, take $\zeta_0 = a$. Then if the derivative $\mu'(a)$ exists and is finite and if $|\nu| < d(r-\rho)$, then we need to show that

$$\frac{1}{2\pi\rho} \int\limits_{|\psi(z;a)|=\rho} P(z;\zeta)d\mu(z) \to \mu'(a)$$

at $z \to \zeta$ (or by radius $\rho \to r$). Without loss of generality, we can assume that $\mu'(a) = 0$; otherwise we would consider $d\mu(\zeta) - \mu'(a)(d\zeta + A(\zeta)d\bar{\zeta})$ instead of $d\mu(\zeta)$, and

$$\frac{1}{2\pi r} \int\limits_{|\psi(\zeta;a)|=r} P(z;\zeta)|d\zeta + A(\zeta)d\bar{\zeta}|$$

equals $\mu'(a)$.

Let for an arbitrary $\varepsilon > 0$ the number exists $\delta > 0$ such that $|\mu(\zeta)| \leq \varepsilon|\psi(\zeta;a)|$ for arc $l_\delta \subset \partial L(a;r)$, where $l_\delta = \{\zeta : \psi(\zeta;a) = re^{it}, -\delta \leq t \leq \delta\}$. If $r-\rho$ is very close to zero, so 2η is much less than δ, then

$$\frac{1}{2\pi r} \int\limits_{|\psi(\zeta;a)|=r} P(z;\zeta)|d\mu(\zeta)| = o(r) + \frac{1}{2\pi r}\int\limits_{l_\delta} P(z;\zeta)|d\mu(\zeta)|,$$

where $o(r) \to 0$ is in radius at $z \to \zeta$. Integrating

$$\frac{1}{2\pi r}\int\limits_{l_\delta} P(z;\zeta)|d\mu(\zeta)| = \frac{1}{2\pi r}\int\limits_{l_\delta} \frac{r^2 - |\psi(z;a)|^2}{|\psi(\zeta;z)|^2}|d\mu(\zeta)|$$

in parts, we get an integrated member (which will be $o(r)$) plus

$$\frac{1}{2\pi r}\int\limits_{l_\delta} \frac{\partial P(\zeta;z)}{\partial \psi(\zeta;a)}\mu(\zeta)d\psi(\zeta;a) + \frac{1}{2\pi r}\int\limits_{l_\delta} \frac{\partial P(\zeta;z)}{\partial \bar{\psi}(\zeta;a)}\mu(\zeta)d\bar{\psi}(\zeta;a)$$

$$= \frac{1}{2\pi r}\int\limits_{l_\delta} \frac{-2\psi(z;a)}{\psi^2(\zeta;z)}\mu(\zeta)d\psi(\zeta;a) + \frac{1}{2\pi r}\int\limits_{l_\delta} \frac{-2\bar{\psi}(z;a)}{\bar{\psi}^2(\zeta;z)}\mu(\zeta)d\bar{\psi}(\zeta;a).$$

When we evaluate this integral in absolute terms, it looks like this:

$$\left|\frac{1}{2\pi r}\int\limits_{l_\delta} \frac{-2\psi(z;a)}{\psi^2(\zeta;z)}\mu(\zeta)d\psi(\zeta;a) + \frac{1}{2\pi r}\int\limits_{l_\delta} \frac{-2\bar{\psi}(z;a)}{\bar{\psi}^2(\zeta;z)}\mu(\zeta)d\bar{\psi}(\zeta;a)\right| \leq \frac{1}{2\pi r}\int\limits_{l_\delta} \frac{4|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\mu(\zeta)||d\psi(\zeta;a)|.$$

Assuming (without loss of generality!) that $\eta > 0$, let's split the last integral into three by $l_\delta = l_\delta^{(1)} \cup l_\delta^{(2)} \cup l_\delta^{(3)}$, where $l_\delta^{(1)} = \{-\delta \leq t \leq 0\}, l_\delta^{(2)} = \{0 \leq t \leq 2\eta\}, l_\delta^{(3)} = \{2\eta \leq t \leq \delta\}$.

$$\frac{1}{2\pi r}\left[\int\limits_{l_\delta^{(1)}} + \int\limits_{l_\delta^{(2)}} + \int\limits_{l_\delta^{(3)}}\right] \frac{4|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\mu(\zeta)||d\psi(\zeta;a)| = I_1 + I_2 + I_3.$$

Then

$$|I_2| \leq \frac{\varepsilon}{\pi r}\int\limits_{l_\delta^{(2)}} \frac{4\eta}{(r-\rho)^2}\varepsilon|\psi(\zeta;a)||d\psi(\zeta;a)| \leq \frac{4\varepsilon\eta^3}{\pi r(r-\rho)^2} \leq \frac{4\varepsilon\eta^3}{\pi(r-\rho)^3} \leq \frac{4}{\pi}d^3\varepsilon,$$

since $0 \leq \eta \leq d(r-\rho)$. For $2\eta \leq |\psi(\zeta;a)| \leq \delta$, inequalities $|f(\zeta)| \leq \varepsilon|\psi(\zeta;a)| \leq 2\varepsilon(|\psi(\zeta;a)|-\eta)$ are fulfilled, hence

$$|I_3| \leq \frac{\varepsilon}{\pi r}\int\limits_{l_\delta^{(3)}} \frac{4|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\psi(\zeta;a)||d\psi(\zeta;a)| = \frac{\varepsilon}{\pi r}\int\limits_{l_\delta^{(4)}} \frac{4|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\psi(\zeta;a)||d\zeta+A(\zeta)d\bar{\zeta})| \leq$$

$$\leq \frac{\varepsilon}{\pi r}\int\limits_{l_\delta} \frac{4|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\psi(\zeta;a)||d\zeta+A(\zeta)d\bar{\zeta})|,$$

where $l_\delta^{(4)} = \{\eta \leq t \leq \delta-\eta\} \subset l_\delta$. This last integral is integrated in parts (in the direction opposite to our original integration in parts!), which gives

$$\frac{\varepsilon}{\pi r}\left[o(r)+\int\limits_{l_\delta} P(z;\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|\right] = \varepsilon+o(r).$$

Similarly $|I_1| \leq \frac{\varepsilon}{2}+o(r)$. Therefore, $|I_1+I_2+I_3| \leq \left(\frac{4d^3}{\pi}+\frac{3}{2}\right)\varepsilon+o(r)$ at $z \to \zeta$ (by radius $\rho \to r$), and since $\varepsilon > 0$ is an arbitrary number, the proof is complete.

Remark 4. The monotony of the kernel $P(z;\zeta)$ at each of the cases $\{z : Re\psi(z;a) < 0\}$ and $\{z : Re\psi(z;a) > 0\}$ is the main point on which the above proof "works".

Corallary 1. (corollary analog of Fatou's theorem). If $\zeta_0 \in \partial L(a;r)$, and the derivative $\mu'(\zeta_0)$ exists and is infinite, then for

$$u(z) = \frac{1}{2\pi r}\int\limits_{|\psi(z;a)|=r} P(z;\zeta)d\mu(\zeta),$$

the ratio $u(z) \to \mu'(\zeta_0)$ is fulfilled at $z \to \zeta_0$ (by radius).

Remark 5. Thus, even when the derivative $\mu'(\zeta_0)$ is infinite, we still have

$$u(z) \to \mu'(\zeta_0),$$

when z radially tends to ζ_0.

Proof. (Proofs corollary analog of Fatou's theorem). Take $\zeta_0 = a$ and suggest that $\mu'(a) = \infty$. For every $\forall E > 0$, let's choose the number $\delta > 0$ so small that $|\mu(\zeta)| \geq E|\psi(\zeta;a)|$ is for l_δ. Then, reasoning in the same way as in the proof of the previous theorem, we get

$$u(z) = o(r)+\frac{1}{2\pi r}\int\limits_{l_\delta} \frac{-2\psi(z;a)}{\psi^2(\zeta;z)}\mu(\zeta)|d\zeta+A(\zeta)d\bar{\zeta}|.$$

When we evaluate this integral in absolute values, too, it looks like this:

$$o(r)+\frac{1}{2\pi r}\int\limits_{l_\delta} \frac{2|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\mu(\zeta)||d\zeta+A(\zeta)d\bar{\zeta}| \geq o(r)+\frac{E}{\pi r}\int\limits_{l_\delta^{(2)}\cup l_\delta^{(3)}} \frac{2|\psi(z;a)|}{|\psi(\zeta;z)|^2}|\psi(\zeta;a)||d\zeta+A(\zeta)d\bar{\zeta}|.$$

Doing the inverse integration in parts, we see that the last integral is $o(r)+E$ at z, sufficiently close to ζ in radius.

Scholia 1. Is it possible to replace "$z \to \zeta_0$" radially in the formulation of the investigation just proved by $z \underset{\triangleleft}{\to} \zeta_0$? It is possible if $u(z) \geq 0$ is $L(a;r)$, that is, if measure μ is positive.

Summing up the above, we introduce the following notation:

Notation 1. Function $u(\zeta)$ is called the (angular or non-tangent) boundary function for function $u(z)$; we will often write

$$u(\zeta) = \lim_{\substack{z \to \zeta \\ \triangleleft}} u(z)$$

almost everywhere.

CONCLUSION

In this paper, we investigate the approximation to the boundary value of the domain in the non-tangential direction for $A(z)-$analytic functions. It is in this direction that Fatou's scientific work on holomorphic functions in the classical case is devoted. In this scientific paper, these classical works are extended for the class $A(z)-$analytic functions.

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