AIP Conference Proceedings

RESEARCH ARTICLE | MARCH 11 2024

Asymptotically optimal lattice cubature formulas with a regular boundary layer in the space $H_p^\mu(\Omega)$

Ozodjon Jalolov

AIP Conf. Proc. 3004, 060028 (2024)
https://doi.org/10.1063/5.0199854

Asymptotically Optimal Lattice Cubature Formulas with a Regular Boundary Layer in the Space $H_p^{\mu}(\Omega)$

Ozodjon Jalolov[a)]

Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan

[a)] *Corresponding author: o_jalolov@mail.ru, o.i.jalolov@buxdu.uz*

Abstract. The so-called functional approach turned out to be very efficient in the study of various issues arising in the theory of approximate integration and partial differential equations and related branches of analysis.
The essence of this approach (if we confine ourselves to the example of a boundary value problem for a differential equation) is that the differential equation with the boundary conditions is implemented as an operator acting in a specially selected functional space; the required information is obtained from the properties of this operator.
S.L. Sobolev developed an algorithm for constructing cubature formulas, which he called formulas with a regular boundary layer. He proved the asymptotic optimality of these formulas and the upper-bound estimate of the norm of the error functional in space $U_2^m(\Omega)$, setting the principal term.
The purpose of this study is to obtain a lower estimate (i.e., a lower bound) for any error functional of lattice cubature formulas for spaces $H_p^{\mu}(\Omega)$ and determine the asymptotical optimality of cubature formulas with a regular in the sense of Sobolev boundary layer in space $H_p^{\mu}(\Omega)$.

INTRODUCTION

S.L. Sobolev in [1, 2, 3] developed an algorithm for constructing cubature formulas, which he called formulas with a regular boundary layer in space $U_2^m(\Omega)$. The same result was obtained in spaces $H_2^{\mu}(\Omega)$ in [4, 5]; in [6], the validity of similar results for spaces $H_p^{\mu}(\Omega)$ was stated. When estimating the norm of the error functional of such formulas, the researchers used the explicit form of the extremal function of periodic functional $\ell_\infty(x) = 1 - h^n \sum_{\gamma} \delta(x - hH\gamma)$ in these spaces and the Hilbert property of such spaces. In the case of Banach spaces, the study of the behavior of such formulas is a very difficult task. In [7], the best approximation of the integral over the period of periodic functions of several variables was considered using a finite sum - a linear combination of function values at points of a given regular lattice. Similar results were obtained in the S.L. Sobolev space (see [8, 9, 10, 11, 12, 13, 14]).

1) Notation and preliminary information.

E_n is the n-dimensional real Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $z = (z_1, z_2, \ldots, z_n)$, $\ldots$ $\alpha, \beta, \gamma, \ldots$ are the vectors with integer coordinates $\alpha_i, \beta_i, \gamma_i = 0, \pm1, \pm2, \ldots . (i = 1, 2, \ldots n)$, $\quad |\alpha| = \alpha_1 + \alpha_2 + \ldots . + \alpha_n$.

The iiner product of n-dimensional vectors x and y is denoted by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

C denotes the space of continuous functions with the norm
$\|f(x)|C\| = \max_{x} |f(x)|$
F is the Fourier transform operator,
$F[f(x)](\xi) = \int_{E_n} f(x) e^{-2\pi i x \xi} dx = \int_{E_n} f(x_1, \ldots, x_n) e^{-2\pi i (x_1 \xi_1 + \ldots + x_n \xi_n)} dx_1 \ldots dx_n.$
$F^{-1}[f(x)](\xi) = \int_{E_n} f(x) e^{-2\pi i x \xi} dx$, where $i = \sqrt{-1}$.
For absolutely integrable functions $f(x)$ and $\varphi(x)$, the convolution is defined as:

$$f(x) * \varphi(x) = \int_{E_n} f(x-y) \varphi(y) dy$$

In what follows, we use the basic concepts of the theory of generalized functions [15].

The space S is taken as the space of basic functions, consisting of infinitely differentiable functions decreasing to infinity with all derivatives faster than any negative power of $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

International Scientific and Practical Conference on "Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)"
AIP Conf. Proc. 3004, 060028-1–060028-13; https://doi.org/10.1063/5.0199854
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

Spaces of generalized function over S, as usual, are denoted by S' [15]. The action of the generalized function $\ell(x)$ on the main function $f(x)$ is denoted by $< \ell(x), f(x) >$.

Let H be the matrix of order $n \times n$, $detH = 1$. Ordinary function $f(x)$ defined in E_n is called a periodic function with the leading matrix of periods $H(\overset{\uparrow}{h}^{(1)}, \ldots \overset{\uparrow}{h}^{(n)})$ (H-periodic), if for any $\beta \in Z^n$

$$f(x + H\beta) = f(x),$$

where each period $\overset{\uparrow}{h}^{(k)}$, $k = 1, 2, ..., n$ is a column vector:

$$\overset{\uparrow}{h}^{(k)} = \begin{bmatrix} \overset{\uparrow}{h}_1^{(k)} \\ \vdots \\ \overset{\uparrow}{h}_n^{(k)} \end{bmatrix}$$

Consider the space of points E_n; we identify all points differing by vectors $H\beta$, $\beta \in Z^n$. Z^n is the set of all vectors with integer coordinates. The resulting manifold of equivalent points is an n-dimensional torus θ. We call this torus the fundamental domain for periodic functions. With several cuts, such a torus can be turned into a simply connected region, and in different ways. One of the ways leads to a parallelepiped. Any domain mapped uniquely to the entire torus we call a fundamental domain in E_n. Denoting the characteristic function of the points of domains Ω_0 by $\varepsilon_{\Omega_0}(x)$, we write the necessary and sufficient conditions for Ω_0 to be the fundamental domain in E_n in the following form:

$$\sum_{\beta \in Z^n} \varepsilon_{\Omega_0}(x + H\beta) = 1.$$

Thus, a connection between matrix H and domain Ω_0, is, generally speaking, multi-valued.

Here, from fundamental domains defined by matrix H, we consider only a parallelepiped.

If $u(x)$ is a generalized or ordinary H-periodic function, and $\varphi(x) \in \tilde{C}^\infty$, is H - periodic function, then

$$< u(x), \varphi(x) > \overset{def}{=} \int_{\Omega_0} u(x)\varphi(x)dx.$$

2) The spaces $\tilde{H}_p^\mu(\Omega_0)$.

Let $\mu(x)$ be a continuous line, whose growth is not higher than the power law. The space $\tilde{H}_p^\mu(\Omega_0)$ is defined as the space of H - periodic generalized functions with the norm

$$\left\| f(x) | \tilde{H}_p^\mu(\Omega_0) \right\| = \left\{ \int_{\Omega_0} \left| \sum_\gamma \mu(H_\gamma^{-1}) \hat{f}(\gamma) e^{-2\pi i(\gamma, H_x^{-1})} \right|^p dx \right\}^{\frac{1}{p}}$$

for $1 \leq p < \infty$

$$\left\| f(x) | \tilde{H}_p^\mu(\Omega_0) \right\| = \sup_\gamma \left\{ \left| \hat{f}(\gamma) \right| \mu(H_\gamma^{-1}) \right\}$$

for $p = \infty$.

Obviously, the space $\tilde{H}_p^\mu(\Omega_0)$ is isometrically isomorphic to space $L_p(\Omega_0)$.

3) Weight functions.

Let $1 \leq p < \infty$. Denote by $B(n, p)$ (see [16]) the class of functions $\mu(\xi) = \mu(\xi_1, \xi_2, ..., \xi_n) \in C^\infty$ such that for some constant $m = m(\mu)$

$$\mu_p^p(\mu(\xi + n)/\mu(\xi)) \leq \eta \left(1 + |\eta|^2\right)^{m/2}$$

is true for any $n \in E_n$ and the same is true for $\mu^{-1}(\xi)$. It follows from the definition that functions $\mu(\xi)$, $\mu^{-1}(\xi)$ are multipliers in the space S, i.e., $\mu(\xi) \cdot \varphi(\xi) \in S$ once $\varphi(\xi) \in S$ since $\mu(\xi) \in C^\infty$ and $\mu^{-1}(\xi), \mu(\xi) \leq \left(1 + |\xi|^2\right)^{m/2}$, which follows from $\mu_p^p \in L_\infty$.

4) The space $H_p^\mu(E_n)$.

We say that the generalized function $u(x) \in S'$ belongs to space $H_p^\mu(E_n)$ if

$$\nu(x) = F^{-1}\{\mu(\xi)F[u(x)](\xi)\}(x) \in L_p(E_n).$$

Introducing the norm

$$\left\|u(x)|H_p^\mu(E_n)\right\| = \left\|\nu(x)|L_p(E_n)\right\|,\ u(x) \in H_p^\mu(E_n)$$

we get a space isometrically isomorphic to space $L_p(E_n)$.

5) Space $H_p^\mu(\Omega)$.

Let Ω be a bounded domain with a sufficiently good boundary $\partial\Omega$ in E_n. Denote the closure of the set $C_0^\infty(\Omega)$ in norm $\left\|\boldsymbol{\cdot}|H_p^\mu(E_n)\ \right\|$ by $\overset{0}{H}{}_p^\mu(\Omega)$ and introduce the space

$$H_p^\mu(\Omega) = H_p^\mu(E_n)/\overset{0}{H}{}_p^\mu(E_n\backslash\overline{\Omega})$$

with the norm

$$\left\|u(x)|H_p^\mu(\Omega)\right\| = \inf\left\|u^c(x)|H_p^\mu(E_n)\right\|, \quad u(x) \in H_p^\mu(\Omega),$$

where the lower bound is taken over all extensions of the element $u(x) \in H_p^\mu(\Omega)$ up to the element $u^c(x) \in H_p^\mu(E_n)$. Then $H_p^\mu(\Omega)$ becomes a Banach space.

STATEMENT OF THE PROBLEM

1) Cubature formulas.

Cubature formulas are formulas of the following form:

$$\int_\Omega f(x)dx \approx \sum_{\lambda=1}^N C_\lambda f(x^{(\lambda)}). \tag{1}$$

Here Ω is a bounded domain with a fairly good boundary $\partial\Omega$, C_λ are the coefficients (or weights), $x^{(\lambda)}$ are the nodes, N is the number of nodes.

Here we consider cubature formulas with nodes located on the lattice $\{q_0 + A\gamma;\ \gamma \in Z^n\}$, where q_0 is the fixed vector and γ runs through all Z^n - the set of integer vectors, A denotes matrix $\det A \neq 0$.

To the cubature formula (1), we assign the functional

$$\ell(f) = \int_\Omega f(x)dx - \sum_{\lambda=1}^N C_\lambda f(x^{(\lambda)}), \tag{2}$$

the so-called error functional.

This functional corresponds to the generalized function

$$\ell(x) = \varepsilon_\Omega(x) - \sum_{\lambda=1}^N C_\lambda \delta(x - x^{(\lambda)}), \tag{3}$$

where $\varepsilon_\Omega(x)$ is the characteristic function of domain Ω, $\delta(x - x^{(\lambda)})$ is the delta function concentrated at point $x^{(\lambda)}$.

2) Cubature formulas in the space of periodic functions $\tilde{H}_p^\mu(\Omega_0)$.

It is known that the optimal formula in these spaces has the following form:

$$\int_{\Omega_0} f(x)dx \approx \sum_{\substack{0\le\lambda_i<N_i\\ i=1,2,..,n}} C_\lambda f(h\lambda),$$

where Ω_0 is the unit cube, $h > 0$, is the small parameter, $N = h^{-n}$, i.e. $N = \frac{mes\Omega}{h^n}$, $C_\lambda = C_0$ are constant. Their expressions are determined, but not given here. Note that the formula is called optimal when the norm of the error functional is the least:

$$\inf_{C_\lambda} \left\| \ell(x) | \tilde{H}_p^\mu(\Omega_0) \right\|.$$

The norm of the error functional of the optimal cubature formula has the following form:

$$\left\| \overset{0}{\ell}(x) | \tilde{H}_p^{\mu *}(\Omega_0) \right\| = \left\{ \int_{\Omega_0} \left| \frac{1}{\mu(0)} - C_0 \sum_\gamma \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{q}}. \tag{4}$$

It is also known that the formula of rectangles

$$\int_{\Omega_0} f(x)dx \approx h^n \sum_{\substack{0 \le \lambda_i < N_i \\ i=1,2,..,n}} f(h\lambda) \tag{5}$$

is asymptotically optimal, i.e. the ratio of the norm of the cubature formula of rectangles to the norm of the optimal cubature formula tends to 1 as $h \to 0$.

The norm of the error functional of the cubature formula of rectangles has the following form:

$$\left\| \ell_{rec}(x) | \tilde{H}_p^{\mu *}(\Omega_0) \right\| = \left\{ \int_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{q}} \tag{6}$$

We are interested in the extremal function of the error functional of the cubature formula of rectangles in the space $\tilde{H}_p^\mu(\Omega_0)$, i.e. the function on which the maximum value of the error functional is reached. The extremal function of the error functional of the cubature formula of rectangles $\ell_{rec}(x)$ has the following form:

$$u_\infty(x) = \sum_{\gamma \neq 0} \frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} * \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^{q-1} sign \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \tag{7}$$

This function is h-periodic in each variable. The following function

$$u(x) = \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^{q-1} sign \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \tag{8}$$

is also h-periodic in each variable (hE, where E is the identity matrix) or $H = hE$-periodic function.

In [16], an upper bound was obtained for the norm of the error functional with regular in the sense of S.L. Sobolev boundary layer in spaces $H_p^\mu(\Omega)$.

The main task of this study is to obtain a lower estimate (i.e., a lower bound) for any error functional of lattice cubature formulas for spaces $H_p^\mu(\Omega)$:

$$(mes\Omega)^{\frac{1}{q}} \left\{ \int_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{q}} [1 + o(1)] \le \left\| \ell(x) | \tilde{H}_p^{\mu *}(\Omega) \right\| \tag{9}$$

Since $\left\| \ell_{opt}(x) | H_p^{\mu *}(\Omega) \right\| \le \left\| \ell(x) | H_p^\mu(\Omega) \right\|$ it suffices to show a lower bound for the functional of the optimal cubature formula, i.e. for $\ell_{opt}(x)$. A function

$$u(x) = \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^{q-1} \cdot sign \sum_{\gamma \neq 0} \frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} \tag{10}$$

is an extremal function of the functional

$$< \sum_{\gamma \neq 0} \frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}, f(x) >_{\Omega}, \quad f(x) \in \tilde{L}_p(\Omega_0)$$

This functional takes zero values in the subspace of constants of space $\tilde{L}_p(\Omega_0)$- the space of periodic functions summable with power p over the cube. Therefore, it reaches its maximum value in subspace $\tilde{L}_p(\Omega_0)$, i.e. on functions $\hat{f}[0]$ with zero coefficients.
Therefore, $u(x)$ has the following form:

$$u(x) = \sum_{\gamma \neq 0} \hat{u}[\gamma] e^{2\pi i h^{-1}(\gamma,x)} \tag{11}$$

where $\hat{u}[\gamma] = \int\limits_{\Omega_0} u(x) e^{2\pi i h^{-1}(\gamma,x)} dx$, $\hat{u}\,[0] = 0$.
The function $u_\infty(x)$, (like h-periodic one), can be represented as a Fourier series:

$$u_\infty(x) = \sum_{\gamma \neq 0} \frac{\hat{u}[\gamma] \cdot e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} \text{ , } \hat{u}_\infty\,[0] = 0 \tag{12}$$

$u_\infty(x)$ is a continuous function.

LOWER BOUND FOR THE NORM OF THE ERROR FUNCTIONAL OF LATTICE CUBATURE FORMULAS IN $H_p^\mu(\Omega)$

In [6] and [16], an upper bound was obtained for the norm of the error functional with regular in the sense of Sobolev boundary layer in spaces $H_p^\mu(\Omega)$, i.e. the inverse inequality to inequality (9) for the error functional of cubature formulas $\ell_{r.b.l}(x)$ with a boundary layer regular in the sense of Sobolev. In this article, the authors obtain a lower bound for an arbitrary error functional of lattice cubature formulas in spaces $H_p^\mu(\Omega)$.

Theorem 1. *Let* $1 < p < \infty$, $\mu(\xi) \in B_{n,p}^*$ *and* $\mu(-\xi) = \mu(\xi)$ *then for any error functional of lattice cubature formulas in spaces* $H_p^\mu(\Omega)$ *the following estimate (see. [*6, 16*]) is valid*

$$\left\| \ell(x) | H_p^{\mu *}(\Omega) \right\| \geq (mes\Omega)^{\frac{1}{q}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{q}} [1 + o(1)]. \tag{13}$$

The proof is based on three lemmas. Since the norm of the error functional of the optimal lattice cubature formula satisfies the following inequality

$$\left\| \ell_{opt}(x) | H_p^{\mu *}(\Omega) \right\| \leq \left\| \ell(x) | H_p^{\mu *}(\Omega) \right\|$$

then it is suffice to prove the theorem for the error functional of the optimal cubature formula.

Let us build a function that will allow us to get a lower bound. For simplicity, without loss of generality, we assume that $\mu(0) = 1$.

To do this, consider the following function

$$\overset{0}{u}(x) = \sum_{\gamma \neq 0} \hat{u}[\gamma] e^{2\pi i h^{-1}(\gamma,x)} - C(h) = u(x) - C(h)$$

where $C(h) = \sum\limits_{\gamma \neq 0} \frac{\hat{u}[\gamma]}{\mu(h^{-1}\gamma)}$.

We construct the following function $\vartheta(x) = \upsilon(x) * \left[\overset{0}{u}(x) \cdot \varepsilon_{\Omega'}(x) \right]$, where $\Omega^{'}$ is the domain of the sets of all cubes with edges of length h that intersect with $\overline{\Omega}$. Let us pose another condition on domain Ω- the quantity $O(h)$ is the measure of the boundary layer of the thickness divisible by h . This condition is also posed when obtaining an upper bound for the norm of the error functional with a regular boundary layer.

Let us estimate norm $\vartheta(x)$ in $H_p^{\mu}(\Omega)$:

$$\left\|\vartheta(x)|H_p^{\mu}(\Omega)\right\| = \left\{\int\limits_{E_n}\left|F^{-1}\left[\mu(\xi)\cdot F\left[\vartheta(x)\right]\right]\right|^p dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int\limits_{E_n}\left|F^{-1}\left[\mu(\xi)\cdot F\left[\upsilon(x)*\left[\overset{0}{u}(x)\cdot\varepsilon_{\Omega'}(x)\right]\right]\right]\right|^p dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int\limits_{E_n}\left|F^{-1}\left[\mu(\xi)\cdot\frac{1}{\mu(\xi)}F[\overset{0}{u}(x)\cdot\varepsilon_{\Omega'}(x)]\right]\right|^p dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int\limits_{E_n}\left|\overset{0}{u}(x)\cdot\varepsilon_{\Omega'}(x)\right|^p dx\right\}^{\frac{1}{p}} = \left\{\int\limits_{\Omega'}\left|\overset{0}{u}(x)\right|^p dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int\limits_{\Omega'}|u(x)-C(h)|^p dx\right\}^{\frac{1}{p}} \le \left\{\int\limits_{\Omega'}|u(x)|^p dx\right\}^{\frac{1}{p}} + |C(h)|\cdot\left(mes\Omega^{'}\right)^{\frac{1}{p}}$$

$$= \left\{\int\limits_{\Omega'}\left|\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^{q-1}\cdot sign\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^p\right\}^{\frac{1}{p}} + |C(h)|\cdot\left(mes\Omega^{'}\right)^{\frac{1}{p}}(1+O(h))$$

$$= \left\{\sum_{\beta\in B}\int\limits_{\Omega_{h,\beta}}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}} + |C(h)|\cdot\left(mes\Omega^{'}\right)^{\frac{1}{p}}(1+O(h))$$

$$= \left\{\sum_{\beta\in B}h^n\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}} + |C(h)|\cdot\left(mes\Omega^{'}\right)^{\frac{1}{p}}(1+O(h))$$

$$\begin{aligned} &= \left\{\sum_{\beta\in B}h^n\right\}^{\frac{1}{p}}\left\{\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}} + |C(h)|\cdot\left(mes\Omega^{'}\right)^{\frac{1}{p}}(1+O(h)) \\ &= \left(mes\Omega^{'}\right)^{\frac{1}{p}}(1+O(h))\left\{\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}} + |C(h)|\cdot\left(mes\Omega^{'}\right)^{\frac{1}{p}}(1+O(h)). \end{aligned} \tag{14}$$

Here B is the set of such β for which all cubes $\Omega_{h,\beta}$ intersect with domain $\overline{\Omega}$:
$\Omega_{h,\beta}\bigcap\overline{\Omega}=\varnothing$ and $\Omega_{h,\beta}=\{x : h\beta_k\le x_k < h(\beta_k+1)\}$.
The following lemma is true

Lemma 1. *The following equality holds*

$$-C(h) = \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \tag{15}$$

To prove Lemma 1, we use the following lemmas.
First, we consider Lemma 2.
Lemma 2. *The following equality is true*

$$\upsilon(x) * \overset{0}{u}(x) = \sum_{\gamma \neq 0} \frac{\hat{u}[\gamma]}{\mu(h^{-1}\gamma)} e^{2\pi i h^{-1}(\gamma,x)} - C(h) \tag{16}$$

Proof. Let $\varphi(x) \in S$ then using Parseval's equality and bearing in mind that $F^{-1}\left[\frac{1}{\mu(\xi)}\delta(\xi - h\gamma)\right] = \frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}$, we obtain

$$\begin{aligned} &< \upsilon(x) * \overset{0}{u}(x), \varphi(x) >=< F[\upsilon(x) * \overset{0}{u}(x)](\xi), F[\varphi(x)](\xi) > \\ &=< F[\upsilon(x)](\xi) * F[\overset{0}{u}(x)](\xi), F[\varphi(x)](\xi) > \end{aligned}$$

$$\begin{aligned} &=< \frac{1}{\mu(\xi)} \cdot \left[\sum_{\gamma \neq 0} \hat{u}(\gamma) F\left[e^{2\pi i h^{-1}(\gamma,x)}\right] - F\left[C(h)\right] \right],\ F[\varphi(x)](\xi) > \\ &=< \frac{1}{\mu(\xi)} \cdot \sum_{\gamma \neq 0} \hat{u}(\gamma)\delta(\xi - h\gamma) - \frac{1}{\mu(\xi)} \cdot C(h)\delta(\xi),\ F[\varphi(x)](\xi) > \end{aligned}$$

$$\begin{aligned} &=< \sum_{\gamma \neq 0} \frac{1}{\mu(\xi)} \cdot \hat{u}(\gamma)\delta(\xi - h\gamma) - \frac{C(h)}{\mu(\xi)} \cdot \delta(\xi),\ F[\varphi(x)](\xi) > \\ &=< \sum_{\gamma \neq 0} \frac{\hat{u}(\gamma)}{\mu(h^{-1}\xi)} \cdot e^{2\pi i h^{-1}(\gamma,x)} - C(h),\ \varphi(x) >, \end{aligned} \tag{17}$$

this proves Lemma 2.

We introduce notation

$$u_\infty(x) = \sum_{\gamma \neq 0} \frac{\hat{u}[\gamma] e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} = \upsilon(x) * u(x)$$

and

$$\overset{0}{u}_\infty(x) = \sum_{\gamma \neq 0} \frac{\hat{u}[\gamma] e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} - C(h).$$

Functions $u_\infty(x)$ and $\overset{0}{u}_\infty(x)$ are extremal functions for the functional

$$1 - h^n \sum_{\gamma} \delta(x - h\gamma). \tag{18}$$

Then, we prove the following lemma
Lemma 3. *The following equality holds for the functional* $1 - h^n \sum\limits_{\gamma} \delta(x - h\gamma)$

$$\left[1 - \sum_{\gamma \neq 0} h^n \delta(x - h\gamma) \right] * \upsilon(x) = \sum_{\gamma \neq 0} \frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}. \tag{19}$$

Proof. Let $\varphi \in S$. Then, using Parseval's equality, we have, firstly

$$< F\left[\sum_{\gamma}\delta\left(\frac{x}{h}-\gamma\right)\right](\xi),\ F[\varphi(x)](\xi) >=< \sum_{\gamma}\delta\left(\frac{x}{h}-\gamma\right),\varphi(x) >$$

$$= h^n < \sum_{\gamma}\delta\left(y-\gamma\right),\varphi(hy) >=< h^n F\left[\sum_{\gamma}\delta\left(y-\gamma\right)\right](\xi),\ F[\varphi(hy)](\xi) >$$
$$= h^n < \sum_{\gamma}\delta\left(y-\gamma\right), F[\varphi(hy)](\xi) > .$$

Since

$$F[\varphi(hy)](\xi) = \int_{E_n}\varphi(hy)e^{2\pi i\xi y}dy = h^{-n}\int_{E_n}\varphi(z)e^{2\pi i\xi z h^{-1}}dz = h^{-n}F[\varphi(z)](h^{-1}\xi).$$

We obtain

$$< F\left[\sum_{\gamma}\delta\left(\frac{x}{h}-\gamma\right)\right](\xi),\ F[\varphi(x)](\xi) >= h^n < \sum_{\gamma}\delta\left(y-\gamma\right), F[\varphi(x)](h^{-1}y) >$$
$$= h^n < \sum_{\gamma}\delta\left(\frac{z}{h}-\gamma\right), F[\varphi(x)](z)h^{-n} >=< \sum_{\gamma}\delta\left(\frac{z}{h}-\gamma\right), F[\varphi(x)](z) > .$$

From this equality follows

$$F\left[\sum_{\gamma}\delta\left(\frac{x}{h}-\gamma\right)\right](\xi) = \sum_{\gamma}\delta\left(\frac{\xi}{h}-\gamma\right). \tag{20}$$

Then we obtain

$$< F\left[1-\sum_{\gamma}h^n\delta\left(x-h\gamma\right)\right](\xi),\ F[\varphi(x)](\xi) >=< 1-h^n\sum_{\gamma}\delta\left(x-h\gamma\right),\varphi(x) >$$
$$=< 1-\sum_{\gamma}\delta\left(\frac{x}{h}-\gamma\right),\varphi(x) >= h^n < 1-\sum_{\gamma}\delta\left(y-\gamma\right),\varphi(hy) >$$

$$= h^n < F\left[1-\sum_{\gamma}\delta\left(y-\gamma\right)\right](\xi),\ F[\varphi(hy)](\xi) >= h^n < \delta(\xi)-\sum_{\gamma}\delta\left(\xi-\gamma\right),\ F[\varphi(hy)](\xi) >$$

$$= h^n < \sum_{\gamma\neq 0}\delta\left(\xi-\gamma\right),\ F[\varphi(hy)](\xi) >= h^n < \sum_{\gamma\neq 0}\delta\left(\xi-\gamma\right),\ h^{-n}F[\varphi(z)](h^{-1}\xi) >$$

$$=< \sum_{\gamma\neq 0}\delta\left(\xi-\gamma\right),\ F[\varphi(z)](h^{-1}\xi) >= h^n < \sum_{\gamma\neq 0}\delta\left(x-h\gamma\right),\ F[\varphi(z)](x) >$$
$$=< \sum_{\gamma\neq 0}\delta\left(x-h^{-1}\gamma\right),\ h^{-n}F[\varphi(z)](x) > .$$

Hence

$$F\left[1-\sum_{\gamma}h^n\delta\left(x-h\gamma\right)\right](\xi) = \sum_{\gamma\neq 0}\delta\left(x-h^{-1}\gamma\right) \tag{21}$$

With (21), we obtain

$$< \left[1-\sum_{\gamma} h^n \delta\left(x-h\gamma\right)\right] * \upsilon(x), \varphi(x) >$$
$$=< F\left[1-\sum_{\gamma} h^n \delta\left(x-h\gamma\right)\right](\xi)\cdot F[\upsilon(x)](\xi), F\left[\varphi(x)\right](\xi) >$$

$$=< \sum_{\gamma\neq 0} \delta\left(\xi-h^{-1}\gamma\right)\cdot\frac{1}{\mu(\xi)}, F[\varphi(x)](\xi) >=< \sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}, \varphi(x) > . \tag{22}$$

This equality implies the proof of Lemma 3.

Since functions $u_\infty(x)$ and $\overset{0}{u}_\infty(x)$ are extremal functions for the functional $1-h^n\sum\limits_{\gamma}\delta(x-h\gamma)$, then using Lemma 3 we have

$$< 1-h^n\sum_{\gamma}\delta(x-h\gamma), u_\infty(x) >_{\Omega_0} =< 1-\sum_{\gamma} h^n\delta(x-h\gamma), \upsilon(x)*u(x)>_{\Omega_0}$$
$$=< \left[1-\sum_{\gamma\neq 0} h^n\delta(x-h\gamma)\right]*\upsilon(x), u(x)>_{\Omega_0} =< \left[\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right], u(x)>_{\Omega_0}$$

$$=< \left[\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right], \left|\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^{q-1} sign\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)} >_{\Omega_0} = \int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i h^{-1}(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx. \tag{23}$$

Equality (23) proves that

$$\left\|1-h^n\sum_{\gamma}\delta\left(x-h\gamma\right)|\tilde{H}_p^{\mu*}(\Omega_0)\right\|^q = \int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx. \tag{24}$$

Now let us prove another equality:

$$\begin{aligned}&< 1-h^n\sum_{\gamma}\delta\left(x-h\gamma\right), \overset{0}{u}_\infty(x)>_{\Omega_0} =< 1, \overset{0}{u}_\infty(x)>_{\Omega_0} - < h^n\sum_{\gamma}\delta\left(x-h\gamma\right), \overset{0}{u}_\infty(x)>_{\Omega_0}\\ &= \int\limits_{\Omega_0}\overset{0}{u}_\infty(x)dx = -C(h).\end{aligned} \tag{25}$$

Here we have taken into account that $\overset{0}{u}_\infty(h\gamma)=0$ and $\int\limits_{\Omega_0} u_\infty(x)dx=0$.

With equality

$$< 1-h^n\sum_{\gamma}\delta\left(x-h\gamma\right), \overset{0}{u}_\infty(x)>_{\Omega_0} =< 1-h^n\sum_{\gamma}\delta\left(x-h\gamma\right), u_\infty(x)>_{\Omega_0} \tag{26}$$

we obtain

$$-C(h) = \int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h\gamma)}\right|^q dx.$$

Lemmas 2 and 3 imply the proof of Lemma 1.

From (14) and (15) it follows

$$\left\|\vartheta(x)|H_p^{\mu}(\Omega)\right\| = (mes\ \Omega)^{\frac{1}{p}}(1+O(h))\left\{\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}}$$
$$+(mes\ \Omega)^{\frac{1}{p}}(1+O(h))\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx$$

$$= (mes\ \Omega)^{\frac{1}{p}}(1+o(h))\left\{\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}}\left(1+\left\{\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{q}}\right)$$
$$= (mes\ \Omega)^{\frac{1}{p}}(1+o(1.)\left\{\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{p}}. \tag{27}$$

Here we mean that $\left\{\int\limits_{\Omega_0}\left|\sum\limits_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx\right\}^{\frac{1}{q}} = o(1)\,,\ \frac{1}{p}+\frac{1}{q}=1.$

Next, we show that

$$< \ell(x), \vartheta(x) >= (mes\ \Omega)^{\frac{1}{p}}(1+o(1))\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx. \tag{28}$$

We modify $\vartheta(x)$ as:

$$\vartheta(x) = \upsilon(x) * \vartheta(x) = \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{\Omega'}(x)\right]$$
$$= \upsilon(x) * \left[\overset{0}{u}(x)\cdot\left[\varepsilon_{\Omega'}(x)+\varepsilon_{E_n/\Omega'}(x)-\varepsilon_{E_n/\Omega'}(x)\right]\right]$$

$$= \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{E_n}(x)\right] - \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{E_n/\Omega'}(x)\right]$$
$$= \upsilon(x)\cdot\overset{0}{u}(x) - \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{E_n/\Omega'}(x)\right]$$
$$= \overset{0}{u}_{\infty}(x) - \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{E_n/\Omega'}(x)\right]$$

So

$$< \ell(x),\ \vartheta(x) >=< \ell(x),\ \overset{0}{u}_{\infty}(x) > - < \ell(x),\ \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{E_n/\Omega'}(x)\right] >$$
$$=< \varepsilon_{\Omega}(x) - \sum_{\beta\in B'}\overset{0}{C}_{\beta}\delta(x-h\beta),\ \overset{0}{u}_{\infty}(x)\cdot\varepsilon_{\Omega'}(x) > - < \ell(x),\ \upsilon(x) * \left[\overset{0}{u}(x)\cdot\varepsilon_{E_n/\Omega'}(x)\right] > .$$

Since $\overset{0}{u}_{\infty}(h\beta)=0$, then the first term is

$$< \varepsilon_{\Omega}(x) - \sum_{\beta\in B'}\overset{0}{C}_{\beta}\delta(x-h\beta),\ \overset{0}{u}_{\infty}(x) >= \int\limits_{\Omega}\overset{0}{u}_{\infty}(x) = -C(h)\cdot mes\ \Omega = mes\Omega\int\limits_{\Omega_0}\left|\sum_{\gamma\neq 0}\frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)}\right|^q dx. \tag{29}$$

Let us show that the second term is:

$$< \ell_{onm}(x) * \upsilon(x), \overset{0}{u}(x)\varepsilon_{E_n/\Omega'}(x) >= o(-C(h)). \tag{30}$$

Indeed, firstly

$$< \ell_{opt}(x) * \upsilon(x), \overset{0}{u}(x)\varepsilon_{E_n/\Omega'}(x) >= \int\limits_{E_n/\Omega'} [\ell_{opt}(x) * \upsilon(x)][u(x) - C(h)]dx$$
$$= \int\limits_{E_n/\Omega'} [\ell_{opt}(x) * \upsilon(x)]u(x)dx - C(h) \int\limits_{E_n/\Omega'} [\ell_{opt}(x) * \upsilon(x)]dx$$

$$= \sum_{\beta' \in R/B} \int\limits_{\Omega_{h,\beta'}} [\ell_{opt}(x) * \upsilon(x)]u(x)dx - C(h) \sum_{\beta' \in R/B} \int\limits_{h,\beta'} [\ell_{opt}(x) * \upsilon(x)]dx$$
$$\leq \sum_{\beta' \in R/B} \left\{ \int\limits_{\Omega_{h,\beta}} \left|\ell_{opt}(x) * \upsilon(x)\right|^q dx \right\}^{\frac{1}{q}} \left\{ \int\limits_{\Omega_{h,\beta}} |u(x)|^p dx \right\}^{\frac{1}{p}} + C(h) \sum_{\beta' \in R/B} h^{\frac{n}{p}} \left\{ \int\limits_{\Omega_{h,\beta}} \left|\ell_{opt}(x) * \upsilon(x)\right|^q dx \right\}^{\frac{1}{q}}. \tag{31}$$

Because of the h-periodicity of $u(x)$, all
$\left\{ \int\limits_{\Omega_{h,\beta}} |u(x)|^p dx \right\}^{\frac{1}{p}}$ are equal to each other, and equal to
$\left\{ h^n \int\limits_{\Omega_0} \left| \sum\limits_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}}$
So, from (31) we obtain

$$< \ell_{opt}(x) * \upsilon(x), \overset{0}{u}(x)\varepsilon_{E_n/\Omega'}(x) >\leq \left\{ h^n \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}}$$
$$\cdot \sum_{\beta' \in R/B} \left\{ \int\limits_{\Omega_{h,\beta'}} \left|\ell_{opt}(x) * \upsilon(x)\right|^q dx \right\}^{\frac{1}{q}} + |C(h)|\, h^{\frac{n}{p}} \sum_{\beta' \in R/B} \left\{ \int\limits_{\Omega_{h,\beta}} \left|\ell_{opt}(x) * \upsilon(x)\right|^q dx \right\}^{\frac{1}{q}}$$
$$\left\{ h^n \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}} \left\{ \int\limits_{E_n/\Omega'} \left|\ell_{opt}(x) * \upsilon(x)\right|^q dx \right\}^{\frac{1}{q}} + |C(h)|\, h^{\frac{n}{p}} \left\{ \int\limits_{E_n/\Omega'} \left|\ell_{opt}(x) * \upsilon(x)\right|^q dx \right\}^{\frac{1}{q}}$$

$$= h^{\frac{n}{p}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}} \left\{ \int\limits_{E_n} \left| \ell_{opt}(x) * \upsilon(x) \right|^q dx \right\}^{\frac{1}{q}} + |c(h)|\, h^{\frac{n}{p}} \left\{ \int\limits_{E_n} \left| \ell_{opt}(x) * \upsilon(x) \right|^q dx \right\}^{\frac{1}{q}}$$

$$\leq h^{\frac{n}{p}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}} \left\{ \int\limits_{E_n} \left| \ell_{p.n.c}(x) * \upsilon(x) \right|^q dx \right\}^{\frac{1}{q}} + |c(h)|\, h^{\frac{n}{p}} \left\{ \int\limits_{E_n} \left| \ell_{p.n.c}(x) * \upsilon(x) \right|^q dx \right\}^{\frac{1}{q}}$$

$$\leq h^{\frac{n}{p}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}} (mes\Omega)^{\frac{1}{q}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{q}} (1 + o(1)) \tag{32}$$

$$+ |C(h)|\, h^{\frac{n}{p}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{p}} (mes\Omega)^{\frac{1}{q}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\}^{\frac{1}{q}} (1 + o(1))$$

$$= h^{\frac{m}{p}} (mes\Omega)^{\frac{1}{q}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx \right\} + h^{\frac{m}{p}} |C(h)|\, (mes\Omega)^{\frac{1}{q}} \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i(\gamma,x)}}{\mu(h^{-1}\gamma)} \right|^q dx$$

$$= h^{\frac{m}{p}} (mes\Omega)^{\frac{1}{q}} |C(h)| + h^{\frac{m}{p}} (mes\Omega)^{\frac{1}{q}} |C(h)|^2 = h^{\frac{m}{p}} |C(h)| (1 + |C(h)|) (mes\Omega)^{\frac{1}{q}} = o(-|C(h)|).$$

The proof of Theorem 1 follows from (29), (30), (31), and (32).

Note that the following theorem was proved in [17] for a cubature formula with a regular boundary layer.

Theorem 2. *If $1 < p < \infty, \mu(\xi) \subset B(n,p)$, then in space $H_p^{\mu}(\Omega)$, the norm of the error functional of a cubature formula with the regular in the sense of S.L.Sobolev boundary layer satisfies the following inequality*

$$\left\| \ell(x) | H_p^{\mu *}(\Omega) \right\| \leq (mes\Omega)^{\frac{1}{q}} \left\{ \int\limits_{\Omega_0} \left| \sum_{\gamma \neq 0} \frac{exp\left(2\pi i h^{-1} H^{-1} x\right)}{\mu(\gamma H^{-1})} \right|^q dx \right\}^{\frac{1}{q}} + O(h^{m+1}), \tag{33}$$

as $h \to 0$. Here H is matrix $n \times n$, $|H| = 1$, Ω_0 is the fundamental domain defined by matrix H.

From the proved Theorem 1 and the results given in [5, 17], in particular, from Theorem 2, follows

Theorem 3. *If $1 < p < \infty$, $\mu(\xi) \in B^*_{n,p}$ and $\mu(-\xi) = \mu(\xi)$, then the cubature formula with the regular in the sense of S.L.Sobolev boundary layer is asymptotically optimal in space $H_p^{\mu}(\Omega)$.*

CONCLUSION

To estimate the norm of the error functional of a cubature formula with the regular in the sense of Sobolev boundary layer in space $H_p^{\mu}(\Omega)$, the authors used the explicit form of the extremal functions $u_\infty(x)$ and $\overset{0}{u}_\infty(x)$ of the periodic functional $1 - h^n \sum\limits_{\gamma} \delta(x - h\gamma)$ in this space and the Hilbert property of such spaces. In the case of Banach spaces, the study of the behavior of such formulas is a very difficult task.

In [17], an upper bound was obtained for the norm of the error functional with the regular in the sense of Sobolev boundary layer in space $H_p^{\mu}(\Omega)$, i.e., the inverse inequality to the inequality for the error functional of the cubature formulas $\ell_{p.n.c}(x)$ with the regular in the sense of Sobolev boundary layer.

In this paper, the authors obtained a lower bound for an arbitrary error functional of lattice cubature formulas in spaces $H_p^{\mu}(\Omega)$. Since the norm of the error functional of the optimal lattice cubature formula satisfies inequality $\left\| \ell_{opt}(x) | H_p^{\mu *}(\Omega) \right\| \leq \left\| \ell(x) | H_p^{\mu *}(\Omega) \right\|$, it suffices to prove the theorem for the error functional of the optimal cubature formula. The developed function $v(x)$ allowed the authors to obtain the lower bound.

Since, by virtue of (1), the authors obtained the upper and lower bounds for the norm of the error functional with the regular in the sense of Sobolev boundary layer in space $H_p^{\mu}(\Omega)$, this implies the asymptotic optimality of the cubature formula with the regular in the sense of Sobolev boundary layer in this space.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the Institute of Mathematics named V.I.Romanovskiy for making convenient research facilities. We much appreciate the reviewers for their thoughtful comments and efforts toward improving our manuscript.

REFERENCES

1. S. L. Sobolev, *Lectures on the theory of cubature formulas (in Russian)*, Vol. part 1-2 (Novosibirsk, 1965).
2. S. L. Sobolev, *Introduction to the theory of cubature formulas (in Russian)* (Nauka, Moscow., 1974) p. 808.
3. S. L. Sobolev, "Cubature formula with a regular boundary layer," DAN USSR **3(163)**, 587–590 (1965).
4. V. D. Charushnikov, "Asymptotics of the error of cubature formulas with regular in the sense of S.L.Sobolev boundary layer," DAN USSR **5(170)**, 1032–1034 (1966).
5. V. D. Charushnikov, "Asymptotic optimality of cubature formulas with regular in the sense of S.L.Sobolev boundary layer," DAN USSR **2(179)**, 297–299 (1967).
6. T. Kh. Sharipov, "Estimation of the norm of the error functional of cubature formulas in space $H_p^\mu(\omega)$," Questions of Computational and Applied Mathematics **issue 14** (1972).
7. M. D. Ramazanov, "Optimal error functional over periodic functions from space $\tilde{H}_p^\mu$," Siberian Math. J. **XIII, No. 1** (1972).
8. A. R. Hayotov and S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in a Hilbert space." Problems of computational and applied mathematics **4(28)**, 73–84 (2020).
9. A. R. Hayotov, S. Jeon, and C.-O. Lee, "On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$." Journal of Computational and Applied Mathematics **372**, 112713 (2020).
10. O. I. Jalolov, "Weight optimal order of convergence cubature formulas in Sobolev space," AIP Conference Proceedings **2365**, 020014 (2021), https://doi.org/10.1063/5.0057015.
11. S.S.Babaev and A.R.Hayotov, "Optimal interpolation formulas in the space $W_2^{(m,m-1)}$," Calcolo **56**, doi.org/10.1007/s10092–019–0320–9 (2019).
12. A. R. Hayotov and S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in $W_2^{(m,m-1)}$ space," AIP Conference Proceedings **2365**, 020021 (2021), https://doi.org/10.1063/5.0057127.
13. S. Babaev, J. Davronov, A.Abdullaev, and S.Polvonov, "Optimal interpolation formulas exact for trigonometric functions," AIP Conference Proceedings **2781**, 020064 (2023), https://doi.org/10.1063/5.0144754.
14. A. Hayotov, S. Babaev, N.Olimov, and Sh.Imomova, "The error functional of optimal interpolation formulas in $W_{2,\sigma}^{(2,1)}$ space," AIP Conference Proceedings **2781**, 020044 (2023), https://doi.org/10.1063/5.0144752.
15. L. R. Volevich and B. P. Paneyakh, "Some spaces of generalized functions and embedding theorems," Achievements in Math. Sci. 20 **issue I**, 3–74 (1965).
16. T. Kh. Sharipov, "An upper bound for the norm of the error functional of cubature formulas with regular in the sense of S.L.Sobolev boundary layer in spaces $h_p^\mu(\omega)$," DAN USSR **No.1(202)**, 51–53 (1972).
17. O. I. Jalolov, "An upper bound for the norm of the error functional of a cubature formula of Hermite type in S.L.Sobolev space." Problems of computational and applied mathematics **No.3**, 70–78 (2017).