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RIESZ'S THEOREM FOR $A(z)$ -ANALYTIC FUNCTIONS

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RESUME

We consider $A(z)$ -analytic functions in case when $A(z)$ is antianalytic function. In this paper, an analogue of theorem Riesz's is proved for $A(z)$ -analytic functions from the Hardy class.

Key words: $A(z)$ -analytic function, Hardy class, $A(z)$ -lemniscate, Taylor series, Blaschke product, Riesz's theorem for $A(z)$ -analytic functions.

Introduction

$A(z)$ -analytic functions. Let $A(z)$ be antianalytic function, i.e. $\frac{\partial A}{\partial \bar{z}} = 0$ in the domain $D \subset \mathbb{C}$ and there is a constant $c < 1$ such that $|A(z)| \leq c$ for all $z \in D$. The function $f(z)$ is said to be $A(z)$ -analytic in the domain D if for any $z \in D$, the following equality holds:

$$\frac{\partial f}{\partial \bar{z}} = A(z) \frac{\partial f}{\partial z}. \quad (1)$$

We denote by $O_A(D)$ the class of all $A(z)$ -analytic functions defined in the domain D . Since an antianalytic function is infinitely smooth, then $O_A(D) \subset C^\infty(D)$ (see [2]). In this case, the following takes place:

Theorem 1. (see [3], analogue of Cauchy integral theorem). *If $f \in O_A(D) \cap C(\bar{D})$, where $D \subset \mathbb{C}$ is a domain with smooth ∂D , then*

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $\xi \in D$ is a fixed point in it. Since the function $\bar{A}(z)$ is analytic, the integral

$$I(z) = \int_{\gamma(\xi, z)} \overline{A(\tau)} d\tau$$

is independent of the path of integration; it coincides with the antiderivative $I'(z) = \bar{A}(z)$. Consider the function

$$K(z, \xi) = \frac{1}{2\pi i} \frac{1}{z - \xi + I(z)},$$

where $\gamma(\xi, z)$ is a smooth curve which connects the points $\xi, z \in D$ [5].

Theorem 2. (see [5]). $K(z, \xi)$ is an $A(z)$ -analytic function outside of the point $z = \xi$, i.e. $K(z, \xi) \in O_A(D \setminus \{\xi\})$. Moreover, at $z = \xi$ the function $K(z, \xi)$ has a simple pole.

Remark 1. (see [5]). If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function

$$\psi(\xi, z) = z - \xi + I(z),$$

although well defined in D , may have other isolated zeros except ξ : $\psi(\xi, z) = 0$ for except $z \in P \setminus \{\xi, \xi_1, \xi_2, \dots\}$. Consequently, $\psi \in O_A(D)$, $\psi(\xi, z) \neq 0$ when $z \notin P$ and $K(z, \xi)$ is an $A(z)$ -analytic function only in $D \setminus P$, it has poles at the points of P . Due to this fact we consider the class of $A(z)$ -analytic functions only in convex domains.

According to [5], Theorem 1.2, the function $\psi(\xi, z)$ is an $A(z)$ -analytic function.

The following set is an open subset of D :

$$L(a, r) = \{z \in D : |\psi(a, z)| < r\}.$$

For sufficiently small $r > 0$, this set compactly lies in D (we denote it by $L(a, r) \subset\subset D$) and contains the point a . The set $L(a, r)$ is called an $A(z)$ -lemniscate centered at the point a . The lemniscate $L(a, r)$ is a simply-connected set (see [5]).

Theorem 3. (see [4], Cauchy's integral formula). Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be an arbitrary subdomain with a smooth or piecewise smooth ∂G . Then for any function $f(z) \in O_A(G) \cap C(\bar{G})$, the following formula holds:

$$f(z) = \int_{\partial G} f(\xi) K(z, \xi) (d\xi + A(\xi) d\bar{\xi}), \quad z \in G. \quad (2)$$

Note that from formula (2) it follows that if $f(z) \in O_A(L(a, r)) \cap C(\bar{L}(a, r))$, where $L(a, r) \subset\subset D$ is a fixed $A(z)$ -lemniscate, then in $L(a, r)$ the function $f(z)$ is expanded in a Taylor series:

$$f(z) = \sum_{k=0}^{\infty} c_k \psi^k(a, z), \quad (3)$$

where $c_k = \frac{1}{2\pi i} \int_{|\psi(a, \xi)|=\rho} \frac{f(\xi)}{(\psi(a, \xi))^{k+1}} (d\xi + A(\xi) d\bar{\xi}), 0 < \rho < r, k = 0, 1, 2, \dots$.

Angular limits and Hardy classes for $A(z)$ -analytic functions. Let $L(a, r) \subset\subset D$ and $f(z) \in O_A(L(a, r))$. We define the concepts of angular and radial limits of $A(z)$ -subharmonic and $A(z)$ -analytic functions in lemniscate $L(a, r)$. The radial limits of the function $f(z)$ at some point $\zeta \in \partial L(a, r)$ are denoted as $f^*(\zeta)$ and the angular limits are denoted as $f_{\triangleleft}^*(\zeta)$ (see [6]).

In the classical case of the disk $U = \{w \in \mathbb{C} : |w| < 1\} \subset \mathbb{C}_w$, the limit by the radius $\tau_{\zeta} = \{w = t\zeta, 0 \leq t \leq 1, \zeta \in \partial U$ of the function $g(w)$,

$$g^*(\zeta) = \lim_{w \rightarrow \zeta, w \in \tau_{\zeta}} g(w)$$

is called the radial limit, and the limit by the angle $\triangleleft \subset U$, ending at the point $\zeta \in \triangleleft$, is called the angular limit,

$$g_{\triangleleft}^*(\zeta) = \lim_{w \rightarrow \zeta, w \in \triangleleft_{\zeta}} g(w).$$

Since lemniscate $L(a, r)$ is a simply connected domain with a real analytic boundary, then according to Riemann's theorem there exists a conformal map $\chi(z) : U \rightarrow L(a, r)$, which is also conformal in some neighborhood of closure $\bar{U}$. Let χ maps the boundary point $\lambda \in \partial U$ to the boundary point $\zeta \in \partial L(a, r)$. Then the curve $\gamma_\zeta = \chi(\tau_\lambda)$ has the property that it connects points a, ζ and is perpendicular to all lines of level $\partial L(a, \rho) = \{|\psi(a, z)| = \rho\}, 0 < \rho \leq r$. In the theory of $A(z)$ -analytic functions, the curve $\gamma_\zeta = \chi(\tau_\lambda)$ plays the role of the radial direction, and the image of the angle $\chi(\angle)$ plays the role of the angular set at the point $\zeta \in \partial L(a, r)$. We will denote this angle by $\angle = \angle_\zeta$. The limit $f^*(\zeta) = \lim_{z \rightarrow \zeta, z \in \gamma_\zeta} f(z)$ is called the radial limit, and $f^*_\angle(\zeta) = \lim_{z \rightarrow \zeta, z \in \angle_\zeta} f(z)$ is the angular limit of the function $f(z)$ at the point $\zeta \in \partial L(a, r)$ (see [6]).

Now we will show the smoothness of the boundary of lemniscate $L(a, r)$. For this, we take automorphism $\chi^{-1}(z) : \bar{L}(a, r) \rightarrow \bar{U}$ by Riemann's theorem. Let there be some neighborhood $V = \{\psi(a, \zeta) = re^{i\theta}, |\theta| < \varepsilon\}$ for $\forall \varepsilon > 0$. Also has $\chi^{-1}(V) \subset \partial U$ and $\chi^{-1}(\zeta_0) = \lambda_0 \in \partial U$. Further, there is a diffeomorphism $\pi = -i \ln \chi^{-1}(\zeta) : V \rightarrow [-1; 1]$. This diffeomorphism represents all boundary points of the differentiability of the function $f^*(\zeta)$ and $f^*_\angle(\zeta)$ (see [8]).

Next we introduce the Hardy class for $A(z)$ -analytic functions:

Definition 1. (see [6]). *The Hardy class $H^p, p > 0$, for $A(z)$ -analytic functions is the set of all functions $f(z)$ such that its averages*

$$\frac{1}{2\pi\rho} \int_{|\psi(a,z)|=\rho} |f(z)|^p |dz + A(z)d\bar{z}| \quad (4)$$

are uniformly bounded for $\rho < r$, i.e. $\sup_{\rho < r} \left\{ \frac{1}{2\pi\rho} \int_{|\psi(a,z)|=\rho} |f(z)|^p |dz + A(z)d\bar{z}| \right\} < \infty$.

The Hardy class for $A(z)$ -analytic functions in the domain $L(a, r)$ is denoted as $H^p_A(L(a, r))$. The norms in them are defined by the formula (see [6]):

$$\|f\|_{H^p_A} = \sup_{|\psi(a,z)| < r} \left(\frac{1}{2\pi\rho} \int_{|\psi(a,z)|=\rho} |f(z)|^p |dz + A(z)d\bar{z}| \right)^{\frac{1}{p}} < \infty.$$

Further, from the inequality $b^q < b^p + 1, 0 < q < p, b \geq 0$ we conclude that $f \in H^p_A$ follows $f \in H^q_A$, i.e. $H^p_A \subset H^q_A$ for all p and q . Let us define a class of bounded functions

$$H^\infty_A(L(a, r)) = \left\{ f(z) \in O_A(L(a, r)) : \sup_{|\psi(a,z)| < r} \{|f(z)|\} < \infty \right\}.$$

The norm in $H^\infty_A(L(a, r))$ is defined as $\|f(z)\|_{H^\infty_A} = \sup_{z \in L(a, r)} \{|f(z)|\}$ (see [6]).

The Fatou's theorems and Cauchy's integral formula for Hardy class H^1_A .

Now, we will consider the Fatou's theorem for the class of functions H^1_A :

Theorem 4. (see [6], the Fatou's theorem for the class of functions H^1_A). *If $f(z) \in H^1_A(L(a, r))$, then the angular limit*

$$f^*_\angle(\zeta) = \lim_{z \rightarrow \zeta, z \in \angle_\zeta} f(z)$$

exists and is finite for almost all $\zeta \in \partial L(a, r)$, except, perhaps, the points of some set E of measure zero.

The following statements follow from Theorem 4:

Theorem 5. (see [6]). If $f(z) \in H_A^1(L(a, r))$, then $f^*(\zeta) \in L_A^1(\partial L(a, r))$. As $\rho \rightarrow r$

$$\int_{|\psi(a, z)|=\rho} f(z) |dz + A(z) d\bar{z}| \longrightarrow \int_{|\psi(a, \zeta)|=r} f^*(\zeta) |d\zeta + A(\zeta) d\bar{\zeta}| \quad (5)$$

and

$$\int_{|\psi(a, z)|=\rho} |f(z) - f^*(\zeta)| |dz + A(z) d\bar{z}| \longrightarrow 0. \quad (6)$$

According to Cauchy integral formula (2) for lemniscates $L(a, r)$

$$f(z) = \frac{1}{2\pi i} \int_{|\psi(a, \xi)|=\rho} f(\xi) K(\xi, z) (d\xi + A(\xi) d\bar{\xi}),$$

we conclude that

$$f(z) = \frac{1}{2\pi i} \int_{|\psi(a, \zeta)|=r} f^*(\zeta) K(\zeta, z) (d\zeta + A(\zeta) d\bar{\zeta}). \quad (7)$$

This is the Cauchy integral formula for functions of H_A^1 .

Let the function $f(z) \in O_A(L(a, r))$ and $a_1, a_2, a_3, \dots$ be the zeros of the function in this domain, $r_n = |\psi(a, a_n)|$. If

$$M = \sup_{0 < \rho < r} \left\{ \frac{1}{2\pi \rho} \int_{\partial L(a, \rho)} |f(z)| |dz + A(z) d\bar{z}| \right\}$$

then

$$\sum_{n=1}^{\infty} (r - |\psi(a, a_n)|)$$

is bounded and the Blaschke product

$$B(z) = \sum_{n=1}^{\infty} r \cdot \frac{|\psi(a, a_n)|}{\psi(a, a_n)} \frac{\psi(a, a_n) - \psi(a, z)}{r^2 - \bar{\psi}(a, a_n) \psi(a, z)}$$

is $A(z)$ -analytic in $L(a, r)$, $f(z) = B(z)G(z)$, where the function $G(z)$ is $A(z)$ -analytic and has no zeros at $|\psi(a, z)| < r$ (see [7]).

Also, from formula (7) it follows that if $f(z) \in H_A^1(L(a, r))$, then there exists a Blaschke product $B(z)$ and a function $G(z) \in H_A^1(L(a, r))$ that has no zeros in $L(a, r)$, such that $f(z) = B(z)G(z)$.

Therefore, for almost all $\zeta \in \partial L(a, r)$, the limit function $B_{\zeta}^*(\zeta)$ at $z \rightarrow \zeta_{\zeta}$ exists from Theorem 4.

Main result

Theorem 6. (Riesz's theorem for $A(z)$ -analytic functions). *If $f(z) \in H^p(L(a, r))$, then*

$$\int_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)| |dz + A(z)d\bar{z}| \rightarrow 0$$

at $\rho \rightarrow r$, where $f^*(\zeta)$ - there is almost everywhere an angular value of the function $f(z)$.

Proof. If $B(z)$ is the Blaschke product constructed from the zeros of the function $f(z)$, then $f(z) = B(z)G(z)$, where $G(z) \in H^p$ and $G(z)$ have no zeros in $L(a, r)$. For $\rho < r$ and $0 < p < 1$

$$\begin{aligned} \int_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|^p |dz + A(z)d\bar{z}| &\leq \int_{|\psi(a,z)|=\rho} |B(z)|^p |f(z) - f^*(\zeta)|^p |dz + A(z)d\bar{z}| + \\ &+ \int_{|\psi(a,z)|=\rho} |B(z) - B^*(\zeta)|^p |f^*(\zeta)|^p |dz + A(z)d\bar{z}| \leq \\ &\leq \int_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|^p |dz + A(z)d\bar{z}| + \int_{|\psi(a,z)|=\rho} |B(z) - B^*(\zeta)|^p |f^*(\zeta)|^p |dz + A(z)d\bar{z}|. \end{aligned}$$

For $p \geq 1$, similar inequalities hold, with the only difference being that the p^{nd} degree roots are extracted from the integrals.

Now, from the fact that $|B(z)| \leq 1$, $B(z) \rightarrow B^*(\zeta)$ is almost everywhere for $\rho \rightarrow r$ and $|G^*(\zeta)|^p \in L_A^1$, it follows by Lebesgue's theorem on majorized convergence that

$$\int_{|\psi(a,z)|=\rho} |B(z) - B^*(\zeta)|^p |f^*(\zeta)|^p |dz + A(z)d\bar{z}| \rightarrow 0 \text{ is for } \rho \rightarrow r.$$

So we see that it is enough to show the following fact:

$$\int_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|^p |dz + A(z)d\bar{z}| \rightarrow 0 \text{ at } \rho \rightarrow r$$

for functions $G(z) \in H^p$ that have no zeros in $L(a, r)$.

If $p \geq 1$, then this is simple, since according to the previous subsection

$$G(z) = \frac{1}{2\pi\rho} \int_{|\psi(a,\zeta)|=r} P(z, \zeta) G^*(\zeta) |dz + A(z)d\bar{z}|,$$

because it is obviously $G(z) \in H_A^1(L(a, r))$, where $P(z, \zeta) = \frac{r^2 - |\psi(a,z)|^2}{|\psi(\zeta, z)|^2}$ is the Poisson kernel. By Theorem 5 the function is $G^*(\zeta) \in L_A^p(L(a, r))$ if $G(z) \in H_A^p(L(a, r))$; therefore, since the kernel $P(z, \zeta)$ is an approximate unit,

$$\int_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|^p |dz + A(z)d\bar{z}| \rightarrow 0,$$

if $\rho \rightarrow r$.

Let's assume $p \geq \frac{1}{2}$. Using Zygmund's clever trick, let us set $h(z) = \sqrt{G(z)}$; the square root in this case is well defined, since the function $G(z)$ never vanishes $L(a, r)$. Then $h(z) \in H_A^{2p}(L(a, r))$ and $2p \geq 1$. We have

$$\int_{|\psi(a,z)|=\rho} |G(z) - G^*(\zeta)|^p |dz + A(z)d\bar{z}| = \int_{|\psi(a,z)|=\rho} |(h(z) - h^*(\zeta))(h(z) + h^*(\zeta))|^p |dz + A(z)d\bar{z}|,$$

which, according to the Schwarz inequality, does not exceed

$$\begin{aligned} & \sqrt{\int_{|\psi(a,z)|=\rho} |h(z) - h^*(\zeta)|^{2p} |dz + A(z)d\bar{z}|} \cdot \sqrt{\int_{|\psi(a,z)|=\rho} |h(z) + h^*(\zeta)|^{2p} |dz + A(z)d\bar{z}|} \leq \\ & \leq C' \sqrt{\int_{|\psi(a,z)|=\rho} |h(z) - h^*(\zeta)|^{2p} |dz + A(z)d\bar{z}|}, \end{aligned}$$

where C' does not depend on ρ , since $h(z) \in H_A^{2p}(L(a, r))$. But due to the fact that $2p \geq 1$, and what was proved above, the last expression tends to zero at $\rho \rightarrow r$. Therefore, it is proved that the required result is valid for $p \geq \frac{1}{2}$.

If now $p \geq \frac{1}{4}$, then, again introducing the auxiliary function $h = \sqrt{G}$, we will reason as above, using the result just proven. $\triangleright$

First of all, we note that the proof in case $p = 2$ is elementary. Indeed, if the function $f(z) = \sum_{n=0}^{\infty} c_n \psi^n(a, z)$ belongs to the space H_A^2 , then

$$\int_{|\psi(a,z)|=\rho} |f(z)|^2 |dz + A(z)d\bar{z}| = \pi \rho \sum_{n=0}^{\infty} |c_n|^2 \rho^{2n},$$

due to absolute convergence and orthogonality, therefore $\sum_{n=0}^{\infty} |c_n|^2$ is bounded.

According to the Riesz-Fischer theorem, there exists a function $f^*(\zeta) \in L_A^2(\partial L(a, r))$ with a Fourier series $\sum_{n=0}^{\infty} c_n \psi^n(a, z) = \sum_{n=0}^{\infty} c'_n e^{in\theta}$, where $\psi(a, z) = \rho e^{i\theta}$. Now, by direct calculation it is easy to check that for $\rho < r$

$$f(z) = \frac{1}{2\pi r} \int_{|\psi(a,\zeta)|=r} P(z, \zeta) f^*(\zeta) |dz + A(z)d\bar{z}|$$

(using series expansion), so that in fact

$$f(z) \rightarrow f^*(\zeta) \text{ almost everywhere at } z \rightarrow \zeta_{\triangleleft}$$

and

$$\int_{|\psi(a,z)|=\rho} |f(z) - f^*(\zeta)|^2 |dz + A(z)d\bar{z}| \rightarrow 0$$

at $\rho \rightarrow r$.

Using this result we can prove the theorem of F. and M. Riesz as follows. Let's take any measure $\mu(\zeta)$ by $\partial L(a, r)$, such that

$$\int_{|\psi(a, \zeta)|=r} \psi^n(a, \zeta) |d\mu(\zeta)| = 0, \quad n \in \mathbb{N},$$

and put

$$f(z) = \frac{1}{2\pi r} \int_{|\psi(a, \zeta)|=r} P(z, \zeta) |d\mu(\zeta)|$$

for $0 \leq \rho < r$. As is easy to see,

$$f(z) = \sum_{n=0}^{\infty} c_n \psi^n(a, z) = \sum_{n=0}^{\infty} c'_n e^{in\theta},$$

where $c_n = \frac{1}{2\pi r} \int_{|\psi(a, z)|=\rho} \psi^n(a, z) (dz + A(z)d\bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} d\mu(\theta) = c'_n$; therefore, the function is $f(z) \in O_A(L(a, r))$ and according to the previous subsection, $f(z) \in H_A^1(L(a, r))$. As at the beginning of the proof of the theorem, we can represent the function $G(z)$ as $f(z) = B(z)G(z)$, where $B(z)$ is the Blaschke product, and $h(z) \in H_A^1(L(a, r))$, and the function $h(z)$ has no zeros in $L(a, r)$. Therefore, there exists an $A(z)$ -analytic function in $L(a, r)$ such that $h(z) = f^2(z)$; it is almost obvious that $f(z) \in H_A^2(L(a, r))$. Now we use a special case of the theorem just proven: let's put $G^*(\zeta) = B^*(\zeta)(f^*(\zeta))^2$; then $G^*(\zeta) \in L_A^1(\partial L(a, r))$. We have

$$\begin{aligned} \int_{|\psi(a, z)|=\rho} |f(z) - f^*(\zeta)| |dz + A(z)d\bar{z}| &\leq \int_{|\psi(a, z)|=\rho} |B(z) - B^*(\zeta)| |f^*(\zeta)|^2 |dz + A(z)d\bar{z}| + \\ &+ \int_{|\psi(a, z)|=\rho} |B(z)| |f^2(z) - (f^*(\zeta))^2| |dz + A(z)d\bar{z}|. \end{aligned}$$

At $\rho \rightarrow r$, the first integral on the right-hand side tends to zero by Lebesgue's theorem on dominated convergence. The second does not exceed

$$\int_{|\psi(a, z)|=\rho} |f^2(z) - (f^*(\zeta))^2| |dz + A(z)d\bar{z}|,$$

and this last value tends to zero at $\rho \rightarrow r$ (we reason in the same way as when using Zigmund's method, taking into account the already proven fact that $\int_{|\psi(a, z)|=\rho} |f(z) - f^*(\zeta)|^2 |dz + A(z)d\bar{z}| \rightarrow 0$

at $\rho \rightarrow r$). Finally,

$$\int_{|\psi(a, z)|=\rho} |G(z) - G^*(\zeta)|^2 |dz + A(z)d\bar{z}| \rightarrow 0, \quad \text{at } \rho \rightarrow r.$$

But according to the previous subsection at $\rho \rightarrow r$

$$G(z) (dz + A(z)d\bar{z}) \rightarrow d\mu(z).$$

Consequently, $d\mu(\zeta) = G^*(\zeta) (d\zeta + A(\zeta)d\bar{\zeta})$ and the measure μ are absolutely continuous, which proves the theorem of F. and M. Riesz.

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REZYUME

Ushbu maqolada $A(z)$ funksiyani antianalitik bo'lgan holda $A(z)$ -analitik funksiyalarni qaraymiz. Ushbu maqolada biz Xardi sinfidagi $A(z)$ -analitik funksiyalar uchun Riesz teoremasining analogini isbotlaymiz.

Kalit so'zlar: $A(z)$ -analitik funksiya, Xardi sinfi, $A(z)$ -lemniskata, Teylor qatori, Blyashke ko'paytmasi, $A(z)$ -analitik funksiyalar uchun Riss teoremasi.

РЕЗЮМЕ

Мы рассматриваются $A(z)$ -аналитические функции в случае, когда $A(z)$ - анти-аналитическая функция. В данной работе доказывается аналог теоремы Рисса для $A(z)$ -аналитических функций из класса Харди.

Ключевые слова: $A(z)$ -аналитическая функция, класс Харди, $A(z)$ -lemniskata, ряд Тейлора, произведение Бляшке, теорема Рисса для $A(z)$ -аналитических функций.