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WEIGHTED OPTIMAL ORDER OF CONVERGENCE OF THE HERMITIAN TYPE CUBATURE FORMULAS IN SOBOLEV SPACE

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In the study of various questions arising in the theory of approximate integration and partial differential equations and related departments of analysis, the so-called Functional approach turned out to very fruitful. Until now, cubature formulas have been considered, with the help of which a definite integral of a function is approximately calculated when the values of this function are known at individual points-nodes of the cubature formula. Bat more general cubature formulas are possible, which include both the values of the function and the values of its derivatives of one order or another. If we know not only the values of the function $f(x)$ at some points of the region Ω but also the values of its derivatives of one order or another, then it is natural that with the correct use of all these data we can expect a more accurate result than in the case of using only the values of the function. In this paper we investigatea weighted cubature formula of the Hermitian type in function spaces $L_2^{(m)}, L_p^{(m)}, \bar{L}_2^{(m)}$ of S.L. Sobolev for the functions defined in the n - dimensional unit cube and obtain an upper estimate for the norm of error functionals of weighted cubature formulas. The basis of the Bakhvalov theorem it is proved that considered cubature formulas of the Hermitian type are optimal on order of convergence in these spaces. It turns out that space $\bar{L}_p^{(m)}$ has some advantages. Indeed, the advantage is that, first, for the norm of the error functional of cubature formulas in space $\bar{L}_p^{(m)}$, the computational operations are much less than in $L_p^{(m)}$, and secondly, the norms of the error functional of cubature formulas given in spaces $\bar{L}_p^{(m)}$ and $L_p^{(m)}$ have the same order of convergence to zero at $N \to \infty$.

Keywords: weighted cubature formula, error functional, Sobolev space, function spaces, generalized function.

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1 Introduction

In many research papers examined the properties of optimal approximations of linear functional(see, for instance, [1–21]. In these papers the problem of optimality with respect to a certain space is investigated. Most of them are discussed in the Sobolev space [1]. Consider the cubature formula of the form

$$\int\limits_{K_n} p(x)f(x)dx \approx \sum_{|\alpha|\leqslant t} \sum_{\lambda=1}^{N} (-1)^{|\alpha|} C_\lambda{}^{(\alpha)} f^{(\alpha)}\left(x^{(\lambda)}\right) \tag{1}$$

in the space $L_2^{(m)}(K_n)$, where K_n is a n- dimensional unit cube,

$$K_n = \{(x_1, x_2...x_n)\,|0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1, ..., 0 \leqslant x_n \leqslant 1\},$$

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n, (\alpha) = (\alpha_1, \alpha_2, ..., \alpha_n),$$

$x^{(\lambda)} = \left({x_1}^{(\lambda_1)}, {x_2}^{(\lambda_2)}, ..., {x_n}^{(\lambda_n)}\right)$ and $\int\limits_{K_n} p(x)dx < \infty$, $0 \leqslant t \leqslant m$, $m = m_1 + m_2 + \cdots + m_n$.

A generalized function

$$\ell_N^{(\alpha)}(x) = p(x)\,\varepsilon_{K_n}(x) - \sum_{|\alpha| \leqslant t} \sum_{\lambda=1}^{N} C_\lambda^{(\alpha)} \delta^{(\alpha)}\left(x - x^{(\lambda)}\right) \tag{2}$$

is called a error functional of the cubature formula (1),

$$< \ell_N^{(\alpha)}, f >= \int\limits_{K_n} p(x) f(x) dx - \sum_{|\alpha| \leqslant t} \sum_{\lambda=1}^{N} (-1)^{|\alpha|} C_\lambda^{(\alpha)} f^{(\alpha)}(x^{(\lambda)}), \tag{3}$$

is an error of the cubature formula (1), $p(x)$ is a weight function, $\varepsilon_{K_n}(x)$ is characteristic function of K_n, $C_\lambda^{(\alpha)}$ and $x^{(\lambda)}$ are coefficients and nodes of the cubature formula (1) , $\delta(x)$ is the Dirac delta- function.

Definition 1. The space $L_2^{(m)}(K_n)$ is defined as the space of functions, given on the n- dimensional unit cube K_n and having all the generalized derivatives of order m, square summable in norm [1]:

$$\left\| f / L_2^{(m)}(K_n) \right\| = \left\{ \int\limits_{K_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} [D^\alpha f]^2 dx \right\}^{\frac{1}{2}} \tag{4}$$

with the inner product $(f, \varphi)_{L_2^{(m)}(K_n)} = \int\limits_{K_n} \sum\limits_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f \cdot D^\alpha \varphi dx$,

where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $dx = dx_1 dx_2 ... dx_n$ and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$,

$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$.

Definition 2. The cubature formula of the form (1) is called asymptotically optimal, if for the norms of error functionals the following holds

$$\lim_{N \to \infty} \frac{\left\| \ell_N^{(\alpha)a,o} / L_2^{(m)^*}(K_n) \right\|}{\left\| \ell_N^{(\alpha)o} / L_2^{(m)^*}(K_n) \right\|} = 1 \tag{5}$$

Here, $\ell_N^{(\alpha)o}(x)$ and $\ell_N^{(\alpha)a,o}(x)$ are error functionals of optimal and asymptotically optimal cubature formulas of the form (1), respectively.

Definition 3. The cubature formula of the form (1) is called an optimal order of convergence, if for the norm of its error functional following holds

$$\lim_{N \to \infty} \frac{\left\| \ell_N^{(\alpha)o,n} / L_2^{(m)^*}(K_n) \right\|}{\left\| \ell_N^{(\alpha)o} / L_2^{(m)^*}(K_n) \right\|} < \infty \tag{6}$$

Here $\ell_N^{(\alpha)o,n}(x)$ is error functional of optimal order of convergence cubature formulas (1).

In this paper we consider the problem of the order of convergence of norms $\left\| \ell_N^{(\alpha)} / L_2^{(m)^*}(K_n) \right\|$, $\left\| \ell_N^{(\alpha)} / L_p^{(m)^*}(K_n) \right\|$ and $\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)^*}(K_n) \right\|$ of the error functional with an increase in the number of its nodes. The results, which we obtain here, are to the arbitrary distribution of points.

2 Optimal in order of convergence of weight cubature formulas in the space $L_2^{(m)}(K_n)$

Here we explore weighted cubature formulas, which are optimal for the convergence order.

We have the following:

Lemma 1. If the error functional (2) of cubature formula (1) satisfies the conditions

$$\ell_N{}^{(\alpha)}(x) = \ell_{N_1}^{(\alpha_1)}(x_1)\cdot\ell_{N_2}^{(\alpha_2)}(x_2)\cdot\dots\cdot\ell_{N_n}^{(\alpha_n)}(x_n) \tag{7}$$

and

$$\left\|\ell_{N_i}^{(\alpha_i)}/L_2^{(m_i)^*}(0,1)\right\| \leqslant c_i\frac{1}{N_i^{m_i}},\ \ \left(i=\overline{1,n}\right),\ \ c_i\ \ \textit{are constants}, \tag{8}$$

that is

$$\left\|\ell_{N_i}^{(\alpha_i)}/L_2^{(m_i)^*}(0,1)\right\| \leqslant c_i O\left(h_i^{m_i}\right),\ \left(i=\overline{1,n}\right),\ \ h_i=\frac{1}{N_i} \tag{9}$$

then

$$\left\|\ell_N^{(\alpha)}/L_2^{(m)^*}(K_n)\right\| \leqslant c\cdot\frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}}, \tag{10}$$

is c constant, or

$$\left\|\ell_N^{(\alpha)}/L_2^{(m)^*}(K_n)\right\| \leqslant c\cdot O\left(h_1^{m_1}\right)\cdot O\left(h_2^{m_2}\right)\cdot\ldots\cdot O\left(h_n^{m_n}\right) \tag{11}$$

where

$$\ell_{N_i}^{(\alpha_i)}(x) = p(x_i)\,\varepsilon_{[0,1]}(x_i) - \sum_{\alpha_i\leqslant t_i}\sum_{\lambda_i=1}^{N_i} C_{\lambda_i}^{(\alpha_i)}\delta^{(\alpha_i)}\left(x_i - x_i^{(\lambda_i)}\right),\ p(x)=\prod_{i=1}^{n}p_i(x_i),\ c=\prod_{i=1}^{n}c_i$$

and $m = m_1 + m_2 + \dots + m_n,\ \ m_i \geqslant 1,\ \ i=\overline{0,1}$.

We are conducting proof by mathematical induction.

Suppose $n=2$, then $x=(x_1,x_2)$, $|\alpha|=\alpha_1+\alpha_2$, $m=m_1+m_2$, $dx=dx_1dx_2$, $f(x)=f(x_1,x_2)$, $p(x)=p_1(x_1)\cdot p_2(x_2)$ and $\ell_N^{(\alpha)}(x)=\ell_{N_1}^{(\alpha_1)}(x_1)\cdot\ell_{N_2}^{(\alpha_2)}(x_2)$.

If we assume in (4) $n=1$, then

$$\left\|f/\bar{L}_{\cdot\,2}^{(m_i)}(0,1)\right\| = \left\{\int_0^1\left[\frac{d^{m_i}}{d{x_i}^{m_i}}f(x_i)\right]^2 dx_i\right\}^{\frac{1}{2}},\ \ (\overline{1,n}) \tag{12}$$

Thus, we have

$$\left|< \ell_N^{(\alpha)}(x_1,x_2), f(x_1,x_2) >\right| = \left|< \ell_{N_1}^{(\alpha_1)}(x_1), < \ell_{N_2}^{(\alpha_2)}(x_2), f(x_1,x_2) >>\right| \leqslant$$

$$\leqslant \left\|\ell_{N_2}^{(\alpha_2)}/L_2^{(m_2)^*}(0,1)\right\|.\left\|< \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2) > /L_2^{(m_2)}(0,1)\right\| \tag{13}$$

Taking into account (12), (13) we compute the following norm [12]:

$$\left\|< \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2) > /L_2^{(m_2)}(0,1)\right\| =$$

$$\left\{\int_0^1\left|\frac{d^{m_2}}{dx_2^{m_2}} < \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2) >\right|^2 dx_2\right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_0^1 \left| < \ell_{N_1}^{(\alpha_1)}(x_1), \frac{d^{m_2}}{dx_2^{m_2}} f(x_1, x_2) > \right|^2 dx_2 \right\}^{\frac{1}{2}} \leqslant$$

$$\leqslant \left\{ \int_0^1 \left[\left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| \frac{d^{m_2}}{dx_2^{m_2}} f(x_1, x_2)/L_2^{(m_1)}(0,1) \right\| \right]^2 dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(0,1) \right\| \cdot \left\{ \int_0^1 \left\{ \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x_1, x_2) \right]^2 dx_1 \right\} dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(0,1) \right\| \cdot \left\{ \int_0^1 \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x) \right]^2 dx \right\}^{\frac{1}{2}} =$$

$$= c' \left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| f(x)/L_2^{(m)}(K_2) \right\|, \tag{14}$$

c' is a constant, $m = m_1 + m_2$, $K_2 = \{(x_1, x_2) \,|\, 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$. Thus, from (13) and (14) we obtain

$$\left| < \ell_N^{(\alpha)}(x_1, x_2), f(x_1, x_2) > \right| \leqslant$$

$$\leqslant c' \left\| \ell_{N_2}^{(\alpha_2)}/L_2^{(m_2)^*}(0,1) \right\| \cdot \left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| f(x)/L_2^{(m)}(K_2) \right\|, \tag{15}$$

Taking into account (4) from (15) we obtain

$$\left\| \ell_N^{(\alpha)}(x)/L_2^{(m)^*}(K_2) \right\| \leqslant c' \left\| \ell_{N_2}^{(\alpha_2)}/L_2^{(m_2)^*}(0,1) \right\| \cdot \left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(0,1) \right\|. \tag{16}$$

Using (8), from (16) we have

$$\left\| \ell_N^{(\alpha)}/L_2^{(m)^*}(K_2) \right\| \leqslant c' \cdot c_1 \cdot c_2 \frac{1}{N_1^{m_1} N_2^{m_2}}$$

That is

$$\left\| \ell_N^{(\alpha)}/L_2^{(m)^*}(K_2) \right\| \leqslant c_3' O(h_1^{m_1}) O(h_2^{m_2}) \tag{17}$$

where $c_3' = c' \cdot c_1 \cdot c_2$

Now suppose that (10) is valid for $n = k$, then from the above calculations we obtain [22]:

$$\left| < \ell_N^{(\alpha)}(x), f(x) > \right| = \left| < \ell_N^{(\alpha)}(x_1, x_2, ..., x_k), f(x_1, x_2, ..., x_k) > \right| =$$

$$= \left| < \ell_{N_k}^{(\alpha_k)}(x_k), < \ell_{N_{k-1}}^{(\alpha_{k-1})}(x_{k-1}), ..., < \ell_{N_2}^{(\alpha_2)}(x_2), < \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1, x_2, .., x_k) > ... > \right| \leqslant$$

$$\leqslant \left\| \ell_{N_k}^{(\alpha_k)}(x_k)/L_2^{(m_k)^*}(0,1) \right\| \cdot \left\| \ell_{N_{k-1}}^{(\alpha_{k-1})}(x_{k-1})/L_2^{(m_{k-1})^*}(0,1) \right\| ...$$

$$. \left\| < \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1, x_2, ..., x_k) > /L_2^{(m_1)^*}(0,1) \right\| \leqslant$$

$$\leqslant \left\| \ell_{N_k}^{(\alpha_k)}(x_k)/L_2^{(m_k)^*}(0,1) \right\| ... \left\| \ell_{N_1}^{(\alpha_1)}(x_1)/L_2^{(m_1)^*}(x_1) \right\| \cdot c'' \left\| f(x)/L_2^{(m)}(K_k) \right\| \tag{18}$$

From (18), considering (13), we have

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)^*}(K_k) \right\| \leqslant c'' \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_2^{(m_1)^*}(0,1) \right\| \ldots \left\| \ell_{N_k}^{(\alpha_k)}(x_k) / L_2^{(m_k)^*}(0,1) \right\| \tag{19}$$

Then, referring to (8), from (19) we obtain

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)^*}(K_k) \right\| \leqslant c'' \cdot c_1 \cdot c_2 \cdot \ldots \cdot c_k \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_k^{m_k}}, \tag{20}$$

or, considering (9), from (20) we have

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)^*}(K_k) \right\| \leqslant c'' \cdot d_k \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_k^{m_k}\right),$$

where $d_k = \prod\limits_{i=1}^{k} c_i$.

Using the validity of assertion $n = k$, we prove that the assertion is valid when $n = $ $= k+1$. Thus, when $n = k+1$, and taking account of (4) and (19), we obtain

$$\left| < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_1, x_2, \ldots, x_{k+1}), f(x_1, x_2, \ldots, x_{k+1}) > \right| =$$

$$\left| < \ell_{N_1}^{(\alpha_1)}(x_1), < \ell_{N_2}^{(\alpha_2)}(x_2), < \ldots, < \ell_{N_k}^{(\alpha_k)}(x_k), < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_{k+1}), f(x_1, x_2, .., x_{k+1}) > \ldots > \right| \leqslant$$

$$\leqslant \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_2^{(m_1)^*}(0,1) \right\| \ldots \left\| \ell_{N_k}^{(\alpha_k)}(x) / L_2^{(m_k)^*}(0,1) \right\| \cdot$$

$$\cdot \left\| < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_{k+1}), f(x_1, x_2, \ldots, x_{k+1}) > / L_2^{(m_{k+1})}(0,1) \right\|$$

$$\cdot \left\| < \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, \ldots, x_{k+1}) > / L_2^{(m_{k+1})}(0,1) \right\| \tag{21}$$

Using (4) and (19) from (21) we obtain

$$\left\| \ell_{N_{k+1}}^{(\alpha_{k+1})} / L_2^{(m_{k+1})^*}(K_{k+1}) \right\| \leqslant c''' \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_2^{(m_1)^*}(0,1) \right\| \ldots$$

$$. \left\| \ell_{N_k}^{(\alpha_k)}(x_k) / L_2^{(m_k)^*}(0,1) \right\| \cdot \left\| \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_{k+1}) / L_2^{(m_{k+1})^*}(0,1) \right\| \tag{22}$$

By using (8), from (22) we have

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)*}(K_{k+1}) \right\| \leqslant c''' \cdot d_{k+1} \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_{k+1}^{m_{k+1}}} \tag{23}$$

where $d_{k+1} = \prod\limits_{i=1}^{k+1} c_i$.

or, taking into account (17), from (23) we obtain

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)*}(K_{k+1}) \right\| \leqslant c''' \cdot d_{k+1} \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_{k+1}^{m_{k+1}}\right).,\tag{24}$$

$c^{''''}$ constants, in conclusion consider, that

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)\,*}(K_n) \right\| \leqslant c \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_n^{m_n}} \tag{25}$$

where $c = c'' \cdot d_{k+1}$, or considering (9), from (25), we have

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)\ *} (K_n) \right\| \leqslant c \cdot O\left(h_1^{m_1}\right) ... O\left(h_n^{m_n}\right)$$

With the help of this lemma it is easy to prove the following theorem.

Theorem 1. The weight cubature formula (1) with the error functional (2) for $N_1 = = N_2 = ... = N_n$, $\prod\limits_{i=1}^{n} N_i = N$ and $m_1+m_2+...+m_n = m$ is optimal in order of convergence in the space $L_2^{(m)}(K_n)$, those, for the norm of the error functional (2) of cubature formula (1) the following holds

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)^*} (K_n) \right\| = O\left(N^{-\frac{m}{n}}\right)$$

Proof. On the basis of Lemma1 under the assumption $N_1 = N_2 = ... = N_n$ we have $N_1 = \sqrt[n]{N}$.
Thus,

$$\prod_{i=1}^{n} N_i^{m_i} = N_1^{m_1+m_2+...+m_n} = N^{\frac{m}{n}} \tag{26}$$

By substituting (26) into (25) we obtain

$$\left\| \ell_N^{(\alpha)} / L_2^{(m)^*} (K_n) \right\| \leqslant c \cdot N^{-\frac{m}{n}} \tag{27}$$

From theorem of N.S. Bakhvalov [23] and from the inequality (27) follows the proof. Theorem 1 is proved.

3 A norm estimation for the error functional of weighted cubature formulas on the space $L_p^{(m)}(K_n)$.

Consider the cubature formula of the form

$$\int\limits_{K_n} p(x) f(x) dx \approx \sum_{|\alpha| \leqslant t} \sum_{\lambda=1}^{N} (-1)^{|\alpha|} C_\lambda{}^{(\alpha)} f^{(\alpha)}\left(x^{(\lambda)}\right) \tag{28}$$

in the Sobolev space $L_p^{(m)}(K_n)$, where K_n is n-dimensional unit cube.

$$K_n = \left\{ (x_1, x_2 ... x_n) \,|0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1, ..., 0 \leqslant x_n \leqslant 1 \right\}$$

$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $(\alpha) = (\alpha_1, \alpha_2, ..., \alpha_n)$, $x^{(\lambda)} = \left(x_1{}^{(\lambda_1)}, x_2{}^{(\lambda_2)}, ..., x_n{}^{(\lambda_n)}\right)$ and $\int\limits_{K_n} p(x) dx < \infty$, $0 \leqslant t \leqslant m$, $m = m_1 + m_2 + \cdots + m_n$,

The generalized function

$$\ell_N^{(\alpha)}(x) = p(x)\, \varepsilon_{K_n}(x) - \sum_{|\alpha| \leqslant t} \sum_{\lambda=1}^{N} C_\lambda^{(\alpha)} \delta^{(\alpha)}\left(x - x^{(\lambda)}\right) \tag{29}$$

is called error *functional of the cubature formula* (28),

$$< \ell_N^{(\alpha)}, f >= \int\limits_{K_n} p(x) f(x) dx - \sum_{|\alpha| \leqslant t} \sum_{\lambda=1}^{N} (-1)^{|\alpha|} C_\lambda^{(\alpha)} f^{(\alpha)}(x^{(\lambda)})$$

is an error of the cubature formula (28), $p(x) \in L_p(K_n)$ is a weight function, $\varepsilon_{K_n}(x)$ is the characteristic function of K_n, $C_\lambda^{(\alpha)}$ and $x^{(\lambda)}$ are coefficients and nodes of the cubature formula (28) and $\delta(x)$ is the Dirac delta function.

Definition 4. The space $L_p^{(m)}(K_n)$ is defined as a space of functions, given on a n-dimensional unit cube K_n and having all the generalized derivatives of order m, summable with a degree p in norm [1]:

$$\left\| f/L_p^{(m)}(K_n) \right\| = \left\{ \int\limits_{K_n} \left\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} [D^\alpha f]^2 \right\}^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}, \tag{30}$$

where $D^\alpha = \frac{\partial^m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$, $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$, $\alpha! = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_n!$. The following is true.

Lemma 2. If for the error functional (29) of the cubature formula (28), the following conditions are fulfiled

$$\ell_N{}^{(\alpha)}(x) = \prod_{i=1}^{n} \ell_{N_i}^{(\alpha_i)}(x_i)$$

where $\ell_{N_i}^{\alpha_i}(x_i) = p_i(x_i)\, \varepsilon_{[0,1]}(x_i) - \sum\limits_{\lambda_i=1}^{N_i} C_{\lambda_i}^{(\alpha_i)} \delta^{(\alpha_i)} \left(x_i - x_i^{(\lambda_i)} \right)$, $p(x) = \prod\limits_{i=1}^{n} p_i(x_i)$
and

$$\left\| \ell_{N_I}^{(\alpha_I)} / L_p^{(m_i)*}(0,1) \right\| \leqslant d_i \frac{1}{N_i^{m_i}} \tag{31}$$

d_i are constants.
That is

$$[\left\| \ell_{N_i}^{(\alpha_i)} / L_p^{(m_i)*}(0,1) \right\| \leqslant d_i\, O(h_i^{m_i}) \tag{32}$$

d_i are constants, $\left(i = \overline{1,n} \right)$, $h_i = \frac{1}{N_i}$
then

$$\left\| \ell_N^{(\alpha)} / L_p^{(m)*}(K_n) \right\| \leqslant d \cdot \frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}} \tag{33}$$

d is a constant, or

$$\left\| \ell_N^{(\alpha)} / L_p^{(m)*}(K_n) \right\| \leqslant d \cdot O(h_1^{m_1}) \cdot O(h_2^{m_2}) \cdot \dots \cdot O(h_n^{m_n})$$

$d = \prod\limits_{i=1}^{n} d_i$ and $m = m_1 + m_2 + \dots + m_n$

Proof. We conduct the method of the mathematical induction.

Suppose $n = 2$ $x = (x_1, x_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $m = m_1 + m_2$, $dx = dx_1 dx_2$, $f(x) = f(x_1, x_2)$, $p(x) = p_1(x_1) \cdot p_2(x_2)$ and $\ell_N^{(\alpha)}(x) = \ell_{N_1}^{(\alpha_1)}(x_1) \cdot \ell_{N_2}^{(\alpha_2)}(x_2)$. If presume in (30) $n = 1$, then

$$\left\| f_i / L_p^{(m_i)}(0,1) \right\| = \left\{ \int\limits_0^1 \left[\left(\frac{d^{m_i}}{dx_i^{m_i}} f(x_i) \right)^2 \right]^{\frac{p}{2}} dx_i \right\}^{\frac{1}{p}}, \quad i = 1, 2, \dots, n. \tag{34}$$

So we have

$$\left| < \ell_N^{(\alpha)}(x_1, x_2), f(x_1, x_2) > \right| = \left| < \ell_{N_1}^{(\alpha_1)}(x_1), < \ell_{N_2}^{(\alpha_2)}(x_2), f(x_1, x_2) >> \right| \leqslant$$

$$\leqslant \left\|\ell_{N_2}^{(\alpha_2)}/L_p^{(m_2)^*}(0,1)\right\| \cdot \left\|< \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2) >/L_p^{(m_2)}(0,1)\right\| \tag{35}$$

We compute the following norm:

$$\left\|< \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2) >/L_p^{(m_2)}(0,1)\right\| =$$

$$\left\{\int_0^1 \left|\frac{d^{m_2}}{dx_2^{m_2}} < \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2) >\right|^p dx_2\right\}^{\frac{1}{p}} = \left\{\int_0^1 \left|< \ell_{N_1}^{(\alpha_1)}(x_1), \frac{d^{m_2}}{dx_2^{m_2}} f(x_1,x_2) >\right|^p dx_2\right\}^{\frac{1}{p}} \leqslant$$

$$\leqslant \left\{\int_0^1 \left[\left\|\ell_{N_1}^{(\alpha_1)}(x_1)/L_p^{(m_1)^*}(0,1)\right\| \cdot \left\|\frac{d^{m_2}}{dx_2^{m_2}} f(x_1,x_2)/L_p^{(m_1)}(0,1)\right\|\right]^p dx_2\right\}^{\frac{1}{p}} =$$

$$= \left\|\ell_{N_1}^{(\alpha_1)}(x_1)/L_p^{(m_1)^*}(0,1)\right\| \cdot \left\{\int_0^1 \left\{\int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1}\partial x_2^{m_2}} f(x_1,x_2)\right]^p dx_1\right\} dx_2\right\}^{\frac{1}{p}} =$$

$$= \left\|\ell_{N_1}^{(\alpha_1)}(x_1)/L_p^{(m_1)^*}(0,1)\right\| \cdot \left\{\int_0^1\int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1}\partial x_2^{m_2}} f(x)\right]^p dx\right\}^{\frac{1}{p}} =$$

$$= c' \left\|\ell_{N_1}^{(\alpha_1)}(x_1)/L_p^{(m_1)^*}(0,1)\right\| \cdot \left\|f(x)/L_p^{(m)}(K_2)\right\| \tag{36}$$

where is a constant.
Thus, from (35) and (36) we obtain

$$\left|< \ell_N^{(\alpha)}(x_1,x_2), f(x_1,x_2) >\right| \leqslant$$

$$\leqslant c' \left\|\ell_{N_2}^{(\alpha_2)}/L_p^{(m_2)^*}(0,1)\right\| \cdot \left\|\ell_{N_1}^{(\alpha_1)}(x_1)/L_p^{(m_1)^*}(0,1)\right\| \cdot \left\|f(x)/L_p^{(m)}(K_2)\right\| \tag{37}$$

From (37), using the definition of norm, we have

$$\left\|\ell_N^{(\alpha)}(x)/L_p^{(m)^*}(K_2)\right\| \leqslant c' \left\|\ell_{N_2}^{(\alpha_2)}/L_p^{(m_2)^*}(0,1)\right\| \cdot \left\|\ell_{N_1}^{(\alpha_1)}(x_1)/L_p^{(m_1)^*}(0,1)\right\|. \tag{38}$$

Taking into account (31), on the basis of (38) we obtain

$$\left\|\ell_N^{(\alpha)}/L_p^{(m)^*}(K_2)\right\| \leqslant c' \cdot c_1 \cdot c_2 \frac{1}{N_1^{m_1} N_2^{m_2}}$$

That is

$$\left\|\ell_N^{(\alpha)}/L_p^{(m)*}(K_2)\right\| \leqslant c_3 O\left(h_1^{m_1}\right) O\left(h_2^{m_2}\right) \tag{39}$$

where $c_3 = c' \cdot c_1 \cdot c_2$.

Now suppose that the inequality (33) holds for $n = k$, then on the basis of the above calculations we obtain

$$\left|< \ell_N^{(\alpha)}(x), f(x) >\right| = \left|< \ell_N^{(\alpha)}(x_1,x_2,...,x_k), f(x_1,x_2,...,x_k) >\right| =$$

$$= \left|< \ell_{N_k}^{(\alpha_k)}(x_k), < \ell_{N_{k-1}}^{(\alpha_{k-1})}(x_{k-1}), ..., < \ell_{N_2}^{(\alpha_2)}(x_2), < \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2,..,x_k) > ... >\right| \leqslant$$

$$\leqslant \left\| \ell_{N_k}^{(\alpha_k)}(x_k) / L_p^{(m_k)^*}(0,1) \right\| \cdot \left\| \ell_{N_{k-1}}^{(\alpha_{k-1})}(x_{k-1}) / L_p^{(m_{k-1})^*}(0,1) \right\| ...$$

$$\cdot \left\| < \ell_{N_1}^{(\alpha_1)}(x_1), f(x_1, x_2, ..., x_k) > / L_p^{(m_1)^*}(0,1) \right\| \leqslant$$

$$\leqslant \left\| \ell_{N_k}^{(\alpha_k)}(x_k) / L_p^{(m_k)^*}(0,1) \right\| ... \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_p^{(m_1)^*}(x_1) \right\| \cdot c'' \left\| f(x) / L_2^{(m)}(K_k) \right\|, \quad (40)$$

where c'' is a constant.

From (40), using the definition for norm of the error functional, we obtain

$$\left\| \ell_N^{(\alpha)} / L_p^{(m)^*}(K_k) \right\| \leqslant c'' \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_2^{(m_1)^*}(0,1) \right\| ... \left\| \ell_{N_k}^{(\alpha_k)}(x_k) / L_p^{(m_k)^*}(0,1) \right\|. \quad (41)$$

Then, using (31), the inequality (41) reduces to

$$\left\| \ell_N^{(\alpha)} / L_p^{(m)*}(K_k) \right\| \leqslant c'' \cdot c \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_k^{m_k}}$$

or

$$\left\| \ell_N^{(\alpha)} / L_p^{(m)*}(K_n) \right\| \leqslant c'' \cdot c \cdot O(h_1^{m_1}) \cdot O(h_2^{m_2}) \cdot \dots \cdot O(h_n^{m_n})$$

where $c = \prod_{i=1}^{n} c_i$ and $m = m_1 + m_2 + ... + m_n$.

Using the validity of Lemma 2 at $n = k$.., we prove that the assertion holds for $n = = k+1$. Taking into account (40), at $n = k+1$ we estimate error of cubature formulas for the form (28)

$$\left| < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_1, x_2, ..., x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > \right| =$$

$$\left| < \ell_{N_1}^{(\alpha_1)}(x_1), < \ell_{N_2}^{(\alpha_2)}(x_2), < ... < \ell_{N_k}^{(\alpha_k)}(x_k), < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_{k+1}), f(x_1, x_2, .., x_{k+1}) > ... > \right| \leqslant$$

$$\leqslant \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_p^{(m_1)^*}(0,1) \right\| ... \left\| \ell_{N_k}^{(\alpha_k)}(x) / L_p^{(m_k)^*}(0,1) \right\| \cdot$$

$$\cdot \left\| < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > / L_p^{(m_{k+1})}(0,1) \right\| \quad (42)$$

Hence, as above, using the definition of norms of functionals, we get

$$\left\| \ell_{N_{k+1}}^{(\alpha_{k+1})} / L_p^{(m_{k+1})^*}(K_{k+1}) \right\| \leqslant c''' \left\| \ell_{N_1}^{(\alpha_1)}(x_1) / L_p^{(m_1)^*}(0,1) \right\| ...$$

$$\cdot \left\| \ell_{N_k}^{(\alpha_k)}(x_k) / L_p^{(m_k)^*}(0,1) \right\| \cdot \left\| < \ell_{N_{k+1}}^{(\alpha_{k+1})}(x_{k+1}), f(x_1, x_2, .., x_{k+1}) > / L_p^{(m_{k+1})^*}(0,1) \right\|. \quad (43)$$

From inequalities (31) and (43) we obtain

$$\left\| \ell_{N_{k+1}}^{(\alpha_{k+1})}{}_N / L_p^{(m_{k+1})*}(K_{k+1}) \right\| \leqslant c''' \cdot c_{k+1} \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_{k+1}^{m_{k+1}}} \quad (44)$$

or

$$\left\| \ell_N^{(\alpha)} / L_p^{(m)*}(K_{k+1}) \right\| \leqslant c''' \cdot c_{k+1} \cdot O(h_1^{m_1}) ... O\left(h_{k+1}^{m_{k+1}}\right),$$

where $c_{k+1} = \prod_{i=1}^{k+1} c_i$.

Summarizing the results obtained, in conclude by noting that

$$\left\|\ell_N^{(\alpha)}/L_p^{(m)*}\left(K_n\right)\right\| \leqslant c\cdot\frac{1}{N_1^{m_1}\cdot N_2^{m_2}...N_n^{m_n}}, \tag{45}$$

or, taking into account (32), from (44) we have
$\left\|\ell_N^{(\alpha)}/L_p^{(m)*}\left(K_n\right)\right\| \leqslant c\cdot O\left(h_1^{m_1}\right)...O\left(h_1^{m_n}\right)$, c is a constant.
Lemma 2 is proved.

With the help of this lemma it is easy to prove the following theorem.

Theorem 2. The weighted cubature formula (28) with the error functional (29) at $N_1 = N_2 = ... = N_n$, $\prod\limits_{i=1}^{n} N_i = N$ and is optimal in order of convergence in the space $L_p^{(m)}\left(K_n\right)$, for the norms of error functionals (29) of the cubature formula (28) we have the equality

$$\left\|\ell_N^{(\alpha)}/L_p^{(m)*}\left(K_n\right)\right\| = O\left(N^{-\frac{m}{n}}\right).$$

Proof. On the basis of Lemma 2 at $N_1 = N_2 = ... = N_n$ we have $N_i = \sqrt[n]{N}$, where $i = 1, 2, ..., n$.

Thus,

$$\prod_{i=1}^{n} N_i^{m_i} = N_1^{m_1+m_2+...+m_n} = N^{\frac{m}{n}}. \tag{46}$$

By substituting (46) into inequality (45), we obtain

$$\left\|\ell_N^{(\alpha)}/L_p^{(m)*}\left(K_n\right)\right\| \leqslant c\cdot N^{-\frac{m}{n}}. \tag{47}$$

From the theorem of N.S. Bakhvalov [23] and the inequality (47) it follows the proof of the theorem. Theorem 2 is proved.

4 Weight cubature formulas in the space $\overline{L}_2^{(m)}\left(K_n\right)$

Multidimensional cubature formulas differ from the quadrature formulas with two features

1. infinitely varied forms of multidimensional areas of integration;
2. rapidly grows number of integration nodes with increasing space dimension.

Problem 2) requires special attention to the construction of the most efficient formulas. Here, we discussthe formula with taking into account this requirement. It is known, that such formulas are called as "practical formulas"by N.S. Bakhvalov [23].

$$\int\limits_{K_n} p(x)f(x)dx \approx \sum_{|\alpha|\leqslant t}\sum_{\lambda=1}^{N}(-1)^{|\alpha|}C_\lambda{}^{(\alpha)}f^{(\alpha)}\left(x^{(\lambda)}\right) \tag{48}$$

in space $\bar{L}_2^{(m)}\left(K_n\right)$, where K_n - n - dimensional unit cube and $p\left(x\right) \in L_2\left(K_n\right)$ is weight function, $K_n = \left\{\left(x_1, x_2...x_n\right)|0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1, ..., 0 \leqslant x_n \leqslant 1\right\}$,$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $(\alpha) = (\alpha_1, \alpha_2, ..., \alpha_n)$,$x^{(\lambda)} = \left(x_1{}^{(\lambda_1)}, x_2{}^{(\lambda_2)}, ..., x_n{}^{(\lambda_n)}\right)$ and $\int\limits_{K_n} p\left(x\right)dx < \infty$, $0 \leqslant t \leqslant m$, $m = m_1 + m_2 + \cdots + m_n$.

In the cubature formula (48) comparable to generalized function

$$\ell_N^{(\alpha)}\left(x\right) = p\left(x\right)\varepsilon_{K_n}(x) - \sum_{|\alpha|\leqslant t}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}\delta^{(\alpha)}\left(x - x^{(\lambda)}\right) \tag{49}$$

and we call it a error functional.

Definition 5. The space $\bar{L}_2^{(m)}(K_n)$ is defined as the space of functions, given on the K_n and the norm of a function, which is determined by the following equation

$$\left\| f/\bar{L}_2^{(m)}(K_n) \right\| = \left\{ \int\limits_{K_n} \left(\frac{\partial^m f(x)}{\partial {x_1}^{m_1} \partial {x_2}^{m_2} ... \partial {x_n}^{m_n}} \right)^2 dx \right\}^{\frac{1}{2}} \tag{50}$$

where $m_1 + m_2 + ... + m_n = m$, $m_i > 0$, $i = \overline{1, n}$
with the scalar product

$$(f, \varphi)_{\bar{L}_2^{(m)}(K_n)} = \int\limits_{K_n} \left(\frac{\partial^m f(x)}{\partial x^m} \right) \left(\frac{\partial^m \varphi(x)}{\partial x^m} \right) dx,$$

where $\partial x^m = \partial x_1^{m_1} \partial x_2^{m_2} ... \partial x_n^{m_n}$, $m = m_1 + m_2 + ... m_n$, $dx = dx_1 dx_2 ... dx_n$.

As it is known [1], the norm of a function in the space $L_2^{(m)}(K_n)$ determined by the formula

$$\left\| f/L_2^{(m)}(K_n) \right\| = \left\{ \int\limits_{K_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha f(x))^2 dx \right\}^{\frac{1}{2}}, \tag{51}$$

where $|\alpha| = \alpha_1 + \alpha_2 + ... \alpha_n$, $\alpha! = \alpha_1! \cdot \alpha_2! \cdot ... \alpha_n!$ and $D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1, ..., x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$.

Suppose that in (51) $n = 2$ and $m = 2$, then we obtain the following

$$\int\limits_{K_2} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\frac{\partial^m f(x)}{\partial x^m} \right)^2 dx = \int\limits_{K_2} \sum_{\alpha_1+\alpha_2=2} \frac{2!}{\alpha! \alpha_2!} \left(\frac{\partial^2 f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right)^2 dx =$$

$$= \int\limits_{K_2} \left[\left(\frac{\partial^2 f(x)}{\partial x_2^2} \right)^2 + \frac{2!}{1! \cdot 1!} \left(\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \right)^2 + \left(\frac{\partial^2 f(x)}{\partial x_1^2} \right)^2 \right] dx. \tag{52}$$

When $n = 2$ and $m = 2$ equality (50) takes the following form:

$$\left\| f/\bar{L}_2^{(2)}\ (K_2) \right\|^2 = \int\limits_{K_2} \left(\frac{\partial^2 f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}} \right)^2 dx \tag{53}$$

Obviously, in the right hand side of (53) is less computing, than in (52), and it follows that the norm of the function in space $\bar{L}_2^{(2)}(K_2)$ the number of computing operations will be much less, than in space $L_2^{(2)}(K_2)$, as in the norm (53), involved only the mixed derivatives.

Now we prove the the following theorem, which is one of the main results of this work.

Theorem 3. If, for the error functional (49) of the weight cubature formula (48) in the space $\bar{L}_2^{(m)}(K_n)$ the following conditions are fulfilled

$$\ell_N{}^{(\alpha)}(x) = \ell_{N_1}^{(\alpha_1)}(x_1) \cdot \ell_{N_2}^{(\alpha_2)}(x_2) \cdot ... \cdot \ell_{N_n}^{(\alpha_n)}(x_n)$$

and

$$\left\| \ell_{N_i}^{(\alpha_i)} / \bar{L}_2^{(m_i)*}(0, 1) \right\| \leqslant c_i \frac{1}{N_i^{m_i}} \tag{54}$$

c_i are constants, that is

$$\left\|\ell_{N_i}^{(\alpha_i)}/\bar{L}_2^{(m_i)^*}(0,1)\right\| \leqslant c_i O\left(h_i^{m_i}\right), \tag{55}$$

c_i are constants, $\left(i=\overline{0,1}\right)$, $h_i=\frac{1}{N_i}$,

then

$$\left\|\ell_N^{(\alpha)}/\bar{L}_2^{(m)^*}(K_n)\right\| \leqslant c\cdot\frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}}., \tag{56}$$

c is a constant, or

$$\left\|\ell_N^{(\alpha)}/\bar{L}_2^{(m)^*}(K_n)\right\| \leqslant c\cdot O\left(h_1^{m_1}\right)\cdot O\left(h_2^{m_2}\right)\cdot ...\cdot O\left(h_n^{m_n}\right) \tag{57}$$

where

$\ell_{N_i}^{(\alpha_i)}(x) = p(x_i)\,\varepsilon_{[0,1]}(x_i) - \sum\limits_{\alpha_i\leqslant t}\sum\limits_{\lambda_i=1}^{N_i} C_{\lambda_i}^{(\alpha_i)}\delta^{(\alpha_i)}\left(x_i-x_i^{(\lambda_i)}\right)$, $p(x)=\prod\limits_{i=1}^{n}p_i(x_i)$, $d=$ $=\prod\limits_{i=1}^{n}d_i$, $m=m_1+m_2+...+m_n$ and m_i is arbitrary $\left(i=\overline{1,n}\right)$, and $m_i\geqslant 1$. $m=m_1+$ $+m_2+...+m_n$ and m_i is arbitrary $\left(i=\overline{1,n}\right)$, and $m_i\geqslant 1$.

Proof. We are conducting proof by mathematical induction. Suppose $n=2$, then $x=(x_1,x_2)$, $|\alpha|=\alpha_1+\alpha_2$, $m=m_1+m_2$, $dx=dx_1dx_2$, $f(x)=f(x_1,x_2)$, $p(x)=$ $=p_1(x_1)\cdot p_2(x_2)$ and $\ell_N^{(\alpha)}(x)=\ell_{N_1}^{(\alpha_1)}(x_1)\cdot\ell_{N_2}^{(\alpha_2)}(x_2)$.

If presume in (50) $n=1$, then

$$\left\|f/\bar{L}_2^{(m_i)}(0,1)\right\| = \left\{\int\limits_0^1\left(\frac{\partial^{m_i}f(x_i)}{\partial x_i^{m_i}}\right)^2 dx_i\right\}^{\frac{1}{2}}.,\ \left(i=\overline{1,n}\right).$$

Thus, we have

$$\left|<\ell_N^{(\alpha)}(x_1,x_2), f(x_1,x_2)>\right| = \left|<\ell_{N_2}^{(\alpha_2)}(x_2), <\ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2)>>\right| \leqslant$$

$$\leqslant\left\|\ell_{N_2}^{(\alpha_2)}/\bar{L}_2^{(m_2)^*}(0,1)\right\|\cdot\left\|<\ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2)>/\bar{L}_2^{(m_2)^*}(0,1)\right\|. \tag{58}$$

We compute the following norm:

$$\left\|<\ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2)>/\bar{L}_2^{(m_2)^*}(0,1)\right\| =$$

$$\left\{\int\limits_0^1\left|\frac{d^{m_2}}{dx_2^{m_2}}<\ell_{N_1}^{(\alpha_1)}(x_1), f(x_1,x_2)>\right|^2 dx_2\right\}^{\frac{1}{2}} =$$

$$=\left\{\int\limits_0^1\left|<\ell_{N_1}^{(\alpha_1)}(x_1), \frac{d^{m_2}}{dx_2^{m_2}}f(x_1,x_2)>\right|^2 dx_2\right\}^{\frac{1}{2}} \leqslant$$

$$\leqslant\left\{\int\limits_0^1\left[\left\|\ell_{N_1}^{(\alpha_1)}(x_1)/\bar{L}_2^{(m_1)^*}(0,1)\right\|\cdot\left\|\frac{d^{m_2}}{dx_2^{m_2}}f(x_1,x_2)/\bar{L}_2^{(m_1)^*}(0,1)\right\|\right]^2 dx_2\right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (0,1) \right\| \cdot \left\{ \int_0^1 \left\{ \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x_1,x_2) \right]^2 dx_1 \right\} dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (0,1) \right\| \cdot \left\{ \int_0^1 \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x) \right]^2 dx \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (0,1) \right\| \cdot \left\| f(x)/\bar{L}_2^{(m)} (K_2) \right\|, \tag{59}$$

where $x = (x_1, x_2)$ and $m = m_1 + m_2$.

Thus, from (58) and (59) we obtain

$$\left| < \ell_N^{(\alpha)} (x_1,x_2), f(x_1,x_2) > \right| \leqslant$$

$$\leqslant \left\| \ell_{N_2}^{(\alpha_2)}/\bar{L}_2^{(m_2)^*} (0,1) \right\| \cdot \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (0,1) \right\| \cdot \left\| f(x)/\bar{L}_2^{(m)} (K_2) \right\|. \tag{60}$$

Considering (50) from (60) we obtain

$$\left\| \ell_N^{(\alpha)} (x)/\bar{L}_2^{(m)^*} (K_2) \right\| \leqslant \left\| \ell_{N_2}^{(\alpha_2)}/\bar{L}_2^{(m_2)^*} (0,1) \right\| \cdot \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (0,1) \right\| \tag{61}$$

Taking into account (54) from (61) we have

$$\left\| \ell_N^{(\alpha)}/\bar{L}_2^{(m)^*} (K_2) \right\| \leqslant c_1 \cdot c_2 \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2}},$$

or

$$\left\| \ell_N^{(\alpha)}/\bar{L}_2^{(m)^*} (K_2) \right\| \leqslant c_3' O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m_2}\right), \tag{62}$$

where $c_3' = c_1 \cdot c_2$.

When $n = k$ we obtain

$$\left| < \ell_N^{(\alpha)} (x), f(x) > \right| = \left| < \ell_N^{(\alpha)} (x_1, x_2, ..., x_k), f(x_1, x_2, ..., x_k) > \right| =$$

$$= \left| < \ell_{N_k}^{(\alpha_k)} (x_k), < \ell_{N_{k-1}}^{(\alpha_{k-1})} (x_{k-1}), ..., < \ell_{N_2}^{(\alpha_2)} (x_2), < \ell_{N_1}^{(\alpha_1)} (x_1), f(x_1, x_2, .., x_k) > ... > \right| \leqslant$$

$$\leqslant \left\| \ell_{N_k}^{(\alpha_k)} (x_k)/\bar{L}_2^{(m_k)^*} (0,1) \right\| \cdot \left\| \ell_{N_{k-1}}^{(\alpha_{k-1})} (x_{k-1})/\bar{L}_2^{(m_{k-1})^*} (0,1) \right\| ...$$

$$. \left\| < \ell_{N_1}^{(\alpha_1)} (x_1), f(x_1, x_2, ..., x_k) > /\bar{L}_2^{(m_1)^*} (0,1) \right\| \leqslant$$

$$\leqslant \left\| \ell_{N_k}^{(\alpha_k)} (x_k)/\bar{L}_2^{(m_k)^*} (0,1) \right\| ... \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (x_1) \right\| \cdot \left\| f(x)/\bar{L}_2^{(m)} (K_k) \right\|. \tag{63}$$

From (63), taking into account (50) we have

$$\left\| \ell_N^{(\alpha)}/\bar{L}_2^{(m)^*} (K_k) \right\| \leqslant \left\| \ell_{N_1}^{(\alpha_1)} (x_1)/\bar{L}_2^{(m_1)^*} (0,1) \right\| ... \left\| \ell_{N_k}^{(\alpha_k)} (x_k)/\bar{L}_2^{(m_k)^*} (0,1) \right\|. \tag{64}$$

Then in view of (54) from (64) we obtain

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_k\right) \right\| \leqslant c \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_k^{m_k}}, \tag{65}$$

where $c = \prod\limits_{i=1}^{k} c_i$ or taking into account (55), from (65) we will have $\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_k\right) \right\| \leqslant c \cdot$ $\cdot\, O\left(h_1^{m_1}\right) \ldots O\left(h_k^{m_k}\right)$.

Using validity of the assertion of theorem 1 for $n = k$, we prove that the assertion is performed when $n = k+1$.

Thus, suppose $n = k+1$, then taking into account (50), from (64) we have

$$\left| < \ell_{N_{k+1}}^{(\alpha_{k+1})}\left(x_1, x_2, \ldots, x_{k+1}\right), f\left(x_1, x_2, \ldots, x_{k+1}\right) > \right| =$$

$$\left| < \ell_{N_1}^{(\alpha_1)}\left(x_1\right), < \ell_{N_2}^{(\alpha_2)}\left(x_2\right), < \ldots < \ell_{N_k}^{(\alpha_k)}\left(x_k\right), < \ell_{N_{k+1}}^{(\alpha_{k+1})}\left(x_{k+1}\right), f\left(x_1, x_2, .., x_{k+1}\right) > \ldots > \right| \leqslant$$

$$\leqslant \left\| \ell_{N_1}^{(\alpha_1)}\left(x_1\right) / \bar{L}_2^{(m_1)^*}(0,1) \right\| \ldots \left\| \ell_{N_k}^{(\alpha_k)}\left(x_k\right) / \bar{L}_2^{(m_k)^*}(0,1) \right\| \cdot$$

$$\cdot \left\| < \ell_{N_{k+1}}^{(\alpha_{k+1})}\left(x_{k+1}\right), f\left(x_1, x_2, \ldots, x_{k+1}\right) > / \bar{L}_2^{(m_{k+1})}(0,1) \right\|. \tag{66}$$

Using (50) and (64) from (66) we obtain

$$\left\| \ell_{N_{k+1}}^{(\alpha_{k+1})} / \bar{L}_2^{(m_{k+1})^*}\left(K_{k+1}\right) \right\| \leqslant \left\| \ell_{N_1}^{(\alpha_1)}\left(x_1\right) / \bar{L}_2^{(m_1)^*}(0,1) \right\| \ldots$$

$$. \left\| \ell_{N_k}^{(\alpha_k)}\left(x_k\right) / \bar{L}_2^{(m_k)^*}(0,1) \right\| \cdot \left\| \ell_{N_{k+1}}^{(\alpha_{k+1})}\left(x_{k+1}\right) / \bar{L}_2^{(m_{k+1})^*}(0,1) \right\| \tag{67}$$

By using (54) from (67) we have

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_{k+1}\right) \right\| \leqslant c_{k+1} \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_{k+1}^{m_{k+1}}}, \tag{68}$$

or, considering (62) and (68) we obtain

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_{k+1}\right) \right\| \leqslant c_{k+1} \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_{k+1}^{m_{k+1}}\right)$$

where $c_{k+1} = \prod\limits_{i=1}^{k+1} c_i$.

Thus obtained inequalities (56) and (57):

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_n\right) \right\| \leqslant c \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_n^{m_n}} \tag{69}$$

$c-$constants, or, considering (55) from (69) we obtain

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_n\right) \right\| \leqslant c \cdot O\left({h_1}^{m_1}\right) \ldots O\left({h_n}^{m_n}\right), \ \ h_i = \frac{1}{N_i}, \ (i = \overline{1, n}), \tag{70}$$

where $c = \prod\limits_{i=1}^{n} c_i$.

If, in (69) or (70) suppose $N = N_1 \cdot N_2 \cdot \ldots \cdot N_n$, $N_1 = N_2 = \ldots = N_n$ and $m_1 + m_2 +$ $+ \ldots + m_n = m$ then we have

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_n\right) \right\| \leqslant c \cdot N^{-\frac{m}{n}}$$

or

$$\left\| \ell_N^{(\alpha)} / \bar{L}_2^{(m)*}\left(K_n\right) \right\| \leqslant c \cdot O\left(h^m\right), \tag{71}$$

c is a constant, $(h = N^{-\frac{1}{n}})$ which was needed of proof. Theorem 3 is proved.

Thus, we obtain an upper estimate for the norm of the error functional (49) for cubature formula (48) in the space $\bar{L}_2^{(m)*}\left(K_n\right)$.

A similar assessment was obtained previously for the norm of the error functional of cubature formula (48) on the quotient space of S.L. Sobolev $L_2^{(m)}\left(K_n\right)$ and as a result we have received the same order of convergence to zero at $N \to \infty$, although the norm of function was defined in different ways, this is confirmed by the inequality is (27), (71).

For illustration, we present an example at $n = 2$.

Suppose

$$f\left(x_1, x_2\right) = e^{a x_1}\left(\frac{1}{2} - b x_2^2\right)^{3/2} \tag{72}$$

Where $a > o$ and $b > 0$.

Obviously, that the derivatives $\frac{\partial^{m-1} f(x_1,x_2)}{\partial x_1^{m-1}}$ and $\frac{\partial f(x_1,x_2)}{\partial x_2}$ are continuous on K_2, but $\frac{\partial^2 f(x_1,x_2)}{\partial x_2} = 3\alpha^{m-2} b^2 e^{a x_1} ({}^1/_2 - b x_2^2)^{-\frac{1}{2}}$ has a feature on K_2. Therefore, from the condition $m = m_1 + m_2$ it is clear that $m_1 = m - 1$ and $m_2 = 1$, in that $m - 1 + 1 = m$. Hence it follows that $f\left(x_1, x_2\right) \in \bar{L}_2^{(m)}\left(K_2\right)$ when $m_1 = m - 1$, $m_2 = 1$ and if $m_2 \geqslant 2$ then $f\left(x_1, x_2\right) \bar{\in} L_2^{(m)}\left(K_2\right)$.

5 Conclusion

In this paper we investigate weighted cubature formula of the Hermitian type in function spaces $L_2^{(m)}, L_p^{(m)}, \bar{L}_2^{(m)}$ of S.L. Sobolev for the functions defined in the n - dimensional unit cube and obtain an upper estimate for the norm of error functionals of weighted cubature formulas. The basis of Bahvalov N.S theorem it is proved that considered cubature formulas of the type Hermit are optimal on order of convergence in these spaces. At the end, it is argued that the weighted cubature formulas in spaces have the same order of convergence, although for the norm of the error functional of these cubature formulas in the space only mixed generalized derivatives are involved and these formulas can be called practical.

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ВЕСОВОЙ ОПТИМАЛЬНЫЙ ПОРЯДОК СХОДИМОСТИ КУБАТУРНЫХ ФОРМУЛ ЭРМИТОВА ТИПА В ПРОСТРАНСТВЕ СОБОЛЕВА

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При изучении различных вопросов, возникающих в теории приближенного интегрирования и дифференциальных уравнений в частных производных и смежных разделах анализа, очень плодотворным оказался так называемый функциональный подход. До сих пор рассматривались кубатурные формулы, с помощью которых приближенно вычисляется определенный интеграл от функции, когда известны значения этой функции в отдельных точках-узлах кубатурной формулы. Возможны более общие кубатурные формулы, которые включают как значения функции, так и значения ее производных того или иного порядка. Если нам известны не только значения функции $f(x)$ в некоторых точках области Ω, но и значения ее производных того или иного порядка, то естественно, что при правильном использовании всех этих данных можно ожидать более точного результата, чем в случае использования только значений функции. В данной работе мы исследуем взвешенную кубатурную формулу эрмитова типа в функциональных пространствах $L_2^{(m)}, L_p^{(m)}, \bar{L}_2^{(m)}$ из С.Л. Соболева для функций, определенных в n-мерном единичном кубе K_n, и получить верхнюю оценку нормы функционалов ошибок взвешенных кубатурных формул. На основе теоремы Бахвалова доказывается, что рассматриваемые кубатурные формулы эрмитова типа оптимальны по порядку сходимости в этих пространствах. Оказывается, пробел $\bar{L}_p^{(m)}$ имеет ряд преимуществ. Действительно, преимущество состоит в том, что, во-первых, для нормы функционала погрешности кубатурных формул в пространстве $\bar{L}_p^{(m)}$ вычислительные операции намного меньше, чем в $L_p^{(m)}$, а во-вторых, нормы функционала погрешности кубатурных формул, заданных в пространствах $\bar{L}_p^{(m)}$ и $L_p^{(m)}$ имеют одинаковый порядок сходимости к нулю в $N \to \infty$.

Ключевые слова: взвешенная кубатурная формула, функционал ошибки, пространство Соболева, функциональные пространства, обобщенная функция.

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