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MUNDARIJA *** СОДЕРЖАНИЕ *** CONTENTS

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GENERALIZATION SMIRNOV'S THEOREM FOR *A*(*z*)-ANALYTIC FUNCTION

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Abstract. *We consider* $A(z)-$*analytic functions in case when* $A(z)$ *is antianalytic function. The paper introduces some classes for* $A(z)-$*analytic functions. Also, a generalized Smirnov theorem for* $A(z)-$*analytic functions is proved and a corollary of this theorem is formulated.*

Key words: $A(z)-$*analytic function,* $A(z)-$*lemniscate, class* E_A^p, C *classes of domains.*

SMIRNOVNING A(z)-ANALITIK FUNKSIYA UMUMIY TEOREMASI

Annotatsiya. *Ushbu maqolada* $A(z)$ *funksiyani antianalitik bo'lgan holda* $A(z)-$*analitik funksiyalarni qaraymiz. Bu maqolada* $A(z)-$*analitik funksiyalar uchun ba'zi sinflar kiritilgan. Shuningdek,* $A(z)-$*analitik funksiyalar uchun Smirnov umumlashgan teoremasi isbotlangan va bu teoremaning natijasi ham keltirilgan.*

Kalit so'zlar: $A(z)-$*analitik funksiya,* $A(z)-$*lemniskata,* E_A^p *sinfi, sohalarning* C *sinfi.*

ОБОБЩЕНИЕ ТЕОРЕМЫ СМИРНОВА ДЛЯ A(z)-АНАЛИТИЧЕСКОЙ ФУНКЦИИ

Аннотация. *Мы рассматриваем* $A(z)-$*аналитические функции в случае, когда* $A(z)$ *антианалитическая функция. В работе вводится некоторых классов для* $A(z)-$*аналитических функций. Также, доказывается обобщённая теорема Смирнова для* $A(z)-$*аналитических функций и сформулировать следствие эта теорема.*

Ключевые слова: $A(z)-$*аналитическая функция,* $A(z)-$*лемниската, класс* E_A^p, C *классов областей.*

Introduction.

1.1. *A*(*z*)-analytic functions. Let $A(z)$ be antianalytic function in the domain $D\subset\mathbb{C}$ and there is a constant $c<1$ such that $|A(z)|\le c$ for all $z\in D.$ The function $f(z)$ is said to be $A(z)-$*analytic* in the domain D if for any $z\in D,$ the following equality holds:

$$\frac{\partial f}{\partial \bar{z}} = A(z)\frac{\partial f}{\partial z}. \tag{1}$$

We denote by $O_A(D)$ the class of all $A(z)-$analytic functions defined in the domain $D.$ Since an antianalytic function is infitely smooth, then $O_A(D)\subset C^\infty(D)$ (see [1]). In this case, the following takes place:

Theorem 1 (see [3], analogue of Cauchy integral theorem). *If* $f\in O_A(D)\cap C(\overline{D}),$ *where* $D\subset\mathbb{C}$ *is a domain with smooth* $\partial D,$ *then*

$$\int_{\partial D} f(z)\left(dz + A(z)d\bar{z}\right) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $\xi \in D$ is a fixed point in it. Since the function $A(z)$ is analytic, the integral

$$I(z) = \int_{\gamma(\xi,z)} \overline{A}(\tau)d\tau$$

is independent of the path of integration; it coincides with the antiderivative $I'(z) = \overline{A}(z)$. Consider the function

$$K(z,\xi) = \frac{1}{2\pi i} \cdot \frac{1}{z - \xi + \overline{\int_{\gamma(\xi,z)} \overline{A}(\tau)d\tau}},$$

where $\gamma(\xi, z)$ is a smooth curve which connects the points $\xi, z \in D$ (see [5]).

Theorem 2 (see [5]). *$K(z,\xi)$ is an $A(z)-$analytic function outside of the point $z=\xi$, i.e. $K(z,\xi) \in O_A(D \setminus \{\xi\})$. Moreover, at $z=\xi$ the function $K(z,\xi)$ has a simple pole.*

Remark 1 (see [5]). *If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function*

$$\psi(\xi, z) = z - \xi + I(z)$$

although well defined in D, may have other isolated zeros except ξ: $\psi(\xi,z)=0$ for except $z \in P = \{\xi, \xi_1, \xi_2, ...\}$. Consequently, $\psi \in O_A(D)$, $\psi(\xi,z) = 0$ when $z \notin P$ and $K(z,\xi)$ is an $A(z)-$analytic function only in $D \setminus P$, it has poles at the points of P. Due to this fact we consider the class of $A(z)-$analytic functions only in convex domains.

According to [5], Theorem 1.2, the function $\psi(\xi, z)$ is an $A(z)-$analytic function. The following set is an open subset of D:

$$L(a,r) = \{z \in D : |\psi(a,z)| < r\}.$$

For suffiently small $r > 0$ this set compactly lies in D (we denote it by $L(a,r) \subset\subset D$) and contains the point a. The set $L(a,r)$ is called an $A(z)-$lemniscate centered at the point a. The lemniscate $L(a,r)$ is a simply-connected set (see [5]).

Theorem 3 (see [4], Cauchy's integral formula). *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be an arbitrary subdomain with a smooth or piecewise smooth ∂G. Then for any function $f(z) \in O_A(G) \cap C(\overline{G})$, the following formula holds:*

$$f(z) = \int_{\partial G} K(\xi, z) f(\xi)\left(d\xi + A(\xi)d\overline{\xi}\right), \quad z \in G. \tag{2}$$

1.2. Angular limits and Hardy classes for *A*(*z*)-analytic functions. Let $L(a,r) \subset\subset D$ and $f(z) \in O_A(L(a,r))$. We define the concepts of angular and radial limits of $A(z)-$analytic functions in lemniscate $L(a,r)$. The radial limits of the function $f(z)$ at some point $\zeta \in \partial L(a,r)$ are denoted as $f^*(\zeta)$ and the angular limits are denoted as $f^*_<(\zeta)$ (see [7]).

In the classical case of the disk $U = \{|w| < 1\} \subset \mathbb{C}_w$, the limit by the $\tau_\zeta = \{w = t\zeta\}$, $0 \le t \le 1$, $\zeta \in \partial U$ of the function $g(w)$,

$$g^*(\zeta) = \lim_{w \to \zeta,\, w \in \tau_\zeta} g(w)$$

is called the radial limit, and the limit by the angle $\angle \subset U$, ending at the point $\zeta \in \partial U$, is called the angular limit,

$$g^*_{\angle}(\zeta) = \lim_{w\to\zeta,\, w\in\angle} g(w).$$

Since lemniscate $L(a,r)$ is a simply connected domain with a real analytic boundary, then according to Riemann's theorem there exists a conformal map $\chi(z): U \to L(a,r)$, which is also conformal in some neighborhood of closure $\bar{U}$. Let χ maps the boundary point $\lambda \in \partial U$ to the boundary point $\zeta \in \partial L(a,r)$. Then the curve $\gamma_\zeta = \chi(\tau_\lambda)$ has the property that it connects points $a,\ \zeta$ and is perpendicular to all lines of level $\partial L(a,\rho) = \{|\psi(a,z)| = \rho\},\ 0<\rho\le r$. In the theory of $A(z)-$analytic functions, the curve γ_ζ plays the role of the radial direction, and the image of the angle $\chi(\angle)$ plays the role of the angular set at the point $\zeta \in \partial L(a,r)$. We will denote this angle by $\angle = \angle_\zeta$. The limit $\lim\limits_{z\to\zeta,\, z\in\gamma_\zeta} f(z) = f^*(\zeta)$ is called the *radial limit*, and is the *angular limit* of the function $f(z)$ at the point $\zeta \in \partial L(a,r)$ (see [7]).

Now we will show the smoothness of the boundary of lemniscate $L(a,r)$. For this, we take automorphism $\chi^{-1}(z): \bar{L}(a,r) \to \bar{U}$ by Riemann's theorem. Let there be some neighborhood $V = \{\psi(a,\zeta) = re^{i\varphi}, |\varphi| < \varepsilon\}$ for $\forall \varepsilon > 0$. Also has $\chi^{-1}(V) \subset \partial U$ and $\chi^{-1}(\zeta_0) = \lambda_0 \in \partial U$. Further, there is a diffeomorphism $\pi = -i \ln \chi^{-1}(\zeta): V \to [-1;1]$. This diffeomorphism represents all boundary points of the differentiability of the function $f^*(\zeta)$ and $f^*_{<}(\zeta)$ (see [8]).

Next, we introduce the Hardy class for $A(z)-$analytic functions:

Definition 1 (see [7]). *The Hardy class* $H_A^p,\ p>0$ *for* $A(z)-$*analytic functions is the set of all functions* $f(z)$ *such that its averages*

$$\frac{1}{2\pi\rho} \int\limits_{|\psi(a,z)|=\rho} |f(z)|^p \left|dz + A(z)d\bar{z}\right| \tag{3}$$

are uniformly bounded for $\rho < r$, *i.e.* $\sup\limits_{\rho<r} \dfrac{1}{2\pi\rho} \int\limits_{|\psi(a,z)|=\rho} |f(z)|^p \left|dz + A(z)d\bar{z}\right| < \infty.$

Let us define a class of bounded functions

$$H_A^\infty(L(a,r)) = \left\{ f(z) \in O_A(L(a,r)):\ \sup_{z\in L(a,r)} \{|f(z)|\} < \infty \right\}.$$

The norm in $H_A^\infty(L(a,r))$ is defined as $\|f\| = \sup\limits_{z\in L(a,r)} \{|f(z)|\}.$

1.3. The Fatou's theorems and Cauchy's integral formula for Hardy class H_A^1. Now, we will consider the Fatou's theorem for the class of functions H_A^1.

Theorem 4 (see [7], the Fatou's theorem for the class of functions H_A^1). *If* $f(z) \in H_A^1(L(a,r))$, *then the angular limit*

$$f^*_S(\zeta) = \lim_{z\to\zeta,\, z\in S} f(z)$$

exists and is finite for almost all $\zeta \in \partial L(a,r),$ *except, perhaps, the points of some set* E *of measure zero.*

The following statements follow from Theorem 4:

Theorem 5 (see [7]). *If* $f(z) \in H_A^1(L(a,r)),$ *then* $f^*(\zeta) \in L_A^1(\partial L(a,r)).$ *As* $\rho \to r$

$$\int_{|\psi(a,z)|=\rho} f(z)\left|dz + A(z)d\bar{z}\right| \to \int_{|\psi(a,\zeta)|=r} f^*(\zeta)\left|d\zeta + A(\zeta)d\bar{\zeta}\right| \qquad (4)$$

and

$$\int_{|\psi(a,z)|=\rho} \left|f(z) - f^*(\zeta)\right|\left|dz + A(z)d\bar{z}\right| \to 0. \qquad (5)$$

According to Cauchy integral formula (2) for lemniscates $L(a,r)$

$$f(z) = \frac{1}{2\pi i}\int_{|\psi(a,\xi)|=\rho} f(\xi)K(\xi,z)\left(d\xi + A(\xi)d\bar{\xi}\right).$$

we conclude that

$$f(z) = \frac{1}{2\pi i}\int_{|\psi(a,\zeta)|=r} f^*(\zeta)K(\zeta,z)\left(d\zeta + A(\zeta)d\bar{\zeta}\right). \qquad (6)$$

This is the Cauchy integral formula for functions of $H_A^1.$

We show a boundary uniqueness theorem for the Hardy class H_A^1 :

Theorem 6 (see [7]). *Let* $f(z) \in H_A^1(L(a,r)).$ *Suppose that for some set* $M \subset \partial L(a,r)$ *of positive measure* $f^*(\zeta) = 0,\ \zeta \in M.$ *Then* $f(z) \equiv 0.$

Analysis results. First, we will introduce one more class. Let $L(a,r) \subset\subset D$ be bounded by a rectifiable Jordan curve $l.$

Definition 2. *A function* $f(z)$ $A(z)-$*analytic in the lemniscate* $L(a,r),$ *belongs to class* $E^p,$ $p > 0,$ *if there exists a sequence of rectifiable curves* l_n *converging to* l *and such that*

$$\exists M > 0, \int_{l_n} \left|f(z)\right|^p \left|dz + A(z)d\bar{z}\right| \le M. \qquad (7)$$

The functions of this class were introduced by V.I.Smirnov and we will designate them as $E_A^p.$

The bounded Jordan rectifiable curves for which the logarithm of the modulus of the derivative of the function that gives a conformal mapping of the boundary lemniscate onto these domains is represented by the Poisson integral, we will call domains of class C (Smirnov).

Now we formulate Smirnov's theorem for $A(z)-$analytic functions:

Theorem 7. *Let* $f(z)$ *belong to class* E_A^p *in the lemniscate* $L(a,r)$ *of classes* $C;$ *then if the function* $f_S^*(\zeta)$ *- the modulus of the angular boundary values of the function* $f(z)$ *is summable on the boundary* $\partial L(a,r)$ *to the power of* $q > p$ *i.e. the integral is*

$$\exists M_1 > 0, \int_{\partial L(a,r)} \left|f(z)\right|^q \left|dz + A(z)d\bar{z}\right| \le M_1,$$

then $f(z)$ *belongs to class* $E_A^q.$

Proof. Let us consider function $F(w)=f\left(\varphi(w)\right)\left(\varphi'(w)\right)^{\frac{1}{p}}$ in lemniscate $\left\{w:\left|\psi\left(w,b\right)\right|<R\right\},$ where $b=\psi(a)$ is the center. As shown from the above, if $f(z)$ belongs to class E_A^p in lemniscate $L(a,r)$ of class $C,$ then for function $f\left(\varphi(z)\right)$ the provision (4) is satisfied. But then for function $F(w)$ (4) also holds. Indeed,

$$\int_E \ln^+\left|F(z)\right|\left|dz+A(z)d\overline{z}\right|\le\int_E \ln^+\left|F\left(\varphi(z)\right)\right|\left|dz+A(z)d\overline{z}\right|+\int_E \ln^+\left|\varphi'(z)\right|^{\frac{1}{q}}\left|dz+A(z)d\overline{z}\right|,$$

since $\varphi'(w)\in H_A^1,$ and for function $f\left(\varphi(z)\right)$ (4) holds, then at $\mu\left(E\right)<\eta,\ E\subset\partial L(a,\rho)$ and $0<\rho<r,$

$$\int_E \ln^+\left|F(\varphi(z))\right|\left|dz+A(z)d\overline{z}\right|<\frac{\varepsilon}{2},\int_E \ln^+\left|\varphi'(z)\right|^{\frac{1}{q}}\left|dz+A(z)d\overline{z}\right|<\frac{\varepsilon}{2}.$$

And we get that

$$\int_E \ln^+\left|F(z)\right|\left|dz+A(z)d\overline{z}\right|<\varepsilon.$$

By the condition of Theorem

$$\int_{\partial L(a,r)} \ln^+\left|f_S^*\left(\zeta\right)\right|\left|d\zeta+A\left(\zeta\right)d\overline{\zeta}\right|<M_1.$$

whence

$$\int_{\partial L(a,r)} \ln^+\left|f_S^*\left(\zeta\right)\right|\left|d\zeta+A\left(\zeta\right)d\overline{\zeta}\right|<M_1.$$

This shows that the function $\left|F\left(\zeta\right)\right|$ is summable to the boundary $\left\{\left|\psi\left(b,w\right)\right|=r\right\}$ to the power $q,$ since $\left|F\left(\zeta\right)\right|^q=\left|f\left(\varphi\left(\zeta\right)\right)\right|^q\left|\varphi'\left(\zeta\right)\right|.$

For function $F(w)$ (4) holds, but then from condition

$$\exists M_2>0,\int_{\left|\psi\left(b,w\right)\right|=R}\left|F\left(\zeta\right)\right|^q\left|d\zeta+A\left(\zeta\right)d\overline{\zeta}\right|<M_2.$$

it follows that $F(w)$ belongs to class H_A^p in domain $\left\{w:\left|\psi\left(w,b\right)\right|<R\right\}.$ Therefore,

$$\int_{\left|\psi\left(b,w\right)\right|=R_1}\left|F(w)\right|^q\left|dw+A\left(w\right)d\overline{w}\right|<M_2,0<R_1<R,$$

whence

$$\int_{\partial L(a,\rho)}\left|f(z)\right|^q\left|dz+A(z)d\overline{z}\right|=\int_{\left|\psi\left(b,\varsigma\right)\right|=R}\left|f\left(\varphi(\varsigma)\right)\right|^q\left|\varphi'\left(\varsigma\right)\right|\left|d\varsigma+A\left(\varsigma\right)d\overline{\varsigma}\right|=\int_{\left|\psi\left(b,\varsigma\right)\right|=R}\left|F\left(\varsigma\right)\right|^q\left|d\varsigma+A\left(\varsigma\right)d\overline{\varsigma}\right|<M_2,$$

which means that $f(z)$ belongs to class E_A^q in the domain $L(a,r).$ In the case where the domain $L(a,r),$ bounded by a straight-rectifiable curve $\gamma,$ is not a domain of class $C,$ we have seen that it is always possible to construct a function $f(z)$ belonging in domain $L(a,r)$ to class $E_A^1,$ whose boundary values are almost everywhere on the boundary $\partial L(a,r)$ equal in absolute value to one, and not included in class E_A^p for $p>1,$ i.e. the theorem just proved is not true in such domains.

From this theorem the following corollary follows:

Corollary 1. *If* $f(z)$ *belongs to class* H_A^p *and* $\left|f_S^*(\zeta)\right|^q$ *is a summable function, then* $f(z)$ *belongs to class* H_A^q.

Conclusion. The theory of H^p and E^p spaces has its origins in discoveries made forty or fifty years ago by such mathematicians as G.H.Hardy, I.I.Privalov, F. and M. Riesz, V.I.Smirnov. Most of this early work is concerned with the properties of individual functions of class E^p and is classical in spirit. Smirnov's theorem was proved in the simple case for $A(z)-$analytic functions in [6]. In this paper we generalized this theorem.

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