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On the Construction of Practically Asymptotically Optimal Weighted Cubature Formulas of Hermite Type in the Sobolev Space $\bar{L}_2^{(m)}(S_n)$

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Abstract—When solving many problems in the theory of approximate integration and differential equations, it is the correct choice of spaces that is the key to success. A very clearly chosen approach was demonstrated in the famous works of Sobolev on the polyharmonic equation. Sobolev posed and solved by the variational method the first boundary value problem for the equation $\Delta^{\ell} u = f$ with boundary conditions on surfaces of various dimensions. Problems of optimization of approximate integration formulas consist in minimizing the norm of the error functional of the formula on selected normalized spaces, and most of them are considered in the Sobolev space. Until now, we have considered cubature formulas, with the help of which a definite integral of a function is approximately calculated when the values of this function at individual points of the nodes of the cubature formula are unknown. But more general cubature formulas are possible, which include both the values of the function and the values of its derivatives of one order or another. If we know not only the values of a function at some points of the n-dimensional unit sphere, but also the values of its derivatives of one order or another, then it is natural that if all these data are used correctly, we can expect a more accurate result than if we use only function values. This paper examines cubature formulas, which require special attention to the construction of the most economical formulas; according to Bakhvalov, such formulas are called practical.

Keywords: generalized function, space, norm, error functional, interpolation formula, extremal function

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INTRODUCTION

The modern formulation of the problem of optimizing approximate integration formulas consists of minimizing the norm of the error functional formula on selected normed spaces (see, e.g., [1–22]).

These works investigate the problem of optimality with respect to some specific space. Many of them are considered in the Sobolev space [1].

The multidimensional cubature formulas differ from the one-dimensional ones in two ways:

(1) The shapes of multidimensional integration domains are infinitely diverse.

(2) The number of integration nodes increases rapidly with increasing space dimension.

Problem (2) requires special attention to constructing the most economical formulas.

In this paper, we consider the formulas taking into account precisely this requirement. As is known [23], Bakhvalov called such formulas practical.

The development of algorithms and construction of cubature formulas for some polyhedra using the proposed method were performed by Voytishek and Blinov [6].

Shadimetov and Mamatova [24] constructed composite lattice optimal cubature formulas in Sobolev space using the variational method. In work [25], they calculated an exact upper bound for the error of the interpolation formula in the Sobolev space. They proved the existence and uniqueness of the optimal interpolation formula that gives the smallest error and presented an algorithm for finding the coefficients of the optimal interpolation formula. By implementing this algorithm, they determined the optimal coefficients.

Let functions $f(\theta)$, defined on the unit sphere S, belong to some Banach space B embedded in the space $C(S)$ of continuous functions on S.

When integrating functions over a sphere, instead of the problem of calculating the integral of a given function, it is convenient to consider the problem of calculating the integral of a function with an average value over the sphere equal to zero.

Because any summable function on a sphere can be represented as the sum of a constant function equal to its mean value over the sphere and a function with a mean value equal to zero, the two problems indicated above are equivalent, for the cubature formulas we are considering integrate a constant exactly.

If some function $f(\theta)$ is defined on the surface of the unit sphere S, then we consider it to be extended to the entire space R^n (except for zero and infinity) and to the spherical layer $S_\delta = \{1-\delta \le r \ge \delta\}$ so that it remains constant on the rays diverging from the center of the sphere.

We denote the continued function by $\overline{f}(x)$ and introduce the notation

$$D^\alpha f(\theta) = \frac{\partial^{|\alpha|}\overline{f}(x)}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}, \tag{1}$$

where the derivatives are taken with respect to the Cartesian coordinates of the point $\theta < S$ in $|\alpha| = \alpha_1 + \dots + \alpha_n$.

By $L_2^{(m)}(S)$ we denote the space of functions that are defined on S and continued as just indicated and for which the norm is defined by the equality

$$\left\| f(\theta) | L_2^{(m)}(S) \right\| = \left\{ \int_{S_\delta} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(D^\alpha f(\theta) \right)^2 d\theta \right\}^{\frac{1}{2}}. \tag{2}$$

1. PROBLEMS FORMULATION

In this paper, we consider formulas called practical-asymptotically optimal weighted cubature formulas of the Hermite type, which, with respect to the norm of the function, are the most economical formulas for use in the Sobolev space.

The works of many authors (see, e.g., [9–11]) are devoted to quadrature and cubature formulas, which include the values of the derivatives of integrable functions. If not only the values of a function $f(x)$ are known at the nodes of cubature formulas, but also the values of its derivatives of some orders, then one can expect a more accurate result than in the case of using only the values of the functions.

Let us consider the error of the Hermite-type weighted cubature formula

$$\int_S p(\theta) f(\theta)\, d\theta \approx \sum_{|\alpha|\le t} \sum_{\lambda=1}^{N} C_\lambda^{(\alpha)} f^{(\alpha)}(\theta^{(\lambda)}) \tag{3}$$

on functions from B:

$$\ell_N^{(\alpha)}[f] = \langle \ell_N^{(\alpha)}, f \rangle = \int_S p(\theta) f(\theta)\, d\theta - \sum_{|\alpha|\le t} \sum_{\lambda=1}^{N} C_\lambda^{(\alpha)} f^{(\alpha)}(\theta^{(\lambda)}) = \int_{R^n} \ell_N^{(\alpha)}(x) f(x)\, dx, \tag{4}$$

$$\ell_N^{(\alpha)}(x) = \delta_S(1-r)\, p(x) - \sum_{|\alpha|\le t} \sum_{\lambda=1}^{N} C_\lambda^{(\alpha)} \delta^{(\alpha)}(x - \theta^{(\lambda)}), \tag{5}$$

$\delta_S(1-r)$ and $\delta\left(x-\theta^{(\lambda)}\right)$ are Dirac delta functions, $C_\lambda^{(\alpha)}$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ are the coefficients and nodes of the cubature formula (1), $\|\theta\| = 1$, $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, $\sum_{\lambda=1}^{N} C_\lambda = \frac{2\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \hat{p}_{0,0}$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $p(\theta) \in L(S)$, $0 \le t \le m$, and $\hat{p}_{0,0}$ is the zero Fourier coefficient $p(\theta)$.

Error (4) of the cubature formula (3) is clearly a functional defined on B by virtue of the nesting assumption $B \to C(S)$ [1], this functional $\ell_N^{(\alpha)}$ (5) is continuous. Therefore, it and its norm are determined by the formula [2]

$$\left\|\ell_N^{(\alpha)} \mid B^*\right\| = \sup_{f \in B, f \neq 0} \frac{\left|< \ell_N^{(\alpha)}, f >\right|}{\|f \mid B\|}.$$

A function $f_0 \in B$, for which the equality

$$\left|< \ell_N^{(\alpha)}, f >\right| = \left\|\ell_N^{(\alpha)} \mid B^*\right\| \cdot \|f_0 \mid B\|,$$

holds is called *extremal*.

Thus, the problem of estimating the error of a weighted cubature formula of the Hermite type on functions from a certain space B is equivalent to calculating the value of the norm of the error functional in the space B^* conjugate to B or, what is the same, finding the extremal function for a given weighted cubature formula of the Hermite type. To solve this problem, for B we take the space $L_2^m(S)$.

Definition 1. The space $L_2^m(S)$ is defined as the space of functions defined on S and having square summable generalized derivatives of order m, the norm of which is determined by the equality

$$\left\|f \mid L_2^m(S)\right\|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m (k+n-2)^m, \tag{6}$$

where $a_{k,\ell} = \int_S f(\theta) Y_{k,\ell}(\theta) d\theta$. Here, $Y_{k,\ell}(\theta)$ is the spherical harmonic of degree k of order ℓ; the index ℓ is obtained by numbering spherical functions of the same degree k and varies within the limits $1 \le \ell \le \sigma(n,k)$, $\sigma(n,k) = (2k+n-2)\frac{(k+n-3)!}{(k-2)!\,k!}$ is the number of linearly independent spherical harmonics of degree k; we assume that $2m > n$.

In work [3], Salikhov proved that if for the function $f(\theta)$ norm (6) is finite, then norm (2) also is finite for it, and vice versa.

The following theorem is true:

Theorem 1. *The norm of the error functional* $\ell_N^{(\alpha)}$ *of the weighted cubature formula of the Hermite type* (3) *over the space* $L_2^{m*}(S)$ *is*

$$\left\|\ell_N^{(\alpha)} \mid L_2^{m*}(S)\right\| = \left\{ \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\hat{p}_{k,\ell} - \sum_{|\alpha| \le t} \sum_{\lambda=1}^{N} C_\lambda^{(\alpha)} Y_{k,\ell}^{(\alpha)}(\theta)\right]^2}{k^m (k+n-2)^m} \right\}^{1/2}.$$

2. METHOD FOR SOLVING THE PROBLEMS

Let the error functionals of one-dimensional weighted quadrature formulas of the Hermite type be given by

$$\ell_{N_i}^{(\alpha_i)}(\theta_i) = p(\theta_i)\,\delta_{\omega_i}(1-r) - \sum_{\alpha_i=1}^{m}\sum_{\lambda_i=1}^{N_i} C_{\lambda_i}^{(\alpha_i)}\delta^{(\alpha_i)}\left(\theta_i - \theta_i^{(\lambda_i)}\right) \quad \left(i = \overline{1,n}\right),$$

and, in the one-dimensional case, for $f_i \in L_2^{(m)}(\omega_i)$ the norm is defined as

$$\left\| f_i \,|\, L_2^{(m)}(\omega_i)\right\| = \left\{\int_{\omega_i}\left(\frac{d^m}{d\theta_i^m} f(\theta_i)\right)^2 d\theta_i\right\}^{\frac{1}{2}},$$

where

$$\omega_i = \begin{cases}[0,2\pi], & i = n;\\ [0,\pi], & i = \overline{1,n-1}.\end{cases}$$

Because the n-dimensional unit sphere is denoted by S_n, the weighted cubature formula of the Hermite type (3) transforms to

$$\int_{S_n} p(\theta)\, f(\theta) d\theta \approx \sum_{|\alpha|\le t}\sum_{\lambda=1}^{N} (-1)^{\alpha}\, C_{\lambda}^{(\alpha)} f^{(\alpha)}\left(\theta^{(\lambda)}\right) \tag{7}$$

over the space $L_2^m(S)$, where S_n is the n-dimensional unit sphere, $\theta = (\theta_1,\theta_2,\ldots,\theta_n)$, $\|\theta\| = 1$ and $\sum_{\lambda=1}^{N} C_\lambda = \frac{2\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)}\hat{p}_{0,0}$, $\delta^{(\alpha)}(\theta)$ is the Dirac delta function, $C_\lambda^{(\alpha)}$ and $\theta^{(\lambda)}$ are the coefficients and nodes of the weighted cubature formula of the Hermite type (7), $\hat{p}_{0,0}$ is the zero Fourier coefficient $p(\theta)$.

The weighted cubature formula of the Hermite type (7) is comparable with the generalized function

$$\ell_N^{(\alpha)}(\theta) = \varepsilon_{S_n}(\theta) p(\theta) - \sum_{|\alpha|\le t}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}\delta^{(\alpha)}(\theta - \theta^{(\lambda)}) \tag{8}$$

and we call it the error functional, where $\delta^{(\alpha)}(\theta)$ is the Dirac delta function, $C_\lambda^{(\alpha)}$ and $\theta^{(\lambda)}$ are the coefficients and nodes of the weighted cubature formula of the Hermite type (7).

Salikhov showed [3] that the space $L_2^m(S)$, in terms of the composition of its elements, coincides with a similar Sobolev space $L_2^{(m)}(S)$ with the norm

$$\left\| f \,|\, L_2^{(m)}(S_n)\right\| = \left\{\int_{S_n}\sum_{|\alpha|=m}\frac{m!}{\alpha!}\left(D^{\alpha} f(\theta)\right)^2 d\theta\right\}^{\frac{1}{2}}, \tag{9}$$

where S_n is the n-dimensional unit sphere, $|\alpha| = \alpha_1 + \alpha_2 + \ldots \alpha_n$, $\alpha! = \alpha_1!\cdot\alpha_2!\cdot\ldots\alpha_n!$, and

$$D^{|\alpha|} f(\theta) = \frac{\partial^{|\alpha|} f(\theta_1,\ldots,\theta_n)}{\partial\theta_1^{\alpha_1}\partial\theta_2^{\alpha_2}\ldots\partial\theta_n^{\alpha_n}}.$$

This means that norms (6) and (9) are equivalent normalizations.

Let us define the norm in $\overline{L}_2^{(m)}(S_n)$.

Definition 2. The space $\bar{L}_2^{(m)}(S_n)$ is defined as the space of functions defined on S_n, and the norm of functions defined by the equality

$$\left\| f(\theta) \,|\, \bar{L}_2^{(m)}(S_n) \right\| = \left\{ \int\limits_{S_n} \left(\frac{\partial^m f(\theta)}{\partial\theta_1^{m_1}\partial\theta_2^{m_2}\ldots\partial\theta_n^{m_n}} \right)^2 d\theta \right\}^{\frac{1}{2}}, \tag{10}$$

where $m_1 + m_2 + \ldots + m_n = m$, $m_i > 0$, $i = \overline{1,n}$, with the scalar product

$$(f(\theta), \varphi(\theta))_{\bar{L}_2^{(m)}(S_n)} = \int\limits_{S_n} \left(\frac{\partial^m f(\theta)}{\partial\theta^m} \right)\left(\frac{\partial^m \varphi(\theta)}{\partial\theta^m} \right) d\theta,$$

where $\partial\theta^m = \partial\theta_1^{m_1}\partial\theta_2^{m_2}\ldots\partial\theta_n^{m_n}$, $m = m_1 + m_2 + \ldots m_n$, $d\theta = d\theta_1 d\theta_2 \ldots d\theta_n$.

In (9) we put $n = 2$ and $m = 2$; hence,

$$\int\limits_{S_\alpha} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\frac{\partial^m f(\theta)}{\partial\theta^m} \right)^2 d\theta = \int\limits_{S_2} \sum_{\alpha_1+\alpha_2=2} \frac{2!}{\alpha_1!\alpha_2!} \left(\frac{\partial^2 f(\theta)}{\partial\theta_1^{\alpha_1}\partial\theta_2^{\alpha_2}} \right)^2 d\theta$$

$$= \int\limits_{S_2} \left[\left(\frac{\partial^2 f(\theta)}{\partial\theta_2^2} \right)^2 + \frac{2!}{1!\cdot 1!} \left(\frac{\partial^2 f(\theta)}{\partial\theta_1\partial\theta_2} \right)^2 + \left(\frac{\partial^2 f(\theta)}{\partial\theta_1^2} \right)^2 \right] d\theta. \tag{11}$$

At $n = 2$ and $m = 2$ equality (10) takes the form

$$\left\| f(\theta) \,|\, \bar{L}_2^{(2)}(S_2) \right\|^2 = \int\limits_{S_2} \left(\frac{\partial^2 f(\theta)}{\partial\theta_1^{m_1}\partial\theta_2^{m_2}} \right)^2 d\theta. \tag{12}$$

It is clear that for the right-hand side of (12) it is necessary to carry out fewer calculations than for (11), and, therefore, for the norm of a function in the space $\bar{L}_2^{(2)}(S_2)$ the number of computational operations is much less than in the space $L_2^{(2)}(S_2)$, because only mixed derivatives participate in norm (12).

Let us prove the following theorem, which is the main result.

Theorem 2. *If the following condition is met for the error functional* (8) *of the weighted cubature formula of the Hermite type* (7) *in the space* $\bar{L}_2^{(m)}(S_n)$:

$$\ell_N^{(\alpha)}(\theta) = \ell_{N_1}^{(\alpha_1)}(\theta_1) \cdot \ell_{N_2}^{(\alpha_2)}(\theta_2) \cdot \ldots \cdot \ell_{N_n}^{(\alpha_n)}(\theta_n)$$

and

$$\left\| \ell_{N_i}^{(\alpha_i)} \,|\, \bar{L}_2^{(m_i)*}(\omega_i) \right\| \le c_i \frac{1}{N_i^{m_i}} \quad \left(i = \overline{1,n}\right), \quad c_i \text{ are constants}, \tag{13}$$

that is,

$$\left\| \ell_{N_i}^{(\alpha_i)} \,|\, \bar{L}_2^{(m_i)*}(\omega_i) \right\| \le c_i O\left(h_i^{m_i}\right) \quad \left(i = \overline{1,n}\right), \quad h_i = \frac{1}{N_i}, \tag{14}$$

then

$$\left\| \ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_n) \right\| \le c \frac{1}{\prod_{i=1}^{n} N_i^{m_i}}, \quad c \text{ are constants}, \tag{15}$$

or

$$\left\| \ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_n) \right\| \le c O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m_2}\right) \cdot \ldots \cdot O\left(h_n^{m_n}\right), \tag{16}$$

where

$$\ell_{N_i}^{(\alpha_i)}(\theta) = \delta_{\omega_i}(1-r) - \sum_{|\alpha|\le t}\sum_{\lambda_i=1}^{N_i} C_{\lambda_i}^{(\alpha_i)}\delta^{(\alpha_i)}\left(\theta_i - \theta_i^{(\lambda_i)}\right),$$

$$c = \prod_{i=1}^{n} c_i, \quad m = m_1 + m_2 + \ldots + m_n, \quad m_i \ge 1, \quad i = \overline{1,n}.$$

and

$$\omega_i = \begin{cases} [0, 2\pi], & i = n; \\ [0, \pi], & i = \overline{1, n-1} \end{cases}.$$

Proof. We use the method of mathematical induction.

Let $n = 2$, then $\theta = (\theta_1, \theta_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $m = m_1 + m_2$, $d\theta = d\theta_1 d\theta_2$, $f(\theta) = f(\theta_1, \theta_2)$, $p(\theta) = p(\theta_1, \theta_2)$, and $\ell_N(\theta) = \ell_{N_1}(\theta_1) \otimes \ell_{N_2}(\theta_2)$.

If we assume that $n = 1$ in (3), then

$$\left\| f(\theta_i) \,|\, \overline{L}_2^{(m_i)}(\omega_i) \right\| = \left\{ \int_{\omega_i} \left(\frac{\partial^{m_i} f(\theta_i)}{\partial \theta_i^{m_i}} \right)^2 d\theta_i \right\}^{\frac{1}{2}} \quad \left(i = \overline{1,n} \right).$$

Thus, we have

$$\begin{aligned} \left| < \ell_N^{(\alpha)}(\theta_1, \theta_2), f(\theta_1, \theta_2) > \right| &= \left| < \ell_{N_2}^{(\alpha_2)}(\theta_2), < \ell_{N_1}^{(\alpha_1)}(\theta_1), f(\theta_1, \theta_2) > \right| \\ &\le \left\| \ell_{N_2}^{(\alpha_2)}(\theta_2) \,|\, \overline{L}_2^{(m_2)*}(\omega_i) \right\| \cdot \left\| < \ell_{N_1}^{(\alpha_1)}(\theta_1), f(\theta_1, \theta_2) > |\, \overline{L}_2^{(m_2)*}(\omega_i) \right\|. \end{aligned} \tag{17}$$

Let us calculate the following norm:

$$\left\| < \ell_{N_1}^{(\alpha_1)}(\theta_1), f(\theta_1, \theta_2) > |\, \overline{L}_2^{(m_2)*}(\omega_i) \right\| = \left\{ \int_{\omega_i} \left| \frac{d^{m_2}}{d\theta_2^{m_2}} < \ell_{N_1}^{(\alpha_1)}(\theta_1), f(\theta_1, \theta_2) > \right|^2 d\theta_2 \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_{\omega_i} \left| < \ell_{N_1}^{(\alpha_1)}(\theta_1), \frac{d^{m_2}}{d\theta_2^{m_2}} f(\theta_1, \theta_2) > \right|^2 d\theta_2 \right\}^{\frac{1}{2}}$$

$$\le \left\{ \int_{\omega_i} \left[\left\| \ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \overline{L}_2^{(m_1)*}(\omega_i) \right\| \cdot \left\| \frac{d^{m_2}}{d\theta_2^{m_2}} f(\theta_1, \theta_2) \,|\, \overline{L}_2^{(m_1)*}(\omega_i) \right\| \right]^2 d\theta_2 \right\}^{\frac{1}{2}}$$

$$= \left\| \ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \overline{L}_2^{(m_1)*}(\omega_i) \right\| \cdot \left\{ \int_{\omega_i} \left\{ \int_{\omega_i} \left[\frac{\partial^{m_1+m_2}}{\partial \theta_1^{m_1} \partial \theta_2^{m_2}} f(\theta_1, \theta_2) \right]^2 d\theta_1 \right\} d\theta_2 \right\}^{\frac{1}{2}}$$

$$= \left\| \ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \overline{L}_2^{(m_1)*}(\omega_i) \right\| \cdot \left\{ \int_{\omega_i} \int_{\omega_i} \left[\frac{\partial^{m_1+m_2}}{\partial \theta_1^{m_1} \partial \theta_2^{m_2}} f(\theta) \right]^2 d\theta \right\}^{\frac{1}{2}}$$

$$= \left\| \ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \overline{L}_2^{(m_1)*}(\omega_i) \right\| \cdot \left\| f(\theta) \,|\, \overline{L}_2^{(m)}(S_2) \right\|, \tag{18}$$

where $\theta = (\theta_1, \theta_2)$ and $m = m_1 + m_2$. Thus, from (17) and (18) we get

$$\begin{aligned} &\left| < \ell_N^{(\alpha)}(\theta_1, \theta_2), f(\theta_1, \theta_2) > \right| \\ &\le \left\| \ell_{N_2}^{(\alpha_2)}(\theta_2) \,|\, \overline{L}_2^{(m_2)*}(\omega_i) \right\| \cdot \left\| \ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \overline{L}_2^{(m_1)*}(\omega_i) \right\| \cdot \left\| f(\theta) \,|\, \overline{L}_2^{(m)}(S_2) \right\|. \end{aligned} \tag{19}$$

Taking into account (10), from (19) we obtain

$$\left\|\ell_N^{(\alpha)}(\theta) \,|\, \bar{L}_2^{(m)*}(\omega_i)\right\| \le \left\|\ell_{N_2}^{(\alpha_2)}(\theta_2) \,|\, \bar{L}_2^{(m_2)*}(\omega_i)\right\| \cdot \left\|\ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \bar{L}_2^{(m_1)*}(\omega_i)\right\|. \tag{20}$$

Due to (13), from (20) we have

$$\left\|\ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_2)\right\| \le c_1 \cdot c_2 \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2}}.$$

According to (14),

$$\left\|\ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_2)\right\| \le c \cdot O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m_2}\right), \quad \text{where} \quad c = \prod_{i=1}^{2} c_i.$$

At $n = k$

$$\left|<\ell_N^{(\alpha)}(\theta), f(\theta)>\right| = \left|<\ell_N^{(\alpha)}(\theta_1, \theta_2, \ldots, \theta_k), f(\theta_1, \theta_2, \ldots, \theta_k)>\right|$$

$$= \left|<\ell_{N_k}^{(\alpha_k)}(\theta_k), <\ell_{N_{k-1}}^{(\alpha_{k-1})}(\theta_{k-1}), \ldots, <\ell_{N_2}^{(\alpha_2)}(\theta_2), <\ell_{N_1}^{(\alpha_1)}(\theta_1), f(\theta_1, \theta_2, \ldots, \theta_k) > \ldots >\right|$$

$$\le \left\|\ell_{N_k}^{(\alpha_k)}(\theta_k) \,|\, \bar{L}_2^{(m_k)*}(\omega_i)\right\| \cdot \left\|\ell_{N_{k-1}}^{(\alpha_{k-1})}(\theta_{k-1}) \,|\, \bar{L}_2^{(m_{k-1})*}(\omega_i)\right\| \ldots$$

$$\ldots \left\|<\ell_{N_1}^{(\alpha_1)}(\theta_1), f(\theta_1, \theta_2, \ldots, \theta_k)> \,|\, \bar{L}_2^{(m_1)*}(\omega_i)\right\|$$

$$\le \left\|\ell_{N_k}^{(\alpha_k)}(\theta_k) \,|\, \bar{L}_2^{(m_k)*}(\omega_i)\right\| \ldots \left\|\ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \bar{L}_2^{(m_1)*}(\omega_i)\right\| \cdot \left\|f(\theta) \,|\, \bar{L}_2^{(m)}(\omega_i)\right\|, \tag{21}$$

$$\omega_i = \begin{cases} [0, 2\pi], & \text{if} \quad i = k; \\ [0, \pi], & \text{if} \quad i = \overline{1, k-1}. \end{cases}$$

Due to (10), from (21) we have

$$\left\|\ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_k)\right\| \le \left\|\ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \bar{L}_2^{(m_1)*}(\omega_i)\right\| \ldots \left\|\ell_{N_k}^{(\alpha_k)}(\theta_k) \,|\, \bar{L}_2^{(m_k)*}(\omega_i)\right\|. \tag{22}$$

Then, due to (13), from (22) we get

$$\left\|\ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_k)\right\| \le \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_k^{m_k}} \tag{23}$$

or, taking into account (14), from (23) we have

$$\left\|\ell_N^{(\alpha)} \,|\, \bar{L}_2^{(m)*}(S_k)\right\| \le c \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_k^{m_k}\right), \quad \text{where} \quad c = \prod_{i=1}^{k} c_i.$$

Using the validity of the statement of the theorem for $n = k$, we are going to prove that the statement is true when $n = k + 1$.

So let $n = k + 1$, then according to (10), from (21) we have

$$\left|<\ell_{N_{k+1}}^{(\alpha_{k+1})}(\theta_1, \theta_2, \ldots, \theta_{k+1}), f(\theta_1, \theta_2, \ldots, \theta_{k+1})>\right|$$

$$\left|<\ell_{N_1}^{(\alpha_1)}(\theta_1), <\ell_{N_2}^{(\alpha_2)}(\theta_2), < \ldots < \ell_{N_k}^{(\alpha_k)}(\theta_k), <\ell_{N_{k+1}}^{(\alpha_{k+1})}(\theta_{k+1}), f(\theta_1, \theta_2, \ldots, \theta_{k+1}) > \ldots >\right|$$

$$\le \left\|\ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \bar{L}_2^{(m_1)*}(\omega_i)\right\| \ldots \left\|\ell_{N_k}^{(\alpha_k)}(\theta_k) \,|\, \bar{L}_2^{(m_k)*}(\omega_i)\right\|$$

$$\times \left\|<\ell_{N_{k+1}}^{(\alpha_{k+1})}(\theta_{k+1}), f(\theta_1, \theta_2, \ldots, \theta_{k+1})> \,|\, \bar{L}_2^{(m_{k+1})}(\omega_i)\right\|. \tag{24}$$

Taking into account (10) and (22), from (24) we obtain

$$\left\|\ell_{N_{k+1}}^{(\alpha_{k+1})} \,|\, \bar{L}_2^{(m_{k+1})*}(S_{k+1})\right\| \le \left\|\ell_{N_1}^{(\alpha_1)}(\theta_1) \,|\, \bar{L}_2^{(m_1)*}(\omega_i)\right\| \ldots$$

$$\ldots \left\|\ell_{N_k}^{(\alpha_k)}(\theta_k) \,|\, \bar{L}_2^{(m_k)*}(\omega_i)\right\| \cdot \left\|\ell_{N_{k+1}}^{(\alpha_{k+1})}(\theta_{k+1}) \,|\, \bar{L}_2^{(m_{k+1})*}(\omega_i)\right\|, \tag{25}$$

where

$$\omega_i = \begin{cases} [0, 2\pi], & \text{if} \quad i = k+1; \\ [0, \pi], & \text{if} \quad i = \overline{1, k}. \end{cases}$$

Using (13), from (25) we have

$$\left\| \ell_{N_{k+1}}^{(\alpha_{k+1})} \,|\, \overline{L}_2^{(m_{k+1})^*}(S_{k+1}) \right\| \le c \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_{k+1}^{m_{k+1}}} \tag{26}$$

or, in view of (14) and (26), we get

$$\left\| \ell_{N_{k+1}}^{(\alpha_{k+1})} \,|\, \overline{L}_2^{(m_{k+1})^*}(S_{k+1}) \right\| \le c \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_{k+1}^{m_{k+1}}\right), \quad \text{where} \quad c = \prod_{i=1}^{k+1} c_i.$$

Thus, we have obtained inequalities (15) and (16).

Next,

$$\left\| \ell_N^{(\alpha)} \,|\, \overline{L}_2^{(m)^*}(S_n) \right\| \le c \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_n^{m_n}}, \qquad c \text{ const}, \tag{27}$$

or, taking into account (16), from (27) we have

$$\left\| \ell_N^{(\alpha)} \,|\, \overline{L}_2^{(m)^*}(S_n) \right\| \le c \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_n^{m_n}\right), \quad h_i = \frac{1}{N_i} \quad (i = \overline{1, n}), \tag{28}$$

where $c = \prod_{i=1}^{n} c_i$.

If in (27) or (28) we assume $N = N_1 \cdot N_2 \cdot \ldots \cdot N_n$, $N_1 = N_2 = \ldots = N_n$ and $m_1 + m_2 + \ldots + m_n = m$, then we get

$$\left\| \ell_N^{(\alpha)} \,|\, L_2^{(m)^*}(S_n) \right\| \le c \cdot N^{-\frac{m}{n}}$$

or

$$\left\| \ell_N^{(\alpha)} \,|\, \overline{L}_2^{(m)^*}(S_n) \right\| \le c \cdot O\left(h^m\right), \quad c \text{ const} \quad \left(h = N^{-\frac{1}{n}} \right),$$

which completes the proof.

Thus, we have obtained an upper bound for the norm of the error functional (8) of the Hermite-type weighted cubature formula (7) in the space $\overline{L}_2^{(m)^*}(S_n)$. The same estimate was previously obtained for the norm of the error functional of the weighted cubature formula of the Hermite type (7) over the quotient Sobolev space $L_2^{(m)}(S_n)$ [9]. As a result, we obtain the same order of convergence to zero for $N \to \infty$, although the function norms are defined in different ways; this is confirmed by inequality (27) and the results of work [23]. Thus, the weighted cubature formulas of the Hermite type (7) over the quotient Sobolev space $L_2^{(m)}(S_n)$ with the error functional (8) are asymptotically optimal. In the words of Bakhvalov, they are practical [23].

To illustrate, we take a specific example: $n = 2$.

Let

$$f(\theta_1, \theta_2) = e^{a\theta_1} \left(\frac{1}{2} - b\theta_2^2 \right)^{3/2}, \quad \text{where} \quad a \ne 0 \quad \text{and } b \ne 0.$$

It is clear that the derivatives $\dfrac{\partial^{m-1} f(\theta_1, \theta_2)}{\partial \theta_1^{m-1}}$ and $\dfrac{\partial f(\theta_1, \theta_2)}{\partial \theta_2}$ are continuous on S_2, but $\dfrac{\partial^2 f(\theta_1, \theta_2)}{\partial \theta_2}$ has a singularity on S_2. Therefore, the condition $m = m_1 + m_2$ immediately leads to $m_1 = m - 1$, $m_2 = 1$, because $m - 1 + 1 = m$. Hence, $f(\theta_1, \theta_2) \in \overline{L}_2^{(m)}(S_2)$ at $m_1 = m - 1$, $m_2 = 1$. Thus, obviously, $\dfrac{\partial^2 f(\theta_1, \theta_2)}{\partial \theta_2}$ is not continuous on S_2, because at $m_1 = m - 2$ and $m_2 = 2$ $f(\theta_1, \theta_2) \in L_2^{(m)}(S_2)$, but the condition $m = m_1 + m_2$ holds, that is, $m - 2 + 2 = m$.

CONCLUSIONS

The problems of approximate calculation of definite integrals with the greatest possible accuracy are connected with the fact that most problems of science and technology are reduced to integral and differential equations, and their solutions are expressed using definite integrals, which, in many cases, cannot be calculated exactly. Many mathematicians have dealt with such problems and there are several methods for constructing optimal cubature formulas. In our research we considered weighted cubature formulas of the Hermite type in the Sobolev space of functions $\overline{L}_2^{(m)}(S_n)$ and obtained an upper bound for the norm of the error functional of the Hermite-type weighted cubature formula. Thus, based on the Bakhvalov theorem, we proved that the considered weighted cubature formulas of the Hermite type are asymptotically optimal on this space. For multidimensional cubature formulas, the number of integration nodes increases rapidly with increasing space dimension. This problem requires special attention to constructing the most economical formulas. In this paper, we considered the formulas taking into account this requirement; Bakhvalov called these formulas practical [23].

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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