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Fatou's Theorem for $A(z)$-Analytic Functions

N. M. Zhabborov[a,]* and B. E. Husenov[b,]**

[a] *Belorussian-Uzbek Joint Intersectoral Institute of Applied Technical Qualifications in Tashkent, Kibraisky district, Tashkent region, 100071 Uzbekistan*

[b] *Bukhara State University, Bukhara, 705018 Uzbekistan*

**e-mail: jabborov61@mail.ru*

***e-mail: husenovbehzod@mail.ru*

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Abstract—We consider $A(z)$-analytic functions in case when $A(z)$ is anti-analytic function. This paper investigates the behavior near the boundary of the derivative of the function, $A(z)$-analytic inside the $A(z)$-lemniscate and with a bounded change of it at the boundary. Thus, this paper introduces the complex Lipschitz condition for $A(z)$-analytic functions and proves Fatou's theorem for $A(z)$-analytic functions.

Keywords: $A(z)$-analytic functions, $A(z)$-lemniscate, "radial" convergence in $A(z)$-lemniscate, the complex Lipschitz condition for $A(z)$-analytic functions, Fatou's theorem for $A(z)$-analytic functions

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INTRODUCTION

Paragraph 1. On a class of $A(z)$-analytic functions.

Solutions of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = A(z)\frac{\partial f}{\partial z} \tag{1}$$

are directly related to quasi-conformal mappings. With respect to the $A(z)$ function, it is measurable and

$$|A(z)| \le c < 1$$

almost everywhere in the domain under consideration $D \subset \mathbb{C}$, where $c = \text{const}$. In the literature, the solutions of Eq. (1) are commonly called $A(z)$-*analytic functions*.

The work of Srebro and Yakubov [1], which established a local theorem of the existence and uniqueness of homeomorphic solutions of degenerate Beltrami equations, is written in geometric terms.

One of the fundamental works in the theory of Beltrami equations is a monograph by Gutlyanskii et al. [2], which considers a geometric approach to the study of the Beltrami equation.

The solutions of Eq. (1), as well as quasi-conformal homeomorphisms in the complex plane $\mathbb{C}$, have been studied in sufficient details. In the introduction we confine ourselves to giving the references ([3–6]) and formulating the following three theorems:

Theorem 1 (see [5]). *For any measurable on the complex plane function* $A(z) : \|A\|_\infty < 1$ *there exists a unique homeomorphic solution* $\psi(z)$ *of Eq.* (1) *which fixes the points* $0, 1, \infty$.

Note that if the function $|A(z)| \le c < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A(z) \equiv 0$ outside D, so Theorem 1 holds for any domain $D \subset \mathbb{C}$.

Theorem 2 (see [3, 4]). *All generalized solutions of Eq.* (1) *have the form* $f(z) = F[\psi(z)]$, *where* $\psi(z)$ *is a homeomorphic solution in Theorem* 1, *and* $F(z)$ *is a holomorphic function in the domain* $\psi(D)$. *Moreover, if a generalized solution* $f(z)$ *has isolated singular points, then the holomorphic function* $F = f \circ \psi^{-1}$ *also has isolated singularities of the same types.*

Theorem 2 implies that an $A(z)$-analytic function f carries out an internal (open) mapping, i.e., it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain $G \subset D$ the maximum of the modulus is reached only on the boundary, i.e., $|f(z)| \le \max_{z \in \partial G} |f(z)|$, $z \in G$. If the function is not zero, then the minimum principle also holds, i.e., $|f(z)| \ge \min_{z \in \partial G} |f(z)|$, $z \in G$ (see [7]).

Theorem 3 (see [6]). *If a function $A(z)$ belongs to the class $C^m(D)$, then every solution f of Eq.* (1) *also belongs, at least, to the same class $C^m(D)$.*

The aim of this paper is to investigate $A(z)$-analytic functions in a special case when the function $A(z)$ is an anti-analytic function in a domain. This paper provides an overview and extends some boundary properties of the class of holomorphic functions, such as [8, 9]. We introduce the angular limit for $A(z)$-analytic functions. Section 1 investigates the behavior near the boundary of the derivative of the function, $A(z)$-analytic inside the $A(z)$-lemniscate and with a bounded change of it at the boundary. Section 2 introduces the complex Lipschitz condition for $A(z)$-analytic functions and proves Fatou's theorem for $A(z)$-analytic functions.

Let $A(z)$ be anti-analytic, i.e., $\frac{\partial A}{\partial z} = 0$, in $D \subset \mathbb{C}$, and such that $|A(z)| \le c < 1$, $\forall z \in D$. We put

$$D_A = \frac{\partial}{\partial z} - \overline{A}(z)\frac{\partial}{\partial \overline{z}}, \quad \overline{D}_A = \frac{\partial}{\partial \overline{z}} - A(z)\frac{\partial}{\partial z}.$$

Then, according to (1), the class $A(z)$-analytic functions in D is characterized by the fact that $\overline{D}_A f = 0$. Since an anti-analytic function is smooth, Theorem 3 implies that $O_A(D) \subset C^\infty(D)$ (see [7]). In this case, the following takes place:

Theorem 4 (analogue of Cauchy's theorem [10]). *If $f \in O_A(D) \cap C(\overline{D})$, where $D \subset \mathbb{C}$ is a domain with rectifiable boundary ∂D, then*

$$\int_{\partial D} f(z)(dz + A(z)d\overline{z}) = 0.$$

Now, we assume that the domain $D \subset \mathbb{C}$ is convex, and $\zeta \in D$ is a fixed point in it. Consider the function

$$K(z,\zeta) = \frac{1}{2\pi i}\frac{1}{z - \zeta + \overline{\int_{\gamma(\zeta,z)} \overline{A(\tau)}d\tau}}, \tag{2}$$

where $\gamma(\zeta, z)$ is a smooth curve which joins points of $\zeta, z \in D$. Since the domain is simply connected and the function $\overline{A}(z)$ is holomorphic, the integral

$$I(z) = \overline{\int_{\gamma(a,z)} \overline{A(\tau)}d\tau}$$

does not depend on a path of integration; it coincides with a primitive, i.e., $I'(z) = \overline{A}(z)$ (see [7]).

Theorem 5 (see [7]). *$K(z,\zeta)$ is an $A(z)$-analytic function outside of the point $z = \zeta$, i.e., $K(z,\zeta) \in O_A(D \backslash \{\zeta\})$. Moreover, at $z = \zeta$ the function $K(z,\zeta)$ has a simple pole.*

Remark 1 (see [7]). If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function

$$\psi(z,\zeta) = z - \zeta + \overline{\int_{\gamma(\zeta,z)} \overline{A(\tau)}d\tau},$$

although well defined in D, may have other isolated zeros except for $\zeta : \psi(z,\zeta) = 0$ for $z \in P \backslash \{\zeta, \zeta_1, \zeta_2, \ldots\}$. Consequently, $\psi \in O_A(D)$, $\psi(z,\varsigma) \ne 0$ when $z \notin P$ and $K(z,\zeta)$ is an $A(z)$-analytic function only in $D \backslash P$, it has poles at the points of P. Due to this fact we consider the class of $A(z)$-analytic functions only in convex domains.

According to Theorem 2, the function $\psi(z,a) \in O_A(D)$ carries out an internal mapping. In particular, the set

$$L(a,r) = \left\{ z \in D : |\psi(z,a)| = \left| z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right| < r \right\}$$

is open in D. For suffiently small $r > 0$ it compactly belongs to D and contains the point a. This set is called an $A(z)$-lemniscate with the center a and denoted by $L(a,r)$. According to the maximum principle the lemniscate $L(a,r)$ is simply connected and to the minimum principle it is connected (see [7]).

Theorem 6 (Cauchy formula [10]). *Let $D \subset \mathbb{C}$ be an arbitrary convex domain and $G \subset D$ be a subdomain, with piecewise smooth boundary ∂G. Then for any function $f(z) \in O_A(G) \cap C(\overline{G})$ we have a formula*

$$f(\zeta) = \int_{\partial G} K(\zeta, z) f(\zeta)(d\zeta + A(\zeta) d\overline{\zeta}), \quad z \in G. \tag{3}$$

Paragraph 2. Class of $A(z)$-harmonic functions.

Let $f = u + iv$.

Theorem 7 (see [11]). *The real part of the $A(z)$-analytic functions of $f(z) \in O_A(D)$ satisfies equation*

$$\Delta_A u = \frac{\partial}{\partial z}\left(\frac{1}{1-|A|^2}\left(\left(1+|A|^2\right)\frac{\partial u}{\partial \overline{z}} - 2A\frac{\partial u}{\partial z} \right) \right) + \frac{\partial}{\partial \overline{z}}\left(\frac{1}{1-|A|^2}\left(\left(1+|A|^2\right)\frac{\partial u}{\partial z} - 2\overline{A}\frac{\partial u}{\partial \overline{z}} \right) \right) = 0 \tag{4}$$

in the domain of D.

In connection with Theorem 7, it is natural to define the $A(z)$-harmonic function as follows.

Definition 1 (see [11]). A double differentiable function $u \in C^2(D)$, $u : D \to \mathbb{R}$ is called $A(z)$-harmonic in the domain D if it satisfies the differential Eq. (4).

The class of $A(z)$-harmonic functions in the domain of D is denoted as $h_A(D)$. Thus, the real part and hence the imaginary part, of the $A(z)$-harmonic function are $A(z)$-harmonic functions in the domain of D. The inverse theorem is also true for simply connected domains.

Theorem 8 (see [11]). *If the function is $u(z) \in h_A(D)$, where D is a simply connected domain, then $f \in O_A(D) : u = \operatorname{Re} f$.*

For $A(z)$-analytic and $A(z)$-harmonic functions, the following Dirichlet problem is naturally considered:

Dirichlet problem (see [11]). *A bounded domain of $G \subset D$ is given and a continuous function of $\omega(\zeta)$ is set at the boundary of ∂G. It is required to find $A(z)$-harmonic in the domain of G continuous on the closure of $\overline{G}$ the function of $u(z) \in h_A(G) \cap C(\overline{G}) : u|_{\partial G} = \omega$.*

Theorem 9 (see [11]) (an analogue of the Poisson formula for $A(z)$-harmonic functions). *If the $\omega(\zeta)$ function is continuous on the boundary of the lemniscate of $L(a,r) \subset D$, then the function*

$$u(z) = \frac{1}{2\pi r} \int_{|\psi(\zeta,a)|=r} \omega(\zeta) \frac{r^2 - |\psi(z;a)|^2}{|\psi(\zeta,z)|^2} |d\zeta + A(\zeta) d\overline{\zeta}| \tag{5}$$

is the solution of the Dirichlet problem in $L(a,r)$.

The $f(\zeta,z) = \dfrac{\psi(a,\zeta) + \psi(a,z)}{\psi(z,\zeta)}$ function is an $A(z)$-analytic function for $z \in L(a,r)$, where $\zeta \in \partial L(a,r)$. Then

$$P(\zeta,z) = \frac{1}{2\pi}(f(\zeta,z) + \overline{f}(z,\zeta)) = \frac{1}{2\pi}\left(\frac{\psi(a,\zeta)+\psi(a,z)}{\psi(a,\zeta)-\psi(a,z)} + \frac{\overline{\psi}(a,\zeta)+\overline{\psi}(a,z)}{\overline{\psi}(a,\zeta)-\overline{\psi}(a,z)} \right)$$
$$= \frac{1}{2\pi}\left(\frac{|\psi(a,\zeta)|^2 - |\psi(a,z)|^2}{|\psi(z,\zeta)|^2} \right) = \frac{1}{2\pi}\left(\frac{r^2 - |\psi(a,z)|^2}{|\psi(z,\zeta)|^2} \right).$$

Formula (5) is called an analogue of the Poisson formula for $A(z)$-harmonic functions.

1. BEHAVIOR NEAR THE BOUNDARY OF THE DERIVATIVE OF THE FUNCTION, $A(z)$-ANALYTIC INSIDE $A(z)$-LEMNISCATE AND WITH A BOUNDED CHANGE AT THE BOUNDARY OF THIS DOMAIN

Consider the function $f(z)$, $A(z)$-analytic inside the lemniscate $L(a;r)$ and continuous in a closed lemniscate $\overline{L}(a;r)$, for which the values $f(\zeta)=\omega(\zeta)+i\nu(\zeta)$ at the boundary $\partial L(a;r)$ form a function with a bounded change. Then we will look at the following proposition:

Proposition 1. *Let* $f\in O_A(L(a,r))\bigcap C(\overline{L}(a,r))$. *The function* $f'(z)$ *tends almost everywhere on the boundary* $\partial L(a,r)$ *to the values* $f'(\zeta)$, *when the point* z *approaches the radius of the point* ζ.

Proof. From (5) we represent f functions in the form of a Poisson integral, i.e.

$$f(z)=u(z)+iv(z),$$

where

$$u(z)=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta,a)|=r}\omega(\zeta)\frac{r^2-|\psi(z,a)|^2}{|\psi(\zeta,z)|^2}\left|d\zeta+A(\zeta)d\overline{\zeta}\right|,$$

$$v(z)=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta,a)|=r}\nu(\zeta)\frac{r^2-|\psi(z,a)|^2}{|\psi(\zeta,z)|^2}\left|d\zeta+A(\zeta)d\overline{\zeta}\right|,$$

is assumed.

By differentiating and integrating in parts, we can easily obtain:

$$\frac{\partial u}{\partial z}=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta,a)|=r}\omega(\zeta)\frac{\partial}{\partial\zeta}\frac{r^2-|\psi(z,a)|^2}{|\psi(\zeta,z)|^2}\left|d\zeta+A(\zeta)d\overline{\zeta}\right|=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta,a)|=r}\frac{r^2-|\psi(z,a)|^2}{|\psi(\zeta;z)|^2}\left|d\omega(\zeta)\right|.$$

From the last relation, according to Theorem 3, it follows that $\frac{\partial\omega}{\partial z}$ tends almost everywhere on the lemniscate boundary to the values of $\omega'(\zeta)$, when the point z approaches the point ζ in radius.

Doing the same, we find:

$$\frac{\partial v}{\partial z}=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta,a)|=r}\nu(\zeta)\frac{\partial}{\partial\zeta}\frac{r^2-|\psi(z,a)|^2}{|\psi(\zeta,z)|^2}\left|d\zeta+A(\zeta)d\overline{\zeta}\right|=\frac{1}{2\pi r}\int\limits_{|\psi(\zeta,a)|=r}\frac{r^2-|\psi(z;a)|^2}{|\psi(\zeta,z)|^2}\left|d\nu(\zeta)\right|,$$

from where, according to the same theorem, it follows that $\frac{\partial v}{\partial z}$ tends almost everywhere on the boundary $\partial L(a;r)$ to the values of $\nu'(\zeta)$, when the point z approaches the point ζ in radius.

Therefore,

$$f'(z)=\frac{\partial f}{\partial z}=\frac{\partial u}{\partial z}+i\frac{\partial v}{\partial z}$$

will tend to the values of $u'(z)+iv'(z)$, when z approaches the point ζ by radius, for all points on the boundary ζ, except for the set of points of measure zero, i.e., there is an open set $U\subset\partial L(a,r)$ on which $\mu\left(\zeta:\lim\limits_{z\to\zeta\in U}f(z)\right)=0$, where $\mu(\zeta)$ is the measure of the lemniscate boundary $\partial L(a,r)$ and $\mu(U)=\int\limits_U\left|d\zeta+A(\zeta)d\overline{\zeta}\right|$. The last expression is converted to the form:

$$f'(\zeta)=\omega'(\zeta)+i\nu'(\zeta),$$

where $f'(\zeta)$ should mean the derivative of the function $f(z)$ at point ζ, the boundary taken relative to the points of this boundary lemniscate $\partial L(a,r)$.

$\square$

Now let's prove Vitali's proposal:

Proposition 2. *If the function $f(z) \in O_A(L(a,r))$ and bounded within a certain sector of the lemniscate $L(a,r)$, tends along its bisector to a certain limit, then it tends to the same limit when the point z approaches the vertex of the sector in an arbitrary way, remaining on the sector internal to the given one.*

Proof. Taking the vertex of the lemniscate sector $L(a,r)$ as the origin, considering the sectors equal to r, the magnitude of the angle 2α, we put

$$f_n(z) = f\left(\frac{z}{2^n}\right), \quad (n \in \mathbb{N}),$$

where $\frac{r}{2} - \varepsilon < |\psi(z,a)| < r + \varepsilon$, $-\alpha < \arg\psi(z,a) < \alpha$.

The sequence of functions $f_n(z)$ is uniformly bounded in this lemniscate $L(a,r)$ and converges to a constant number at all points z satisfying condition

$$\frac{r}{2} - \varepsilon < |\psi(z,a)| < r + \varepsilon, \quad \arg\psi(z,a) = 0.$$

Therefore, according to Vitali's theorem, this sequence must converge to the same constant uniformly in a closed lemniscate $L(a,r)$ defined by inequalities

$$\frac{r}{2} < |\psi(z,a)| < r, \quad -\alpha + \varepsilon < \arg\psi(z,a) < \alpha - \varepsilon.$$

This also means that $f(z)$ tends to our limit when the point approaches the top of the lemniscate $L(a,r)$ sector in any way, remaining in sector

$$0 < |\psi(z,a)| < r, \quad -\alpha + \varepsilon \le \arg\psi(z,a) \le \alpha - \varepsilon,$$

which was required to be proved.

□

So, in the "radial" convergence, taking $\psi(z,a) = \rho e^{i\varphi}$, $\psi(\zeta,a) = re^{i\varphi}$, there will be $\rho \to r$, where $0 < \varphi < 2\pi$, $0 < \rho < r$.

2. FATOU'S THEOREM FOR $A(z)$-ANALYTIC FUNCTIONS

First, we prove Lipschitz conditions for $A(z)$-analytic functions. Let us put $\psi(\zeta_1,a) = re^{i\varphi_1}$, $\psi(\zeta_2,a) = re^{i\varphi_2}$, where $\varphi_1,\varphi_2 \in [0,2\pi]$.

Statement 1. *If a function $f(z) \in O_A(L(a,r)) \bigcap C(\overline{L}(a,r))$ and satisfies the Lipschitz condition*

$$|f(\varphi_1) - f(\varphi_2)| \le E_1|\varphi_1 - \varphi_2|^k, \quad 0 < k \le 1, \tag{6}$$

by $\partial L(a,r)$, then in $\overline{L}(a,r)$ it satisfies the complex Lipschitz condition for $A(z)$-analytic functions

$$|f(\zeta_1) - f(\zeta_2)| < E_2|\psi(\zeta_1,\zeta_2)|^k, \tag{7}$$

where E_1, E_2 are positive numbers.

Proof. It is easy to see that (7) will be proved if we prove two particular inequalities:

$$|f(z_1) - f(z_2)| < E_2|\psi(z_1,z_2)|^k, \tag{8}$$

$$|f(z_1) - f(\xi)| < E_2|\psi(z_1,\xi)|^k, \tag{9}$$

for any points z_1, z_2, ξ of the lemniscate $L(a,r)$. To prove (8), we can assume that $|\varphi_1 - \varphi_2| < \pi$. Now, the function $g(z) = \dfrac{f(z_1) - f(z_2)}{\psi(z,a)}$, where $g(z) \in O_A(L(a,r)) \bigcap C(\overline{L}(a,r))$; therefore, the maximum of its modulus in $\overline{L}(a,r)$ does not exceed the maximum of magnitude $|f(t+\varphi_1) - f(t+\varphi_2)|$ in $[0,2\pi)$, and this maximum in (6) is no greater than $E_1|\varphi_1 - \varphi_2|^k$. Therefore, we have

$$|f(z_1) - f(z_2)| \le E_1 \rho |\varphi_1 - \varphi_2|^k. \tag{10}$$

But

$$\frac{\varphi_1 - \varphi_2}{2} \le \frac{\pi}{2}\left|\sin\frac{\varphi_1 - \varphi_2}{2}\right| = \frac{\pi}{4}|\varphi_1 - \varphi_2|,$$

so from (10) follows (8) with $E_2 = \frac{\pi^k}{2^k}E_1$.

Let us now prove (9). Let us say $0 \le \rho < \rho' < r$. Here, let us put $\psi(\xi, a) = \rho' e^{i\varphi}$. If the Lipschitz condition (6) is satisfied, then the following inequality holds:

$$\exists E > 0, \quad |f(z_1) - f(\xi)| \le \int_{\rho}^{\rho'} |f(t\varphi)|\, dt \le \int_{\rho}^{\rho'} \frac{E}{(r-t)^{1-k}}.$$

This last inequality follows directly from Cauchy's formula for $A(z)$-analytic functions (3). If $\rho' < \frac{r+\rho}{2}$ and therefore $\rho' - \rho < r - \rho'$, then

$$|f(z) - f(\xi)| \le \frac{E}{(r-\rho')^{1-k}} \int_{\rho}^{\rho'} dt \le E\frac{\rho' - \rho}{(\rho' - \rho)^{1-k}} = E(\rho' - \rho)^k.$$

If $\rho' > \frac{r+\rho}{2}$ and therefore $\rho' - \rho > r - \rho'$, then $2(\rho' - \rho) > (r - \rho)$, is also

$$|f(z) - f(\xi)| \le -E\frac{(r-\rho')^k}{k}\Bigg|_{\rho}^{\rho'} < \frac{E}{k}(r-\rho)^k < \frac{E2^k}{k}|\rho' - \rho|^k \le \frac{E}{k}|\rho' - \rho|^k.$$

In both cases, they received (9) with the proper E. The statements are proven.

□

Let us consider a function $f(z)$ $A(z)$-analytic inside the lemniscate $L(a, r)$ and assume it to be bounded; at the same time, we do not make a priori any hypothesis about the existence of limit values of the function for boundary points $\partial L(a, r)$. Fatou's proposal consists of the following statement:

Theorem 10. *The function $f(z) \in O_A(L(a, r))$ bounded inside the lemniscate $L(a, r)$ tends almost everywhere on the boundary $\partial L(a, r)$ to certain values $f(\zeta)$, when point z approaches point ζ along any tangent path.*

Since any nontangent to the boundary path belonging to lemniscate $L(a, r)$ and ending at point ζ_0, $|\psi(a, \zeta_0)| = r$, can be enclosed inside the corner with vertex ζ_0 contained in the lemniscate, the boundary values for all nontangent to the boundary paths inside the lemniscate can be characterized as angular boundary values.

Proof. For proof, consider function

$$\phi(z) = \int_{\gamma(a,z)} f(\vartheta)(d\vartheta + A(\vartheta)d\bar{\vartheta}). \tag{11}$$

Noting that for $|f(z)| < E$ at $|\psi(z, a)| < r$, it is easy to show the existence of boundary values of the function $\phi(z)$ on the boundary, regardless of the path of the point $z \in L(a, r)$. In fact, let l_1 and l_2 be any two paths inside lemniscate $L(a, r)$ connecting point $z = a$ with point ζ border $\partial L(a, r)$.

$$\int_{l_1} f(z)(dz + A(z)d\bar{z}) = \int_{l_2} f(z)(dz + A(z)d\bar{z}). \tag{12}$$

Denoting by C a closed contour formed by a set of lines l_1 and l_2, we reduce the question to the proof of the relation

$$\int_C f(z)(dz + A(z)d\bar{z}) = 0.$$

By connecting two points of contour C with an auxiliary line, lying respectively at l_1 and l_2 arbitrarily close to ζ, we divide C into the sum of two closed lines, and the length λ of one of them (containing point ζ) is arbitrarily small. Accordingly, the integral along C of the function $f(z)$ will be replaced by an integral along the contour of length λ, whence we conclude that

$$\left|\int_C f(z)(dz + A(z)d\overline{z})\right| < E\lambda.$$

Since λ can be made any small positive number,

$$\int_C f(z)(dz + A(z)d\overline{z}) = 0.$$

So, equality (12) is proved, and thus it is found that the function $\phi(z)$ is continuous in a closed lemniscate $\overline{L}(a;r)$.

Further, it is clear from formula (11) that the values of this function on $\partial L(a,r)$ satisfy the complex Lipschitz condition for $A(z)$-analytic functions, i.e.

$$|\phi(\zeta_1) - \phi(\zeta_2)| < E\,|\psi(\zeta_1,\zeta_2)|,$$

where $\zeta_1, \zeta_2 \in \partial L(a,r)$.

Having noticed that the function satisfying the Lipschitz condition for $A(z)$-analytic function will be a fortiori with a bounded change, we can attach the previous section to the $\phi(z)$ proposition. As a result of this proposition $\phi'(z) = f(z)$ tends to a certain limit almost everywhere on the boundary when the point z approaches the radius of the points of the boundary $\partial L(a,r)$.

Note 2. This statement can also be established on the basis of the following considerations.

Due to condition $|f(z)| < E$ and transformations $\psi(z,a) = \rho e^{i\varphi}$, the expression

$$\int_{\gamma(a;z)} f(\vartheta)(d\vartheta + A(\vartheta)d\overline{\vartheta}) = \int_0^{\varphi} f(\varphi)d\varphi,$$

represents a family equally absolutely continuous on the segment $[0,2\pi]$, and, therefore, by virtue of Theorem 3 and the proof of Proposition 1, the function $f(z)$ is represented by the following integral:

$$f(z) = \frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r} f(\zeta)P(\zeta,z)\,|d\zeta + A(\zeta)d\overline{\zeta}|,$$

where $f(z)$ is a bounded change at the boundary $\partial L(a,r)$. We will assume that at any points ζ_0 the boundary is given by the function $\eta(\zeta)$, with a bounded change at the boundary $\partial L(a,r)$ and applying then the Proposition 1:

$$f(z) = \frac{1}{2\pi r}\int_{|\psi(\zeta,a)|=r} P(\zeta,z)\,|d\eta(\zeta)|,$$

where $f(z) \to \eta'(\zeta_0)$, with "radial" convergence $z \to \zeta_0$, from this we learn about the existence of radial boundary values of the function $f(z)$ almost everywhere on the boundary $\partial L(a;r)$.

If we use Proposition 2, we will see that the function $f(z)$ will tend to a certain limit when the point z approaches the point ζ of the boundary along any nontangential paths, for any point ζ for which there is a radial boundary value. Since the latter circumstance takes place almost everywhere on the boundary $\partial L(a,r)$, Fatou's theorem is thus fully proved.

□

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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