RESEARCH ARTICLE | JUNE 08 2023

Weighted optimal order of convergence cubature formulas in Sobolev space $\overline{L}_P^{(m)}(K_n)$ FREE

Ozodjon Jalolov

AIP Conference Proceedings 2781, 020066 (2023)
https://doi.org/10.1063/5.0144837

Weighted Optimal Order of Convergence Cubature Formulas in Sobolev Space $\bar{L}_P^{(m)}(K_n)$

Ozodjon Jalolov[a)]

Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan

[a)]*Corresponding author: o_jalolov@mail.ru*

Abstract. The main area of application of various spaces of generalized functions lies in the theory of differential equations and in the theory of quadrature and cubature formulas. Therefore, it becomes necessary to study spaces of generalized functions, one way or another related to various domains in R^n. The theory of differential equations in the space of generalized functions differs from the theory of these equations in the space of ordinary functions. Deriving these equations and finding their solutions are important in applications. In the study of various questions arising in the theory of approximate integration and partial differential equations and related departments of analysis, the so-called Functional approach turned out to very fruitful.

In this paper we investigate weighted cubature formula in the functional spaces $\bar{L}_p^{(m)}$ of S.L. Sobolev for the functions defined in the n - dimensional unit cube K_n and obtain an upper estimate for the norm of error functionals of weighted cubature formulas. Based on the Bakhvalov theorem it is proved that considered cubature formulas of the optimal on order of convergence in these spaces.

INTRODUCTION

In many research papers examined the properties of optimal approximations of linear functionals [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In these papers the problem of optimality with respect to a certain space are investigated. Most of them are discussed in the Sobolev space [1].

We consider the cubature formula of the form

$$\int_{K_n} p(x) f(x)\, dx \approx \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right). \tag{1}$$

The generalized functional

$$\ell_N(x) = p(x)\, \varepsilon_{K_n}(x) - \sum_{\lambda=1}^{N} C_\lambda\, \delta\left(x - x^{(\lambda)}\right) \tag{2}$$

is called the error functional of the cubature formula (1),

$$< \ell_N, f >= \int_{K_n} p(x) f(x)\, dx - \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right)$$

is an error of the cubature formula (1), $p(x) \in L_p(K_n)$ is a weight function, $\varepsilon_{K_n}(x)$ is the characteristic function of K_n, C_λ and $x^{(\lambda)}$ are the coefficients and the nodes of the cubature formula (1) and $\delta(x)$ is the Dirac delta-function.

Definition 1 *The cubature formula of the form* (1) *is called an optimal order of convergence, if for the norm of its error functional following holds*

$$\lim_{N\to\infty} \frac{\left\| \ell_N^{o,n} | \bar{L}_p^{(m)^*}(K_n) \right\|}{\left\| \ell_N^{o} | \bar{L}_p^{(m)^*}(K_n) \right\|} < \infty. \tag{3}$$

Here $\ell_N^{o,n}(x)$ is error functional of optimal order of convergence cubature formulas (1).

In this paper we consider the problem of the descending order of norm of the error functional $\left\| \ell_N | \bar{L}_P^{(m)^*}(K_n) \right\|$ with an increase in the number of its nodes. The results, which we obtain here, are to the arbitrary distribution of points.

Modern Problems of Applied Mathematics and Information Technology (MPAMIT 2021)
AIP Conf. Proc. 2781, 020066-1–020066-5; https://doi.org/10.1063/5.0144837
Published by AIP Publishing. 978-0-7354-4535-2/$30.00

A NORM ESTIMATION FOR THE ERROR FUNCTIONAL OF WEIGHTED CUBATURE FORMULAS ON THE SPACE $\overline{L}_p^{(m)}$

Definition 2 *The space $\overline{L}_p^{(m)}(K_n)$ is defined as a space of functions, given on a n- dimensional unit cube K_n and having all the generalized derivatives of order m, summable with a degree p in norm [1]:*

$$\left\| f|\overline{L}_p^{(m)}(K_n) \right\| = \left\{ \int_{K_n} \left\{ \left(\frac{\partial^m f(x)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} ... \partial x_n{}^{m_n}} \right)^p \right\} dx \right\}^{\frac{1}{p}}, \tag{4}$$

and the norms of the error functional in the Sobolev space have the following form

$$\left\| f|L_p^{(m)}(K_n) \right\| = \left\{ \left\{ \int_{K_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha f(x))^2 dx \right\}^{\frac{p}{2}} \right\}^{\frac{1}{p}}, \tag{5}$$

where $|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_n$, $\alpha! = \alpha_1! \cdot \alpha_2! \cdot ...\alpha_n!$ and $D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1,...,x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$.

Obviously, in the right hand side of (4) is less computing, than in (5), and it follows that the norm of the function in space $\bar{L}_p^{(m)}(K_n)$ the number of computing operations will be much less, than in the space $L_p^{(m)}(K_n)$, as in the norm (4), involved only the mixed derivatives.

Now we prove the following theorem, which is one of the main results of this work.

The following is true.

Lemma 1 *If for the error functional* (2) *of the cubature formula* (1)*, the following conditions are fulfilled*

$$\ell_N(x) = \ell_{N_1}(x_1) \cdot \ell_{N_2}(x_2) \cdot ... \cdot \ell_{N_n}(x_n),$$

where

$$\ell_{N_i}(x_i) = p_i(x_i)\, \varepsilon_{[0,1]}(x_i) - \sum_{\lambda_i=1}^{N_i} C_{\lambda_i} \delta\left(x_i - x_i^{(\lambda_i)}\right), p(x) = \prod_{i=1}^{n} p_i(x_i)$$

and

$$\left\| \ell_{N_i} | \bar{L}_p^{(m_i)*}(0,1) \right\| \leq d_i \frac{1}{N_i^{m_i}}, \ \ d_i \text{ are constants,} \tag{6}$$

that is

$$\left\| \ell_{N_i} | \bar{L}_p^{(m_i)*}(0,1) \right\| \leq d_i\, O\left(h_i^{m_i}\right), \quad d_i \text{ are constants, } \left(i = \overline{1,n}\right), h_i = \frac{1}{N_i} \tag{7}$$

then

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_n) \right\| \leq d \cdot \frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}}, \ d \text{ is constant,} \tag{8}$$

or

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_n) \right\| \leq d \cdot O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m_2}\right) \cdot ... \cdot O\left(h_n^{m_n}\right),$$

$$d = \prod_{i=1}^{n} d_i \text{ and } m = m_1 + m_2 + ... + m_n.$$

Proof. We conduct the method of the mathematical induction.
Suppose $n=2$, $x=(x_1,x_2)$, $|\alpha|=\alpha_1+\alpha_2$, $m=m_1+m_2$, $dx=dx_1dx_2$, $f(x)=f(x_1,x_2)$, $p(x)=p_1(x_1)\cdot p_2(x_2)$ and $\ell_N(x)=\ell_{N_1}(x_1)\cdot\ell_{N_2}(x_2)$.
If presume in (4) $n=1$, then

$$\left\|f_i|\bar{L}_p^{(m_i)}(0,1)\right\|=\left\{\int\limits_0^1\left[\left(\frac{\partial^{m_i}}{\partial x_i^{m_i}}f(x_i)\right)^p\right]dx_i\right\}^{\frac{1}{p}},\quad i=1,2,...,n. \tag{9}$$

So, we have

$$\begin{aligned}&|<\ell_N(x_1,x_2),f(x_1,x_2)>|=|<\ell_{N_2}(x_2),<\ell_{N_1}(x_1),f(x_1,x_2)>|\\&\le\left\|\ell_{N_2}(x_2)|\bar{L}_p^{(m_2)*}(0,1)\right\|\cdot\left\|<\ell_{N_1}(x_1),f(x_1,x_2)>|\bar{L}_p^{(m_2)}(0,1)\right\|.\end{aligned} \tag{10}$$

We compute the following norm:

$$\begin{aligned}\left\|<\ell_{N_1}(x_1),f(x_1,x_2)>|\bar{L}_p^{(m_2)}(0,1)\right\|&=\left\{\int\limits_0^1\left[\left|\frac{\partial^{m_2}}{\partial x_2^{m_2}}<\ell_{N_1}(x_1),f(x_1,x_2)>\right|^p\right]dx_2\right\}^{\frac{1}{p}}\\&=\left\{\int\limits_0^1\left[\left|<\ell_{N_1}(x_1),\frac{\partial^{m_2}}{\partial x_2^{m_2}}f(x_1,x_2)\right|^p\right]dx_2\right\}^{\frac{1}{p}}\\&\le\left\{\int\limits_0^1\left[\left(\left\|\ell_{N_1}(x_1)|\bar{L}_p^{(m_1)*}(0,1)\right\|\cdot\left\|\frac{\partial^{m_2}}{\partial x_2^{m_2}}f(x_1,x_2)|\bar{L}_p^{(m_1)}(0,1)\right\|\right)^p\right]dx_2\right\}^{\frac{1}{p}}\\&=\left\|\ell_{N_1}(x_1)|\bar{L}_p^{(m_1)*}(0,1)\right\|\cdot\left\{\int\limits_0^1\left\{\int\limits_0^1\left[\left(\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1}\partial x_2^{m_2}}f(x_1,x_2)\right)^p\right]dx_1\right\}dx_2\right\}^{\frac{1}{p}}\\&=\left\|\ell_{N_1}(x_1)|\bar{L}_p^{(m_1)*}(0,1)\right\|\cdot\left\|f(x)|\bar{L}_p^{(m)}(K_2)\right\|.\end{aligned} \tag{11}$$

Thus, from (10) and (11) we obtain

$$|<\ell_N(x_1,x_2),f(x_1,x_2)>|\le\left\|\ell_{N_2}(x_2)|\bar{L}_p^{(m_2)*}(0,1)\right\|\cdot\left\|\ell_{N_1}(x_1)|\bar{L}_p^{(m_1)*}(0,1)\right\|\cdot\left\|f(x)|\bar{L}_p^{(m)}(K_2)\right\|. \tag{12}$$

From (12), using the definition of norm, we have

$$\left\|\ell_N(x)|\bar{L}_p^{(m)^*}(K_2)\right\|\le\left\|\ell_{N_1}(x_1)|\bar{L}_p^{(m_1)*}(0,1)\right\|\cdot\left\|\ell_{N_2}(x_2)|\bar{L}_p^{(m_2)*}(0,1)\right\|. \tag{13}$$

Taking into account (7), on the basis of (13) we obtain

$$\left\|\ell_N|\bar{L}_p^{(m)^*}(K_2)\right\|\le d_1\cdot d_2\frac{1}{N_1^{m_1}N_2^{m_2}}.$$

That is

$$\left\|\ell_N|\bar{L}_p^{(m)*}(K_2)\right\|\le dO\left(h_1^{m_1}\right)O\left(h_2^{m_2}\right), \tag{14}$$

where $d=d_1\cdot d_2$.
Now suppose that the inequality (8) holds for $n=k$, then on the basis of the above calculations we obtain the following results

$$\left\|\ell_N|\bar{L}_p^{(m)*}(K_k)\right\|\le d\cdot\frac{1}{N_1^{m_1}\cdot N_2^{m_2}...N_k^{m_k}},\ \text{where } d=\prod_{i=1}^k d_i \tag{15}$$

or

$$\left\| \ell_N | \bar{L}_P^{(m)^*}(K_k) \right\| \leq d \cdot O\left(h_1^{m_1}\right) ... O\left(h_k^{m_k}\right). \tag{16}$$

Using the validity of lemma at $|< \ell_N(x_1, x_2), f(x_1, x_2) >| \leq \left\| \ell_{N_2}(x_2) | \bar{L}_2^{(m_2)^*}(0,1) \right\|$, we prove that the assertion holds for $n = k+1$. Taking into account (15), at $n = k+1$ we estimate error of cubature formula of the form (1)

$$\begin{aligned} &\left|< \ell_{N_{k+1}}(x_1, x_2, ..., x_{k+1}), f(x_1, x_2, ..., x_{k+1}) >\right| \\ &= \left|< \ell_{N_1}(x_1), < \ell_{N_2}(x_2), <, ... < \ell_{N_k}(x_k), \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > ... >\right| \\ &\leq \left\| \ell_{N_1} | \bar{L}_P^{(m_1)^*}(0,1) \right\| ... \left\| \ell_{N_k} | \bar{L}_P^{(m_k)^*}(0,1) \right\| . \left\| < \ell_{N_{k+1}}(x_{k+1}), f(x_1, ... x_{k+1}) > | \bar{L}_P^{(m_{k+1})^*}(0,1) \right\| \end{aligned} \tag{17}$$

Hence, as above, using the definition of norms of functional, we get

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_{k+1}) \right\| \leq \left\| \ell_{N_1}(x_1) | \bar{L}_p^{(m_1)*}(0,1) \right\| \cdot ... \cdot \left\| \ell_{N_k}(x_k) | \bar{L}_p^{(m_k)*}(0,1) \right\| \cdot \left\| \ell_{N_{k+1}}(x_{k+1}) | \bar{L}_p^{(m_{k+1})*}(0,1) \right\|. \tag{18}$$

From inequalities (6) and (18) we obtain

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_{k+1}) \right\| \leq d_{k+1} \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_{k+1}^{m_{k+1}}}, \tag{19}$$

or

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_{k+1}) \right\| \leq d_{k+1} \cdot O\left(h_1^{m_1}\right) ... O\left(h_{k+1}^{m_{k+1}}\right), \text{ where } d_{k+1} = \prod_{i=1}^{k+1} d_i.$$

Summarizing the results obtained, in conclude by noting that

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_n) \right\| \leq d \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_n^{m_n}}, h_i = \frac{1}{N_i}, \left(i = \overline{1,n}\right), \text{ where } d = \prod_{i=1}^{n} d_i \tag{20}$$

or taking into account (7), from (19) we have

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_n) \right\| \leq d \cdot O\left(h_1^{m_1}\right) ... O\left(h_1^{m_n}\right), d \text{ is constant} \tag{21}$$

or

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_n) \right\| \leq d \cdot O(h^m) \text{ , where } d = \prod_{i=1}^{n} d_i. \tag{22}$$

Lemma is proved.

Thus, we obtain an upper estimate for the norm of the error functional (2) for cubature formula (1) in the space $\bar{L}_p^{(m)*}(K_n)$.

A similar assessment was obtained previously for the norm of the error functional of cubature formula (1) on the quotient space of S.L. Sobolev $L_p^{(m)}(K_n)$ [11] and as a result we have received the same order of convergence to zero as $N \to \infty$, although the norm of function was defined in different ways, this is confirmed by the inequality [11], (22). With the help of this lemma, it is easy to prove the following theorem.

Theorem 1 *The weight cubature formula* (1) *with the error functional* (2) *at* $N_1 = N_2 = ... = N_n$, $\prod_{i=1}^{n} N_i = N$ *and* $m_1 + m_2 + ... + m_n = m$ *is the optimal in order of convergence the space* $\bar{L}_p^{(m)}(K_n)$, *for the norms of the error functionals* (2) *of the cubature formula* (1) *have the equality*

$$\left\| \ell_N | L_p^{(m)*}(K_n) \right\| = O\left(N^{-\frac{m}{n}}\right).$$

Proof. On the basis of Lemma at $N_1 = N_2 = ... = N_n$ we have $N_i = \sqrt[n]{N}$, $i = 1,2,...,n$.
Thus,

$$\prod_{i=1}^{n} N_i^{m_i} = N_1^{m_1+m_2+...+m_n} = N^{\frac{m}{n}}. \tag{23}$$

By substituting (23) into inequality (20), we obtain

$$\left\| \ell_N | \bar{L}_p^{(m)*}(K_n) \right\| \leq C \cdot N^{-\frac{m}{n}}. \tag{24}$$

By N.S. Bakhvalov theorem [14] and the inequality (24) follows the proof of the theorem.

CONCLUSION

In this paper we investigate weighted cubature formula in function space $\bar{L}_p^{(m)}$ of S.L. Sobolev for the functions defined in the n - dimensional unit cube K_n and obtain an upper estimate for the norm of the error functional of weighted cubature formulas. Based on the Bahvalov N.S theorem it is proved that considered cubature formulas are an optimal on order of convergence in these spaces. At the end, it is argued that the weighted cubature formulas in spaces have the same order of convergence, although for the norm of the error functional of these cubature formulas in the space only mixed generalized derivatives are involved and these formulas can be called practical.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the Institute of Mathematics named V.I. Romanovskiy for making convenient research facilities. We much appreciate the reviewers for their thoughtful comments and efforts toward improving our manuscript.

REFERENCES

1. S. L. Sobolev, *Introduction to the theory of cubature formulas (in Russian).* (Nauka, Moscow., 1974).
2. K. Shadimetov, "Construction of weight optimal quadrature formulas in $l_2^{(m)}(0,1)$," Sibirian journal of Computational Matematics **5**, 275–293 (2002).
3. K. M. Shadimetov, *Layyice quadrature and cubature formulas in the Sobolev space. Dissertation of Doctor of Sciences*, DSc dissertation, Tashkent (2002), a full DScTHESIS entry.
4. F. Lanzara, "On optimal quadrature formulae," of Inequality Appl **5**, 201–225 (2000).
5. Kh.M.Shadimetov and A.R.Hayotov, "Optimal quadrature formulas with positive coefficients in $l_2^{(m)}(0,1)$," Journal of Computational and Applied Mathematics **235**, 1114–1128 (2011).
6. A. R. Hayotov, G. Milovanovic, and K. M. Shadimetov, "On an optimal quadrature formula in the sense of sard," Numerical Algorithms **57**, 487–510 (2011).
7. F. A. Nuraliev., "Minimisation the error functionals norm of the hermitian type quadrature formula," Uzbek Matematical journal **1**, 53–64 (2015).
8. Kh.M.Shadimetov and B.S.Daliev, "The extremal function of quadrature formulas for the approximate solution of the generalized abel integral eduation." Problems of computational and applied matematics **2**, 88–96 (2019).
9. A. R. Hayotov and S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in a Hilbert space." Problems of computational and applied mathematics **4(28)**, 73–84 (2020).
10. A. R. Hayotov and S. S. Babaev, "Optimal quadrature formulas for computing of Fourier integrals in $W_2^{(m,m-1)}$ space," AIP Conference Proceedings **2365**, 020021 (2021), https://doi.org/10.1063/5.0057127.
11. I. I. Jalolov, "The algorithm for constructing a differentialoperator of 2nd order and finding afundamental solution," AIP Conference Proceedings **2365**, 020015 (2021), https://doi.org/10.1063/5.0057025.
12. O.I.Jalolov, "Weight optimal order of convergence cubature formulas in sobolev space," AIP Conference Proceedings **2365**, 020014 (2021), https://doi.org/10.1063/5.0057015.
13. S. S. Babaev and A. R. Hayotov, "Optimal interpolation formulas in the space $w_2^{(m,m-1)}$," Calcolo **56**, 25 (2019).
14. N. S. Bahvalov, *Numerical methods (in Russian).* (Moscow,Nauka, 1973) p. 631.