ISSN 1066-369X, Russian Mathematics, 2024, Vol. 68, No. 12, pp. 45–56. © Allerton Press, Inc., 2024.
Russian Text © The Author(s), 2024, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2024, No. 12, pp. 00000–00000.

An Algorithm for Constructing the Fourier Transform of the Function $\overset{\frown\frown}{V}_m(x)$ to Determine a Discrete Analog of a Differential Operator

Ik. I. Jalolov[a] and O. I. Jalolov[b,*,]**

[a] *Tashkent State Transport University, Tashkent, 100167 Republic of Uzbekistan*

[b] *Bukhara State University, Bukhara, 200118 Republic of Uzbekistan*

**E-mail: o_jalolov@mail.ru*

***E-mail: o.i.jalolov@buxdu.uz*

Received January 13, 2024; revised January 13, 2024; accepted March 20, 2024

Abstract—This paper considers the problem of constructing the Fourier transform of a harrow-shaped function to determine a discrete analog of the differential operator, which is used in constructing optimal quadrature formulas in Hörmander space. In addition, the problem of constructing a discrete analog of a specific operator in a particular case is considered.

Keywords: generalized function, Sobolev space, error functional, interpolation formula, extremal function

DOI: 10.3103/S1066369X2470097X

INTRODUCTION

The discrete analog $D_{m,n}[\beta]$ of the following polyharmonic operator plays an important role in constructing optimal quadrature and cubature formulas in the spaces $H_2^{\mu}\left(R^n\right)$ and $L_2^{(m)}\left(R^n\right)$:

$$\Delta^m = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^m .$$

For the first time, Sobolev constructed and studied the properties of inversion of the convolution operator with the function $G_{m,n}[\beta]$, where $G_{m,n}(x)$ is the fundamental solution to the polyharnomic operator, in the space $L_2^{(m)}\left(R^n\right)$, that is, the properties of a function of discrete argument $D_{m,n}[\beta]$ such that satisfies the equality $D_{m,n}[\beta] * G_{m,n}[\beta] = \delta[\beta]$, where $\delta[\beta]$ is unity at $\beta = 0$ and zero at $\beta \neq 0$ [1]. He investigated the properties of the operator $D_{hH}^{(m)}[\beta]$, which is the inversion of the convolution operator with the function $G_{hH}^{(m)}[\beta] = h^n G_m(hH\beta)$ [1]. The problem of constructing the discrete operator $D_{hH}^{(m)}[\beta]$ for an arbitrary n appeared to be very hard.

In the one-dimensional case, a discrete analog of the operator $\frac{d^{2m}}{dx^{2m}}$ was constructed by Zhamolov [2]. However, in this work we wrote the form of this function up to $m+1$ unknown coefficients. In work [3] Shadimetov found these coefficients, thus completely constructing a discrete analog of the operator $\frac{d^{2m}}{dx^{2m}}$. Shadimetov and Khaetov constructed a discrete analog of the differential operators

$$\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$$

and

$$\frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1$$

in their papers [4, 5].

In work [6] Shadimetov and Mamatova used the variational method in the Sobolev space to construct the composite lattice optimal cubature formulas. In work [7], in the space of square summable functions, Gabdulkhaev established the effective sufficient conditions of continuity and compactness of singular integral operators with Cauchy kernels on a segment of the real axis.

In the current work we consider construction of the Fourier transform of the function $\overleftrightarrow{V}_m(x)$ for determining the optimal coefficients of quadrature formulas in the Hörmander space $H_2^{\mu}(R)$.

Definition. The space $H_2^{\mu}(R)$ is defined as a closure of infinitely differentiable functions givene in R and decreasing at infinity faster than any negative exponent in the norm

$$\left\| f \mid H_2^{\mu}(R) \right\| = \left\{ \int_{-\infty}^{\infty} \left| F^{-1}\left[(1+y^2)^{m/2} \cdot F[f(x)](y)\right] \right|^2 dx \right\}^{1/2}.$$

Here, F and F^{-1} are the direct and inverse Fourier transforms:

$$F[f(x)](y) = \int_{-\infty}^{\infty} f(x) e^{2\pi i y x} dx \quad \text{and} \quad F^{-1}[f(x)](y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i y x} dx.$$

When the condition $V_{m/2}(x) = F^{-1}\left[(1+y^2)^{-m/2}\right](x) \in L_2(R)$ is fulfilled, the space $H_2^{\mu}(R)$ is included into the space of continuous functions $C(R)$ [1, 8].

We define the scalar product in $H_2^{\mu}(R)$ as

$$\langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} F^{-1}\left[(1+y^2)^{m/2} \cdot F[\varphi(x)](y)\right] \cdot F^{-1}\left[(1+y^2)^{m/2} \cdot F[\psi(x)](y)\right] dx.$$

1. PROBLEM FORMULATION

In this work we study the problem of constructing the Fourier transform of the function $\overleftrightarrow{V}_m(x)$ for determining a discrete analog $D_m\,[\beta]$ of the differential operator

$$\left(1 - \frac{d^2}{(2\pi\omega)^2\, dx^2}\right)^m$$

[9, 10], which is used in constructing optimal quadrature formulas in the space $H_2^{\mu}(R)$.

This work is a generalization of works [11, 12], that is, we provide an algorithm for constructing the Fourier transform of the function $\overleftrightarrow{V}_m(x)$ for determining a discrete analog of the operator $D_m\,[h\beta]$, which satisfies the equality

$$D_m[\beta] * V_m[\beta] = \delta[\beta], \tag{1}$$

where $V_m(x) = \nu_m(\omega x)$, $\omega > 0$. Here,

$$\nu_m(x) = F^{-1}\left\{\left[1+y^2\right]^m\right\}(x), \quad x = h\beta,$$

where $F^{-1}[\varphi]$ is the inverse Fourier transform of a function φ, $\delta[\beta]$ is the discrete delta function, which is unity at $\beta = 0$ and zero at $\beta \neq 0$, $h = \dfrac{1}{N}$, $N = 2, 3, \ldots, [\beta] = h\beta$.

The operator $D_m[\beta]$ at $m = 3,4$ was constructed in works [11, 12].

Theorem 1. *The operator* $D_3[\beta]$ *satisfying equality* (1) *at* $m = 3$ *is determined by the formula*

$$D_3[\beta] = B\begin{cases} c + \sum_{i=1}^{2}\frac{A_i}{\lambda_i}, & \beta = 0; \\ 1 + \sum_{i=1}^{2} A_i, & |\beta| = 1; \\ \sum_{i=1}^{2} A_1\lambda_1^{|\beta|-1}, & |\beta| \geq 2, \end{cases}$$

where

$$B = \frac{4}{a_1\pi},\ \ a_1 = 16\pi^2h^2\,\mathrm{sh}(2\pi h) - 3\left(\pi h\,\mathrm{ch}(2\pi h) - \frac{1}{2}\mathrm{sh}(2\pi h)\right),\ \ c = -\left(6\,\mathrm{ch}(2\pi)h + \frac{a_2}{a_1}\right),$$

$$a_2 = 6\,\mathrm{ch}(2\pi h)\left(\pi hh2\pi h - \frac{1}{2}\mathrm{sh}(2\pi h)\right) - 3\left(\mathrm{sh}(2\pi h)\,\mathrm{ch}(2\pi h) - 2\pi h\right) - 8\pi^2h^2\,\mathrm{sh}(2\pi h),$$

$$A_i = \frac{\lambda_i^6 - 6b\lambda_i^5 + 3\left(b^2 4 + 1\right)\lambda_i^4 - 4b\left(2b^2 + 3\right)\lambda_i^3 + 3\left(4b^2 + 1\right)\lambda_i^2 - 6b\lambda_i + 1}{\lambda_i^2 - 1},\ \ i = \overline{1,2},$$

$$\lambda_1 = \frac{1}{4}\left[b_1 + \sqrt{b_1^2 - 4b_2 - 8} + \sqrt{\left(b_1 + \sqrt{b_1^2 - 4b_2 + 8}\right)^2 - 16}\right],$$

$$\lambda_2 = \frac{1}{4}\left[b_1 + \sqrt{b_1^2 - 4b_2 - 8} - \sqrt{\left(b_1 + \sqrt{b_1^2 - 4b_2 + 8}\right)^2 - 16}\right],$$

$$b_1 = \frac{a_2}{a_1},\ \ b_2 = \frac{a_3}{a_1},$$

$$a_3 = \left[6\,\mathrm{ch}(2\pi h)\left(\mathrm{sh}(2\pi h)\,\mathrm{ch}(2\pi h) - 2\pi h\right) - 6\left(\pi h\,\mathrm{ch}(2\pi h) - \frac{1}{2}\mathrm{sh}(2\pi h)\right) + 16\pi^2h^2\,\mathrm{sh}\,2\pi h ch 2\pi h\right],$$

$|\lambda_1| < 1$, *and h is a small parameter.*

Theorem 2. *The operator* $D_4[\beta]$ *satisfying equality* (1) *at* $m = 4$ *is determined by the formula*

$$D_4[\beta] = B\cdot\begin{cases} c + \sum_{i=1}^{3}\frac{A_i}{\lambda_i}, & \beta = 0; \\ 1 + \sum_{i=1}^{3} A_i, & |\beta| = 1; \\ \sum_{i=1}^{3} A_i\lambda_i^{|\beta|-1}, & |\beta| \geq 2, \end{cases}$$

where

$$B = \frac{2^2\cdot 3}{\pi^2\cdot a_1},\ \ a_1 = -15a + 120hb - 12\cdot 16\pi h^2 a + 32\pi^2h^3 b,$$

$$c = \frac{a_2}{a_1} + 8b,\ \ a_2 = 90ab - 240h\left(1 + 2b^2\right) + 12\cdot 16\pi^2 a\cdot 2(1 + b) + 32\pi^2h^3\cdot 8\left(b^2 - 1\right),$$

$$A_i = \frac{\lambda_i^8 + b_1'\lambda_i^7 + b_2'\lambda_i^6 + b_3'\lambda_i^5 + b_4'\lambda_i^4 + b_5'\lambda_i^3 + b_6'\lambda_i^2 + b_7' + 1}{\lambda_i^2 - 1},\ \ i = \overline{1,3},$$

$$b_1' = -8b,\ \ b_2' = 4\left(b^2 + 1\right),\ \ b_3' = 8b\left(1 - 4b^2\right),\ \ b_4' = 2\left(24b^2 + 8b^4 + 3\right);$$

where

$$b_1' = b_7',\ \ b_2' = b_6',\ \ b_3' = b_5',$$

λ_i are the roots of the polynomial $P_6(\lambda) = \left(\lambda^6 + b_1\lambda^5 + b_2\lambda^4 + b_3\lambda^3 + b_4\lambda^2 + b_5\lambda + 1\right)$, $i = \overline{1,3}$,

$$b_1 = \frac{a_2}{a_1},\ \ b_2 = \frac{a_3}{a_1},\ \ b_3 = \frac{a_4}{a_1},$$

where

$$a_3 = -45a\left(1+4b^2\right) + 120hb\left(11+4b^2\right) + 12\cdot 16\pi h^2 a(8b-11) + 32\pi^2 h^3 b\left(4b^2-13\right),$$

$$a_4 = 60ab(3+2b) - 120h4\left(1+4b^2\right) + 12\cdot 16\pi h^2 a4\left(1+2b^2+3b\right) + 32\pi^2 h^3 16\left(2-b^2\right),$$

$$\lambda_1 = \frac{1}{2}\left\{\frac{1}{2}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \sqrt{\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)^2 - 4\left(b_2-\left(A+B+\frac{b_1}{3}\right)\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)-3\right)}\right]\right.$$

$$\left. + \sqrt{\frac{1}{4}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \sqrt{\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)^2 - 4\left(b_2-\left(A+B+\frac{b_1}{3}\right)\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)-3\right)}\right]^2 - 4}\right\},$$

$$\lambda_2 = \frac{1}{2}\left\{\frac{1}{2}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \sqrt{\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)^2 - 4\left(b_2-\left(A+B+\frac{b_1}{3}\right)\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)-3\right)}\right]\right.$$

$$\left. - \sqrt{\frac{1}{4}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \sqrt{\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)^2 - 4\left(b_2-\left(A+B+\frac{b_1}{3}\right)\left(b_1-\left(A+B+\frac{b_1}{3}\right)\right)-3\right)}\right]^2 - 4}\right\};$$

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{9} + \frac{q^2}{4}}},\ \ B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{9} + \frac{q^2}{4}}};$$

where $p = -\dfrac{b_1^2}{3} + b_2 - 3$, $q = 2\cdot\dfrac{b_1^3}{9} - \dfrac{b_1(b_2-3)}{3} + 2b_1 - b_3$ *and h is a small parameter.*

2. SOME PROPERTIES OF THE FUNCTION $V_m(x)$

1. The evenness of $\mu^{-1}(\xi) = (1+\xi^2)^{-m}$ leads to the evenness of $V_m(x) = F^{-1}[\mu(\xi)](\omega x)$. It is clear.

2. The function $V_m(x)$ decreases at infinity faster than any negative exponent of $|x|$. Indeed, the functions $\mu(\xi)$ and $\mu^{-1}(\xi)$ are infinitely differentiable functions. The infinite differentiability and summbaility of $\mu^{-1}(\xi)$ and its derivatives result in

$$V_m(x) = \int_{-\infty}^{\infty} (-2\pi i\omega x)^{(\alpha)} \cdot \left[\left(1+\xi^2\right)^{-m}\right]^{(\alpha)} e^{-2\pi i\xi\omega x} d\xi,$$

for any $\alpha \geq 0$. Therefore, $\left|V_m(x)(2\pi i\omega x)^{\alpha}\right| < \infty$.

3. The explicit form of $V_m(x)$. Due to the evenness of $V_m(x)$ we have

$$V_m(x) = \nu_m(\omega x) = \int_{-\infty}^{\infty} \frac{e^{2\pi i y\omega x}}{(1+y^2)^m} dy = 2\int_0^{\infty} \frac{\cos 2\pi\omega y x}{(1+y^2)^m} dy.$$

Next, the following equality is well-known (see [13]):

$$\int_0^\infty \frac{\cos(ay)dy}{(b^2+y^2)^n} = \frac{\pi \cdot e^{-ab}}{(2\cdot b)^{2n-1}(n-1)!} \sum \frac{(2n-k-2)!}{k!(n-k-1)!}(2ab)^k. \tag{2}$$

By putting $a = 2\pi\omega x$ and $n = m$ in this equality, we obtain an explicit expression for the Fourier transform of the function $\mu^{-1}(\xi)$ in elementary functions. Because the left-hand side of Eq. (2) does not change when we replace x by $-x$ due to the evenness of the integrand function and the right-hand side holds only for $x > 0$, in the right-hand side of the last equality we replace x by $|x|$.

Hence,

$$V_m(x) = \nu_m(\omega x) = 2\int_0^\infty \frac{\cos 2\pi\omega y x}{(1+y^2)^m}dy = \frac{\pi\cdot e^{-2\pi\omega|x|}}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(2\pi\omega)^k}{k!(m-k-1)!}|x|^k. \tag{3}$$

3. ALGORITHM FOR CONSTRUCTING THE FOURIER TRANSFORM OF THE FUNCTION $\overline{V}_m(x)$

In what follows, we consider the system from work [14] having the form

$$\sum_{\beta=0}^{N} \overset{0}{C}_\beta V_m(h\alpha - h\beta) = \int_{-\infty}^{\infty} p(y)\cdot \varepsilon_{[0,1]}(y)\cdot V_m(h\alpha - y)dy, \quad \alpha = 0,1,2,\ldots,N. \tag{4}$$

We denote $C[\beta] = C_\beta$ and $V_m^h[\beta] = V_m(h\beta)$ and can write system (4) as a convolution of functions of discrete argument:

$$\overset{0}{C}[\beta] * V_m^h[\beta] = f_m[\beta], \quad \beta = 0,1,\ldots,N, \tag{5}$$

$$\overset{0}{C}[\beta] = 0, \quad h\beta \notin [0,1], \tag{6}$$

where

$$f_m[\beta] = \int_0^1 p(y)\cdot V_m(h\beta - y)dy.$$

We denote the system of equations (5), (6) by the system B.

Consider the corresponding problem.

Problem B. Find a discrete function $C[\beta]$ satisfying the system B at given $f_m[\beta]$.

The main idea of this method consists in replacing the unknown function $C[\beta]$ by the function $u_m^h[\beta] = u_m(h\beta)$, namely, instead of $C[\beta]$ we introduce the unknown function

$$u_m^h[\beta] = V_m^h[\beta] * C[\beta]. \tag{7}$$

Then we need to find an operator $D_m(h\beta) = D_m^h[\beta]$ fulfilling the equality

$$D_m^h[\beta] * V_m^h[\beta] = \delta(h\beta), \tag{8}$$

where $\delta(h\beta) = \begin{cases} 1, & \beta = 0; \\ 0, & \beta \neq 0. \end{cases}$

From (7), taking into account (8),we obtain

$$C[\beta] = D_m^h[\beta] * u_m^h[\beta],$$

where $u_m^h[\beta]$ was determined in work [14], where the following theorem was proved.

Theorem 3. *The function*

$$u_m(h\beta) = \begin{cases} f_m[\beta], & h\beta \in [0,1]; \\ \sum\limits_{\alpha=0}^{N} C_\alpha V_m(h\beta - h\alpha), & h\beta \notin [0,1]. \end{cases}$$

Now, we are going to seek the solution to Eq. (8). According to the theory of periodic generalized functions and Fourier transform, instead of a discrete function $D_m(h\beta)$ it is more convenient to seek the harrow-shaped function

$$\overleftrightarrow{D}_m(x) = \sum_{\beta=-\infty}^{\infty} D_m(h\beta)\delta(x - h\beta).$$

Then Eq. (8) in the class of harrow-shaped functions transforms to the equation

$$\overleftrightarrow{D}_m(x) * \overleftrightarrow{V}_m(x) = \delta(x), \tag{9}$$

where

$$\overleftrightarrow{V}_m(x) = \sum_{\beta=-\infty}^{\infty} V_m(h\beta)\delta(x - h\beta).$$

1 It is well-known (see [1]) that there exists an isomorphism between the calss of harrow-shaped functions and the class of functions of discrete argument. Therefore, instead of $D_m(h\beta)$ it is sufficient to study the function $\overleftrightarrow{D}_m(x)$. We apply the Fourier transform to both parts of Eq. (9) and keep in mind that $F[\varphi(x) * \psi(x)] = F[\varphi] \cdot F[\psi]$ and $F[\delta(x)] = 1$, thus obtaining

$$F\left[\overleftrightarrow{D}_m(x)\right] \cdot F\left[\overleftrightarrow{V}_m(x)\right] = 1.$$

Hence,

$$F\left[\overleftrightarrow{D}_m(x)\right] = \left\{F\left[\overleftrightarrow{V}_m(x)\right]\right\}^{-1}. \tag{10}$$

Let us compute the Fourier transform $F\left[\overleftrightarrow{V}_m(x)\right]$ of the harrow-shaped function $\overleftrightarrow{V}_m(x)$.

It is well-known (see [1]) that

$$\phi_0(x) = \sum_{\beta=-\infty}^{\infty} \delta(x - \beta), \quad \delta(hx) = h^{-1}\delta(x) \quad \text{and} \quad F\left[\sum_{\beta=-\infty}^{\infty} \exp(2\pi i \beta x)\right] = \sum_{\beta=-\infty}^{\infty} \delta(x - \beta).$$

Due to these equalities we obtain

$$\begin{aligned} \overleftrightarrow{V}_m(x) &= \sum_{\beta=-\infty}^{\infty} V_m(h\beta)\delta(x - h\beta) = V_m(x)\sum_{\beta=-\infty}^{\infty} \delta(x - h\beta) \\ &= h^{-1}V_m(x) \cdot \sum_{\beta=-\infty}^{\infty} \delta(xh^{-1} - \beta) = h^{-1}V_m(x)\phi_0(h^{-1}x). \end{aligned} \tag{11}$$

It is also well-known (see [2]) that

$$\begin{aligned} F[\phi_0(h^{-1}x)] &= \sum_{\beta=-\infty}^{\infty} F[\delta(h^{-1}x - \beta)] = h\sum_{\beta=-\infty}^{\infty} F[\delta(x - h\beta)] \\ &= h\sum_{\beta=-\infty}^{\infty} \exp(2\pi i \,\mathrm{ph}\,\beta) = h\sum_{k=-\infty}^{\infty} \delta(hp - \beta) = h\phi_0(hp), \end{aligned}$$

that is,

$$F[\phi_0(h^{-1}x)] = h\phi_0(hp). \tag{12}$$

Therefore, taking into account (11) and (12), we have

$$\begin{aligned} F\left[\overleftrightarrow{V}_m(x)\right] &= F[h^{-1}V_m(x)\cdot\phi_0(h^{-1}x)] \\ &= h^{-1}F[V_m(x)] * F[\phi_0(h^{-1}x)] = h^{-1}F[V_m(x)] * \phi_0(hp)\cdot h. \end{aligned} \tag{13}$$

Because $F[V_m(x)](p) = (1+p^2)^{-m}$, from (20) we get

$$\begin{aligned} F\left[\overleftrightarrow{V}_m(x)\right](p) &= \left(1+p^2\right)^{-m} * \phi_0(hp) = \sum_{\beta=-\infty}^{\infty}\left(1+p^2\right)^{-m} * \delta(hp-\beta) \\ &= h^{-1}\sum_{\beta=-\infty}^{\infty}\frac{1}{\left[1+\left(p-\beta h^{-1}\right)^2\right]^m}. \end{aligned} \tag{14}$$

The function $F\left[\overleftrightarrow{V}_m(x)\right](p)$ is an $N = h^{-1}$-periodic function. Using Eq. (14), we must first determine $\left\{F\left[\overleftrightarrow{V}_m(x)\right](p)\right\}^{-1}$, which also is an N-periodic function, and then expand it into the Fourier series. Consequently, from (10) we have

$$F\left[\overleftrightarrow{D}_m(x)\right](p) = \sum_{\beta=-\infty}^{\infty}\hat{D}_\beta\exp(2\pi i\beta hp), \tag{15}$$

where $\hat{D}_k$ are the Fourier coefficients of the function $F\left[\overleftrightarrow{D}_m(x)\right](p)$, that is,

$$\hat{D}_\beta = \int_0^N F\left[\overleftrightarrow{D}_m(x)\right](p)\exp(2\pi iph\beta)dp.$$

By applying the Fourier inversion formula to equality (15), we arrive at the harrow-shaped function

$$\overleftrightarrow{D}_m(x) = \sum_{\beta=-\infty}^{\infty}\hat{D}_k\delta(x-h\beta).$$

Thus, by definition of the harrow-shaped function, the set of Fourier coefficients is the sought function of discrete argument $D_m(hk)$. To determine the Fourier coefficients $F\left[\overleftrightarrow{D}_m(x)\right]$, we first determine the Fourier coefficients of the function

$$F\left[\overleftrightarrow{V}_m(x)\right](p).$$

To determine $F\left[\overleftrightarrow{V}_m(x)\right](p)$, we prove

Lemma. *The equality holds*:

$$\sum_{\beta=-\infty}^{\infty}\frac{h^{-1}}{\left[1+(p-\beta h^{-1})^2\right]^m} = \sum_{\beta=-\infty}^{\infty}\nu_m(\omega h\beta)\exp(2\pi i\beta hp). \tag{16}$$

Proof. Let φ be the trial function. Then

$$\left(\sum_{\beta=-\infty}^{\infty}\frac{h^{-1}}{\left[1+(p-h^{-1}\beta)^2\right]^m},\varphi(p)\right)=\left(F^{-1}\left[\sum_{\beta=-\infty}^{\infty}\frac{h^{-1}}{\left[1+(p-h^{-1}\beta)^2\right]^m}\right],F^{-1}[\varphi(p)]\right)$$

$$=\left(F^{-1}\left[\frac{1}{(1+p^2)^m}*\phi_0(hp)\right](x),F^{-1}[\varphi(p)](x)\right)$$

$$=\left(F^{-1}\left[\frac{1}{(1+p^2)^m}\right](x)\cdot F^{-1}\left[\phi_0(hp)\right](x),F^{-1}[\varphi(p)](x)\right)$$

$$=(V_m(x)\cdot h^{-1}\phi_0(h^{-1}x),F^{-1}[\varphi(p)](x))=\left(\sum_{k=-\infty}^{\infty}V_m(hk)\exp(2\pi ihkx),\varphi(x)\right).$$

□

The definition of equality of two generalized functions leads to (16), which proves the lemma. Relations (14) and (16) result in

$$F\left[\overleftrightarrow{V}_m(x)\right](p)=\sum_{\beta=-\infty}^{\infty}V_m(h\beta)e^{2\pi ihp\beta}. \tag{17}$$

From (13) we see that $F\left[\overleftrightarrow{V}_m(x)\right](p)>0$ as a sum of positive functions. Therefore, $\left\{F\left[\overleftrightarrow{V}_m(x)\right](p)\right\}^{-1}>0$. According to (10), to determine the Fourier coefficients of the function $F\left[\overleftrightarrow{D}_m(h\beta)\right](p)$, we need to find the Fourier transform of the function $\overleftrightarrow{V}_m(x)$.

The main result of the current work is

Theorem 4. *The Fourier transform of the function* $\overleftrightarrow{V}_m(x)$ *for determination of a discrete analog of the differential operator*

$$\left[1-\frac{1}{(2\pi\omega)^2}\frac{d^2}{dx^2}\right]^m, \tag{18}$$

satisfying equality (10), *has the form*

$$F\left[\overleftrightarrow{V}_m(x)\right](p)=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[\frac{\lambda\left(e^{4\pi\omega h}-1\right)}{-\lambda^2e^{2\pi\omega h}+\lambda\left(e^{4\pi\omega h}+1\right)-e^{2\pi\omega h}}\right.$$

$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\frac{e^{2\pi\omega h}}{e^{2\pi\omega h}-\lambda}\sum_{i=0}^{k}\left(\frac{\lambda}{e^{2\pi\omega h}-\lambda}\right)^i\right.$$

$$\left.\times\sum_{\rho=0}^{i}(-1)^{i-\rho}\frac{i!}{\rho!(i-\rho)!}\rho^k+\frac{\lambda e^{2\pi\omega h}}{\lambda e^{2\pi\omega h}-1}\sum_{i=0}^{k}\left(\frac{1}{\lambda e^{2\pi\omega h}-1}\right)^i\sum_{\rho=0}^{i}(-1)^{i-\rho}\frac{i!}{\rho!(i-\rho)!}\rho^k\right].$$

Proof. Using (17), we compute the Fourier transform of the function $\overleftrightarrow{V}_m(x)$. In formula (3) we put $x=h\beta$; then

$$F\left[\overleftrightarrow{V}_m(x)\right](p)=\sum_{k=-\infty}^{\infty}\frac{\pi\cdot e^{-2\pi\omega h|\beta|}}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega)^k}{k!(m-k-1)!}|h\beta|^k\,e^{2\pi ih\beta p}.$$

Thus,

$$F\left[\overleftarrow{V}_m(\omega x)\right](p)=\sum_{\beta=-\infty}^{\infty}\nu_m(h\beta)e^{2\pi i h\omega\beta p}=\sum_{k=-\infty}^{\infty}\frac{\pi\cdot e^{-2\pi\omega h|\beta|}}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega)^k}{k!(m-k-1)!}|h\beta|^k$$
$$\times e^{2\pi i h\beta p}=\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\sum_{\beta=-\infty}^{\infty}e^{-2\pi\omega h|\beta|}e^{2\pi i h p\beta}|\beta|^k. \tag{19}$$

Hence, by denoting $e^{2\pi hp}=\lambda$, after some manipulations, from (19) we get

$$F\left[\overleftarrow{V}_m(x)\right](p)=\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\sum_{\beta=-\infty}^{\infty}e^{-2\pi\omega h|\beta|}\lambda^\beta|\beta|^k$$
$$=\frac{\pi}{2^{2m-2}(m-1)!}\frac{(2m-2)!}{(m-1)!}\sum_{\beta=-\infty}^{\infty}e^{-2\pi\omega h|\beta|}\lambda^\beta$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\sum_{\beta=-\infty}^{\infty}e^{-2\pi\omega h|\beta|}\lambda^\beta|\beta|^k$$
$$=\frac{\pi}{2^{2m-2}(m-1)!}\frac{(2m-2)!}{(m-1)!}\left[1+\sum_{\beta=-\infty}^{-1}e^{-2\pi\omega h|\beta|}\lambda^\beta+\sum_{\beta=1}^{\infty}e^{-2\pi\omega h|\beta|}\lambda^\beta\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\sum_{\beta=-\infty}^{-1}e^{-2\pi\omega h|\beta|}\lambda^\beta|\beta|^k+\sum_{\beta=1}^{\infty}e^{-2\pi\omega h|\beta|}\lambda^\beta|\beta|^k\right] \tag{20}$$
$$=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[1+\sum_{\beta=1}^{\infty}\left(\frac{\lambda}{e^{2\pi\omega h}}\right)^\beta+\sum_{\beta=1}^{\infty}e^{-2\pi\omega h\beta}\lambda^{-\beta}\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\sum_{\beta=1}^{\infty}\left(\frac{\lambda}{e^{2\pi\omega h}}\right)^\beta\beta^k+\sum_{\beta=1}^{\infty}\left(\lambda^{-\beta}e^{-2\pi\omega h\beta}\right)\beta^k\right]$$
$$=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[1+\sum_{\beta=1}^{\infty}\left(\frac{\lambda}{e^{2\pi\omega h}}\right)^\beta+\sum_{\beta=1}^{\infty}\frac{1}{\left(\lambda e^{2\pi\omega h}\right)^\beta}\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\sum_{\beta=0}^{\infty}\left(\frac{\lambda}{e^{2\pi\omega h}}\right)^\beta\beta^k+\sum_{\beta=0}^{\infty}\left(\frac{1}{\lambda e^{2\pi\omega h}}\right)^\beta\beta^k\right].$$

We apply some formulas from [15] to the first term in (20) and obtain

$$F\left[\overleftarrow{V}_m(x)\right](p)=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[1+\frac{\lambda}{e^{2\pi\omega h}-1}+\frac{1}{\lambda e^{2\pi\omega h}-1}\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\sum_{\beta=0}^{\infty}\left(\frac{\lambda}{e^{2\pi\omega h}}\right)^\beta\beta^k+\sum_{\beta=0}^{\infty}\frac{1}{\left(\lambda e^{2\pi\omega h}\right)^\beta}\beta^k\right]. \tag{21}$$

The following formula holds [16]:

$$\sum_{\gamma=0}^{n-1}q^\gamma\gamma^k=\frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i\Delta^i0^k-\frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i\Delta^i\gamma^k\Big|_{\gamma=n},$$

for $|q|<1$ from (21) we have

$$\sum_{\gamma=0}^{\infty}q^\gamma\gamma^k=\frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i\Delta^i0^k. \tag{22}$$

Due to (22), from (21) we have

$$F\left[\overleftarrow{\overrightarrow{V}}_m(x)\right](p)=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[\frac{\lambda\left(e^{4\pi\omega h}-1\right)}{-\lambda^2e^{2\pi\omega h}+\lambda\left(e^{4\pi\omega h}+1\right)-e^{2\pi\omega h}}\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\frac{e^{2\pi\omega h}}{e^{2\pi\omega h}-\lambda}\sum_{i=0}^{k}\left(\frac{\lambda}{e^{2\pi\omega h}-\lambda}\right)^i\Delta^i0^k\right.\tag{23}$$
$$\left.+\frac{\lambda e^{2\pi\omega h}}{\lambda e^{2\pi\omega h}-1}\sum_{i=0}^{k}\left(\frac{1}{\lambda e^{2\pi\omega h}-1}\right)^i\Delta^i0^k\right],$$

where $\Delta^i\gamma^k$ is the finite difference of order i for γ^k, $\Delta^i0^k=\Delta^i\gamma^k\big|_{\gamma=n}$.

Thus, by applying the formulas from [17] to (23), we have

$$F\left[\overleftarrow{\overrightarrow{V}}_m(x)\right](p)=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[\frac{\lambda\left(e^{4\pi\omega h}-1\right)}{-\lambda^2e^{2\pi\omega h}+\lambda\left(e^{4\pi\omega h}+1\right)-e^{2\pi\omega h}}\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\frac{e^{2\pi\omega h}}{e^{2\pi\omega h}-\lambda}\sum_{i=0}^{k}\left(\frac{\lambda}{e^{2\pi\omega h}-\lambda}\right)^i\Delta^i0^k\right.$$
$$\left.+\frac{\lambda e^{2\pi\omega h}}{\lambda e^{2\pi\omega h}-1}\sum_{i=0}^{k}\left(\frac{1}{\lambda e^{2\pi\omega h}-1}\right)^i\Delta^i0^k\right]$$
$$=\frac{\pi(2m-2)!}{2^{2m-2}[(m-1)!]^2}\left[\frac{\lambda\left(e^{4\pi\omega h}-1\right)}{-\lambda^2e^{2\pi\omega h}+\lambda\left(e^{4\pi\omega h}+1\right)-e^{2\pi\omega h}}\right]$$
$$+\frac{\pi}{2^{2m-2}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(4\pi\omega h)^k}{k!(m-k-1)!}\left[\frac{e^{2\pi\omega h}}{e^{2\pi\omega h}-\lambda}\sum_{i=0}^{k}\left(\frac{\lambda}{e^{2\pi\omega h}-\lambda}\right)^i\right.$$
$$\left.\times\sum_{\rho=0}^{i}(-1)^{i-\rho}\frac{i!}{\rho!(i-\rho)!}\rho^k+\frac{\lambda e^{2\pi\omega h}}{\lambda e^{2\pi\omega h}-1}\sum_{i=0}^{k}\left(\frac{1}{\lambda e^{2\pi\omega h}-1}\right)^i\sum_{\rho=0}^{i}(-1)^{i-\rho}\frac{i!}{\rho!(i-\rho)!}\rho^k\right],$$

which completes the proof.

□

Note that in works [18–21] researchers constructed optimal quadrature formulas and convergence-order optimal, asymptotic optimal, and practical asymptotic optimal cubature formulas in the Sobolev functional spaces.

Because we know that $F\left[\overleftarrow{\overrightarrow{V}}_m(x)\right](p)$, to determine the Fourier coefficients of the function $F\left[\overleftarrow{\overrightarrow{D}}_m(h\beta)\right](p)$, that is, $D_m(x)$, taking into account (10) after expansion of $\left\{F\left[\overleftarrow{\overrightarrow{V}}_m(x)\right](p)\right\}^{-1}$ into the Fourier series, we below find a discrete analog $D_m[\beta]$ of operator (18).

Consider the case $m=1$ and $\omega=1$. For $F\left[\overleftarrow{\overrightarrow{V}}_m(x)\right](p)$ we have

$$F[\overleftarrow{\overrightarrow{V}}_1(\beta)](p)=\pi\left[1+\frac{\lambda}{\exp(2\pi h)-\lambda}+\frac{1}{\lambda\exp(2\pi h)-1}\right]$$
$$=\pi\left[\frac{(\exp(2\pi h)-\lambda)(\lambda\exp(2\pi h)-1)+\lambda(\lambda\exp(2\pi h)-1)+(\exp(2\pi h)-\lambda)}{\lambda\exp(4\pi h)-\exp(2\pi h)-\lambda^2\exp(2\pi h)+\lambda}\right]$$

$$= \pi\left[\frac{-\lambda^2\exp(2\pi h)+\lambda(\exp(4\pi h)+1)-\exp(2\pi h)+\lambda^2\exp(2\pi h)-\lambda+\exp(2\pi h)-\lambda}{-\lambda^2\exp(2\pi h)+\lambda(\exp(4\pi h)+1)-\exp(2\pi h)}\right]$$

$$= \pi\left[\frac{\lambda(\exp(4\pi h)-1)}{-\lambda^2\exp(2\pi h)+\lambda(\exp(4\pi h)+1)-\exp(2\pi h)}\right],$$

where $\lambda = \exp(2\pi iph)$. Therefore,

$$F\left[\overleftrightarrow{V}_1(\beta)\right](p) = \pi\left[\frac{\lambda(\exp(4\pi h)-1)}{-\lambda^2\exp(2\pi h)+\lambda(\exp(4\pi h)+1)-\exp(2\pi h)}\right].$$

Next, we find $\left\{F\left[\overleftrightarrow{V}_1(x)\right](p)\right\}^{-1}$. We have

$$\left\{F\left[\overleftrightarrow{V}_m(x)\right](p)\right\}^{-1} = \frac{1}{\pi}\left[\frac{\lambda^2\exp(2\pi h)-\lambda(\exp(4\pi h)+1)+\exp(2\pi h)}{\lambda(1-\exp(4\pi h))}\right]$$

$$= \frac{1}{\pi}\left[\lambda\frac{\exp(2\pi h)}{1-\exp(4\pi h)} - \frac{\exp(4\pi h)+1}{1-\exp(4\pi h)} + \frac{1}{\lambda}\frac{\exp(2\pi h)}{1-\exp(4\pi h)}\right] = \sum_{\beta=-\infty}^{\infty} D_1[\beta]\exp(2\pi iph\beta).$$

This leads to

Theorem 5. *A discrete analog* $D_m[\beta]$ *of the differential operator* $\left[1-\frac{1}{(2\pi)^2}\frac{d^2}{dx^2}\right]^m$, *satisfying equality* (8), *at* $m = 1$ *has the following form*:

$$D_1[\beta] = \frac{1}{\pi}\begin{cases}\dfrac{\exp(4\pi h)+1}{\exp(4\pi h)-1}, & \beta = 0;\\[2mm] \dfrac{\exp(2\pi h)}{1-\exp(4\pi h)}, & |\beta| = 1.\end{cases}$$

CONCLUSIONS

Discrete analogs of differential operators play an important role in the theory of quadrature and cubature formulas. Sobolev developed an algorithm for determining the optimal coefficients of quadrature and cubature formulas.

In the current work we considered the problem of constructing the Fourier transform of the function $\overleftrightarrow{V}_m(x)$ for determining a discrete analog $D_m[\beta]$ of the differential operator $\left(1-\frac{d^2}{(2\pi\omega)^2\,dx^2}\right)^m$, which is used in constructing optimal quadrature formulas in the space $H_2^{\mu}(R)$.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. S. L. Sobolev, *Introduction to the Theory of Cubature Formulas* (Nauka, Moscow, 1974).

2. Z. Zh. Zhamolov, “On one difference analog of the operator, and its construction,” in *Direct and Inverse Problems for Differential Equations with Partial Derivatives and Their Applications* (Fan, Tashkent, 1978).
3. Kh. M. Shadimetov, “Discrete analog of the differential operator, and its construction,” Vopr. Vychisl. Prikl. Mat. **79**, 22–35 (1985).
4. Kh. M. Shadimetov and A. R. Khaetov, “Construction of the discrete analog of the differential operator,” Uzbekskii Matematicheskii zhurnal, No. 2, 85–95 (2004).
5. A. R. Khaetov, “Construction of the discrete analog of the differential operator, and its properties,” Uzbekskii Matematicheskii zhurnal, No. 3, 81–88 (2009).
6. Kh. M. Shadimetov and N. Kh. Mamatova, “Compound cubature formulas on a lattice,” Russ. Math. **67** (11), 49–63 (2023).
https://doi.org/10.3103/s1066369x23110087
7. B. G. Gabdulkhaev, “Continuity and compactness of singular integral operators,” Russ. Math. **53** (8), 1–7 (2009).
https://doi.org/10.3103/s1066369x09080015
8. L. R. Volevich and B. P. Paneyakh, “Certain spaces of generalized functions and embedding theorems,” Russ. Math. Surv. **20**, 1–73 (1965).
https://doi.org/10.1070/rm1965v020n01abeh004139
9. I. I. Jalolov, “A differential operator of order $2m$ and its fundamental solution,” Uzbek Mathematical Journal **2018** (1), 65–72 (2018).
https://doi.org/10.29229/uzmj.2018-1-6
10. I. I. Jalolov, “The algorithm for constructing a differential operator of 2nd order and finding a fundamental solution,” AIP Conf. Proc. **2365**, 20015 (2021).
https://doi.org/10.1063/5.0057025
11. Ik. I. Jalolov, “Algorithm for constructing the operator $D_3[\beta]$ is given and the explicit form of the operator is shown,” Problemy Vychislitel’noi i Prikladnoi Matematiki, No. 2, 48–52 (2015).
12. I. I. Jalolov, “Algorithm for constructing a discrete analogue $D_4[\beta]$ of a differential operator,” AIP Conf. Proc. **2781**, 20041 (2023).
https://doi.org/10.1063/5.0144834
13. I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* (Nauka, Moscow, 1971).
14. Kh. M. Shadimetov and Ik. I. Jalolov, “Optimal quadrature formulas in the space of Sobolev,” Problemy Vychislitel’noi i Prikladnoi Matematiki, No. 2, 94–102 (2016).
15. V. L. Danilov, A. N. Ivanova, E. K. Isakova, L. A. Lyusternik, G. S. Salekhov, A. N. Khovanskii, L. Ya. Tslaf, and A. R. Yanpol’skii, *Mathematical Analysis: Functions, Limits, Series, and Chain Fractions*, Ed. by L. A. Lyusternik and A. R. Yanpol’skii, Reference Mathematical Library (Fizmatgiz, Moscow, 1961).
16. Kh. M. Shadimetov, “Optimal lattice quadrature and cubature formulas in Sobolev spaces,” Doctoral Dissertation in Physics and Mathematics (Tashkent, 2002).
17. A. O. Gel’fond, *Finite Difference Calculus* (Nauka, Moscow, 1967).
18. A. R. Hayotov and S. S. Boboev, “Optimal quadrature formulas for computing of Fourier integrals in a Hilbert space,” Problemy Vychislitel’noi i Prikladnoi Matematiki, No. 4, 73–84 (2020).
19. A. R. Hayotov, S. Jeon, and C.-O. Lee, “On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$,” J. Comput. Appl. Math. **372**, 112713 (2020).
https://doi.org/10.1016/j.cam.2020.112713
20. O. I. Jalolov, “Weight optimal order of convergence cubature formulas in Sobolev space,” AIP Conf. Proc. **2365**, 20014 (2021).
https://doi.org/10.1063/5.0057015
21. O. I. Jalolov, “Weight optimal order of convergence cubature formulas in Sobolev space $\overline{L}_p^{(m)}(K_n)$,” AIP Conf. Proc. **2781**, 20066 (2023).
https://doi.org/10.1063/5.0144837

Translated by E. Oborin

Publisher’s Note. Allerton Press remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
AI tools may have been used in the translation or editing of this article.

SPELL: 1. calss