AIP Conference Proceedings

scitation.org/journal/apc

Volume 2365

International Uzbekistan-Malaysia Conference on “COMPUTATIONAL MODELS AND TECHNOLOGIES (CMT2020)”

CMT2020

Tashkent, Uzbekistan • 24–25 August 2020

Editors • Rakhmatillo D. Aloev, Kholmat M. Shadimetov, Abdullo R. Hayotov and Mirzoali U. Khudoyberganov

CMT 2020

AIP Publishing

July 2021

INTERNATIONAL UZBEKISTAN-MALAYSIA CONFERENCE ON “COMPUTATIONAL MODELS AND TECHNOLOGIES (CMT2020)”: CMT2020

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Cite as: AIP Conference Proceedings **2365**, 020014 (2021); https://doi.org/10.1063/5.0057015
Published Online: 16 July 2021

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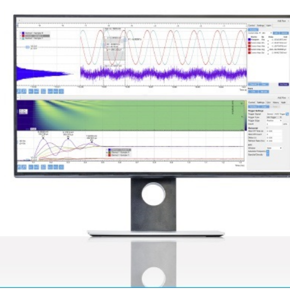

Weight Optimal Order of Convergence Cubature Formulas in Sobolev Space

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Abstract. In this paper we investigate weight cubature formula in function spaces of S.L. Sobolev $L_2^{(m)}, L_p^{(m)}, \bar{L}_2^{(m)}$ for the functions defined in the n - dimensional unit cube K_n and obtain an upper estimate for the norm of error functionals of weight cubature formulas. The basis of theorem N.S. Bahvalov it isproved that consideredviewed cubature formulas are optimal on order of convergence in these spaces.

INTRODUCTION

In many research papers examined the properties of optimal approximations of linear functionals [1-28], and others. In these papers the problem of optimality with respect to a certain space are investigated. Most of them are discussed in the Sobolev space [1]. Consider the cubature formula of the form

$$\int_{K_n} p(x) f(x) dx \approx \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right), \tag{1}$$

in the space $L_2^{(m)}(K_n)$, where K_n is a n- dimensional unit cube. A generalized function

$$\ell_N(x) = p(x)\,\varepsilon_{K_n}(x) - \sum_{\lambda=1}^{N} C_\lambda \delta\left(x - x^{(\lambda)}\right), \tag{2}$$

is called a error functional of the cubature formula (1),

$$< \ell_N, f > = \int_{K_n} p(x) f(x) dx - \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right), \tag{3}$$

is an error of the cubature formula (1), $p(x)$ is a weight function, $\varepsilon_{K_n}(x)$ is characteristic function of K_n, C_λ and $x^{(\lambda)}$ are coefficients and nodes of the cubature formula (1) , $\delta(x)$ is the Dirac delta- function.

Definition 1. The space $L_2^{(m)}(K_n)$ is defined as the space of functions, given on the n- dimensional unit cube K_n and having all the generalized derivatives of order m, square summable in norm [1]:

$$\left\| f \middle/ L_2^{(m)}(K_n) \right\| = \left\{ \int_{K_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} [D^\alpha f]^2 dx \right\}^{\frac{1}{2}}, \tag{4}$$

with the inner product
$(f, \varphi)_{L_2^{(m)}(K_n)} = \int_{K_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f \cdot D^\alpha \varphi dx,$
where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $dx = dx_1 dx_2 ... dx_n$ and $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$.

Definition 2. The cubature formula of the form (1) is called asymptotically optimal, if for the norms of error functional the following equality holds

$$\lim_{N\to\infty} \frac{\left\| \ell_N^{a,o} \middle/ L_2^{(m)^*}(K_n) \right\|}{\left\| \ell_N^{o} \middle/ L_2^{(m)^*}(K_n) \right\|} = 1. \tag{5}$$

International Uzbekistan-Malaysia Conference on "COMPUTATIONAL MODELS AND TECHNOLOGIES (CMT2020)"
AIP Conf. Proc. 2365, 020014-1–020014-14; https://doi.org/10.1063/5.0057015
Published by AIP Publishing. 978-0-7354-4110-1/$30.00

Here, $\ell_N^o(x)$ and $\ell_N^{a,o}(x)$ are error functional of optimal and asymptotically optimal cubature formulas of the form (1), respectively.

Definition 3. The cubature formula of the form (1) is called an optimal order of convergence, if for the norm of its error functional the following holds

$$\lim_{N\to\infty} \frac{\left\| \ell_N^{o,n} \Big/ L_2^{(m)^*}(K_n) \right\|}{\left\| \ell_N^{o} \Big/ L_2^{(m)^*}(K_n) \right\|} < \infty. \tag{6}$$

Here $\ell_N^{o,n}$ is error functional of optimal order of convergence cubature formulas (1). In this paper we consider the problem of the descending order of norm of the error functional $\left\| \ell_N \Big/ L_2^{(m)^*}(K_n) \right\|$, $\left\| \ell_N \Big/ L_p^{(m)^*}(K_n) \right\|$ and $\left\| \ell_N \Big/ \bar{L}_2^{(m)^*}(K_n) \right\|$ with an increase in the number of its nodes. The results, which we obtain here, are to the arbitrary distribution of points.

OPTIMAL IN ORDER OF CONVERGENCE OF WEIGHT CUBATURE FORMULAS IN THE SPACE $L_2^{(m)}(K_n)$

Here we explore weight cubature formulas, which are optimal for the convergence order. We have the following Here we explore weight cubature formulas, which are optimal for the convergence order. We have the following

Lemma 1. If for error functional (2) of cubature formula (1) satisfies the conditions

$$\ell_N(x) = \ell_{N_1}(x_1)\cdot \ell_{N_2}(x_2)\cdot \ldots \cdot \ell_{N_n}(x_n) \tag{7}$$

and

$$\left\| \ell_{N_i} \Big/ L_2^{(m_i)^*}(0,1) \right\| \leq c_i \frac{1}{N_i^{m_i}}, \left(i=\overline{1,n}\right), c_i - constants \tag{8}$$

that is

$$\left\| \ell_{N_i} \Big/ L_2^{(m_i)^*}(0,1) \right\| \leq c_i O\left(h_i^{m_i}\right), \left(i=\overline{1,n}\right), h_i = \frac{1}{N_i} \tag{9}$$

then

$$\left\| \ell_N \Big/ L_2^{(m)^*}(K_n) \right\| \leq c \cdot \frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}}, c - constants \tag{10}$$

or

$$\left\| \ell_N \Big/ L_2^{(m)^*}(K_n) \right\| \leq c \cdot O\left(h_1^{m_1}\right)\cdot O\left(h_2^{m_2}\right)\cdot \ldots \cdot O\left(h_n^{m_n}\right), \tag{11}$$

where

$$\ell_{N_i}(x_i) = p(x_i)\,\varepsilon_{[0,1]}(x_i) - \sum_{\lambda_i=1}^{N_i} C_{\lambda_i}\delta\left(x_i - x_i^{(\lambda_i)}\right),$$

$p(x) = \prod\limits_{i=1}^{n} p_i(x_i)$, $c = \prod\limits_{i=1}^{n} c_i$ and $m = m_1 + m_2 + \ldots + m_n$, $m_i \geq 1$, $i = \left(\overline{1,n}\right)$.
We are conducting proof by mathematical induction.

Suppose $n=2$, then
$x = (x_1, x_2)$, $\quad |\alpha| = \alpha_1 + \alpha_2$, $\quad m = m_1 + m_2$, $\quad dx = dx_1 dx_2$, $f(x) = f(x_1, x_2)$,

$p(x) = p_1(x_1) \cdot p_2(x_2)$ and $\ell_N(x) = \ell_{N_1}(x_1) \cdot \ell_{N_2}(x_2)$.
If we assume in (4) $n = 1$, then

$$\left\| f_i \Big/ \overline{L}_2^{(m_i)}(0,1) \right\| = \left\{ \int_0^1 \left[\frac{d^{m_i}}{d{x_i}^{m_i}} f(x_i) \right]^2 dx_i \right\}^{\frac{1}{2}}, \left(\overline{1,n}\right). \tag{12}$$

Thus, we have

$$|< \ell_N(x_1,x_2), f(x_1,x_2) >| = |< \ell_{N_2}(x_2), < \ell_{N_1}(x_1), f(x_1,x_2) >>| \leq$$

$$\leq \left\| \ell_2(x_2) \Big/ L_2^{(m_2)^*}(0,1) \right\| \cdot \left\| < \ell_{N_1}(x_1), f(x_1,x_2) > \Big/ L_2^{(m_2)}(0,1) \right\|. \tag{13}$$

Taking into account (12), (13) we compute the following norm [13]:

$$\left\| < \ell_{N_1}(x_1), f(x_1,x_2) > \Big/ L_2^{(m_2)}(0,1) \right\| = \left\{ \int_0^1 \left| \frac{d^{m_2}}{dx_2^{m_2}} < \ell_{N_1}(x_1), f(x_1,x_2) > \right|^2 dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_0^1 \left| < \ell_{N_1}(x_1), \frac{d^{m_2}}{dx_2^{m_2}} f(x_1,x_2) > \right|^2 dx_2 \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ \int_0^1 \left[\left\| \ell_{N_1}(x_1) \Big/ L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| \frac{d^{m_2}}{dx_2^{m_2}} f(x_1,x_2) \Big/ L_2^{(m_1)}(0,1) \right\| \right]^2 dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}(x_1) \Big/ L_2^{(m_1)^*}(0,1) \right\| \cdot \left\{ \int_0^1 \left\{ \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x_1,x_2) \right]^2 dx_1 \right\} dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}(x_1) \Big/ L_2^{(m_1)^*}(0,1) \right\| \cdot \left\{ \int_0^1 \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x) \right]^2 dx \right\}^{\frac{1}{2}} =$$

$$= c' \left\| \ell_{N_1}(x_1) \Big/ L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| f(x) \Big/ L_2^{(m)}(K_2) \right\|, c' - constants. \tag{14}$$

Thus, from (13) and (14) we obtain
$|< \ell_N(x_1,x_2), f(x_1,x_2) >| \leq$

$$\leq c' \left\| \ell_{N_2}(x_2) \Big/ L_2^{(m_2)^*}(0,1) \right\| \cdot \left\| \ell_{N_1}(x_1) \Big/ L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| f(x) \Big/ L_2^{(m)}(K_2) \right\|. \tag{15}$$

Taking into account (4) from (15) we obtain

$$\left\| \ell_N \Big/ L_2^{(m)^*}(K_2) \right\| \leq c' \left\| \ell_{N_1}(x_1) \Big/ L_2^{(m_1)^*}(0,1) \right\| \cdot \left\| \ell_{N_2}(x_2) \Big/ L_2^{(m_2)^*}(0,1) \right\|. \tag{16}$$

Using (8), from (16) we have

$$\left\| \ell_N \middle/ L_2^{(m)^*}(K_2) \right\| \leq c' \cdot c_1 \cdot c_2 \frac{1}{N_1^{m_1} N_2^{m_2}},$$

that is

$$\left\| \ell_N \middle/ L_2^{(m)^*}(K_2) \right\| \leq c_3' O\left(h_1^{m_1}\right) O\left(h_2^{m_2}\right), \tag{17}$$

where $c_3' = c' \cdot c_1 \cdot c_2$.

Now suppose that (10) is valid for $n = k$, then from the above calculations we obtain [23]

$|< \ell_N(x), f(x) >| = |< \ell_N(x_1, x_2, ..., x_k), f(x_1, x_2, ..., x_k) >| =$

$= \left|< \ell_{N_k}(x_k), < \ell_{N_{k-1}}(x_{k-1}), ..., < \ell_{N_2}(x_2), < \ell_{N_1}(x_1), f(x_1, x_2, .., x_k) > ... >\right| \leq$

$\leq \left\| \ell_{N_k}(x_k) \middle/ L_2^{(m_k)^*}(0,1) \right\| \cdot \left\| \ell_{N_{k-1}}(x_{k-1}) \middle/ L_2^{(m_{k-1})^*}(0,1) \right\| ...$

$\cdot \left\| < \ell_{N_1}(x_1), f(x_1, x_2, ..., x_k) > \middle/ L_2^{(m_1)^*}(0,1) \right\| \leq$

$$\leq \left\| \ell_{N_k}(x_k) \middle/ L_2^{(m_k)^*}(0,1) \right\| ... \left\| \ell_{N_1}(x_1) \middle/ L_2^{(m_1)^*}(x_1) \right\| \cdot c'' \left\| f(x) \middle/ L_2^{(m)}(K_k) \right\|. \tag{18}$$

From (18), considering (13), we have

$$\left\| \ell_N \middle/ L_2^{(m)^*}(K_k) \right\| \leq c'' \left\| \ell_{N_1}(x_1) \middle/ L_2^{(m_1)^*}(0,1) \right\| ... \left\| \ell_{N_k}(x_k) \middle/ L_2^{(m_k)^*}(0,1) \right\|. \tag{19}$$

Then, referring to (8), from (19) we obtain

$$\left\| \ell_N \middle/ L_2^{(m)^*}(K_k) \right\| \leq c'' \cdot c_1 \cdot c_2 \cdot ... \cdot c_k \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_k^{m_k}}, \tag{20}$$

or, considering (9), from (20) we have

$\left\| \ell_N \middle/ L_2^{(m)^*}(K_k) \right\| \leq c'' \cdot d_k \cdot O\left(h_1^{m_1}\right) ... O\left(h_k^{m_k}\right)$, where $d_k = \prod_{i=1}^{k} c_i$.

Using the validity of assertion $n = k$, we prove that the assertion executed when $n = k+1$. Thus, when $n = k+1$, and taking account of (4) and (19), we obtain

$\left|< \ell_{N_{k+1}}(x_1, x_2, ..., x_{k+1}), f(x_1, x_2, ..., x_{k+1}) >\right| =$

$\left|< \ell_{N_1}(x_1), < \ell_{N_2}(x_2), < ... < \ell_{N_k}(x_k), < \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, .., x_{k+1}) > ... >\right| \leq$

$\leq \left\| \ell_{N_1}(x_1) \middle/ L_2^{(m_1)^*}(0,1) \right\| ... \left\| \ell_{N_k}(x) \middle/ L_2^{(m_k)^*}(0,1) \right\| \cdot$

$$\cdot \left\| < \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > \middle/ L_2^{(m_{k+1})}(0,1) \right\|. \tag{21}$$

Using (4) and (19) from (21) we obtain

$$\left\| \ell_N \middle/ L_2^{(m)^*}(K_{k+1}) \right\| \leq c''' \left\| \ell_{N_1}(x_1) \middle/ L_2^{(m_1)^*}(0,1) \right\| ...$$

$$\cdot \left\| \ell_{N_k}(x_k) \middle/ L_2^{(m_k)^*}(0,1) \right\| \cdot \left\| \ell_{N_{k+1}}(x_{k+1}) \middle/ L_2^{(m_{k+1})^*}(0,1) \right\|. \tag{22}$$

By using (8), from (22) we have

$\left\| \ell_N \middle/ L_2^{(m)*}(K_{k+1}) \right\| \leq c''' \cdot d_{k+1} \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_{k+1}^{m_{k+1}}}$,

where

$$d_{k+1} = \prod_{i=1}^{k+1} c_i, \tag{23}$$

or, taking into account (17), from (23) we obtain

$$\left\| \ell_N \Big/ L_2^{(m)*}\left(K_{k+1}\right) \right\| \leq c''' \cdot d_{k+1} \cdot O\left(h_1^{m_1}\right) ... O\left(h_{k+1}^{m_{k+1}}\right), c''' constants, \tag{24}$$

in conclusion consider, that

$$\left\| \ell_N \Big/ L_2^{(m)\,*}\left(K_n\right) \right\| \leq c \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_n^{m_n}}, \tag{25}$$

where $c = c'' \cdot d_{k+1}$.
or, considering (9), from (25), we have

$$\left\| \ell_N \Big/ L_2^{(m)\,*}\left(K_n\right) \right\| \leq c \cdot O\left(h_1^{m_1}\right) ... O\left(h_n^{m_n}\right).$$

With the help of this lemma it is easy to prove the following theorem.

Theorem 1. The weight cubature formula (1) with the error functional (2) for $N_1 = N_2 = ... = N_n$, $\prod\limits_{i=1}^{n} N_i = N$ and $m_1 + m_2 + ... + m_n = m$ is optimal in order of convergence in the space $L_2^{(m)}\left(K_n\right)$, those, for the norm of the error functional (2) of cubature formula (1) the following holds

$$\left\| \ell_N \Big/ L_2^{(m)^*}\left(K_n\right) \right\| = O\left(N^{-\frac{m}{n}}\right).$$

Proof. On the basis of Lemma1 under the $N_1 = N_2 = ... = N_n$ have $N_1 = \sqrt[n]{N}$.
Thus,

$$\prod_{i=1}^{n} N_i^{m_i} = N_1^{m_1+m_2+...+m_n} = N^{\frac{m}{n}}. \tag{26}$$

By substituting (26) into (25) we obtain

$$\left\| \ell_N \Big/ L_2^{(m)^*}\left(K_n\right) \right\| \leq c \cdot N^{-\frac{m}{n}}. \tag{27}$$

From theorem of N.S. Bahvalov [24] and from the inequality (27) follows the proof.

A NORM ESTIMATION FOR THE ERROR FUNCTIONAL OF WEIGHT CUBATURE FORMULAS ON THE SPACE $L_p^{(m)}\left(K_n\right)$.

Consider the cubature formula of the form

$$\int_{K_n} p\left(x\right) f\left(x\right) dx \approx \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right), \tag{28}$$

(28) in the Sobolev space $L_p^{(m)}\left(K_n\right)$, where K_n is n-dimensional unit cube.
The generalized function

$$\ell_N\left(x\right) = p\left(x\right) \varepsilon_{K_n}\left(x\right) - \sum_{\lambda=1}^{N} C_\lambda\, \delta\left(x - x^{(\lambda)}\right). \tag{29}$$

is called error functional of the cubature formula (28),

$$< \ell_N, f >= \int_{K_n} p\left(x\right) f\left(x\right) dx - \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right)$$

is an error of the cubature formula (28), $p(x) \in L_p(K_n)$ is a weight function, $\varepsilon_{K_n}(x)$ is the characteristic function of K_n, C_λ and $x^{(\lambda)}$ are coefficients and nodes of the cubature formula (28) and $\delta(x)$ is the Dirac delta function.

Definition 4. The space $L_p^{(m)}(K_n)$ is defined as a space of functions, given on a n- dimensional unit cube K_n and having all the generalized derivatives of order m, summable with a degree p in norm [1]:

$$\left\| f \middle/ L_p^{(m)}(K_n) \right\| = \left\{ \int\limits_{K_n} \left\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} [D^\alpha f]^2 \right\}^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}, \tag{30}$$

where $D^\alpha = \frac{\partial^m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, |\alpha| = \sum\limits_{j=1}^{n} \alpha_j,\ \ \alpha! = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_n!.$

The following is true

Lemma 2. If for the error functional (29) of the cubature formula (28), the following conditions are fulfilled

$$\ell_N(x) = \ell_{N_1}(x_1) \cdot \ell_{N_2}(x_2) \cdot \dots \cdot \ell_{N_n}(x_n),$$

where $\ell_{N_i}(x_i) = p_i(x_i)\, \varepsilon_{[0,1]}(x_i) - \sum\limits_{\lambda_i=1}^{N_i} C_{\lambda_i} \delta\left(x_i - x_i^{(\lambda_i)}\right)$, $p(x) = \prod\limits_{i=1}^{n} p_i(x_i)$

and

$$\left\| \ell_{N_i} \middle/ L_p^{(m_i)*}(0,1) \right\| \le d_i \frac{1}{N_i^{m_i}},\ \ d_i - constants, \tag{31}$$

that is

$$\left\| \ell_{N_i} \middle/ L_p^{(m_i)*}(0,1) \right\| \le d_i\, O\left(h_i^{m_i}\right), \quad d_i - constants, \left(i = \overline{1,n}\right), h_i = \frac{1}{N_i} \tag{32}$$

then

$$\left\| \ell_N \middle/ L_p^{(m)*}(K_n) \right\| \le d \cdot \frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}},\ d - constants, \tag{33}$$

or

$$\left\| \ell_N \middle/ L_p^{(m)*}(K_n) \right\| \le d \cdot O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m_2}\right) \cdot \dots \cdot O\left(h_n^{m_n}\right),$$

$d = \prod\limits_{i=1}^{n} d_i$ and $m = m_1 + m_2 + \dots + m_n$.

Proof. We conduct the method of the mathematical induction. Suppose $n = 2$ $x = (x_1, x_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $m = m_1 + m_2$, $dx = dx_1 dx_2$, $f(x) = f(x_1, x_2)$, $p(x) = p_1(x_1) \cdot p_2(x_2)$ and $\ell_N(x) = \ell_{N_1}(x_1) \cdot \ell_{N_2}(x_2)$.

If presume in (30) $n = 1$, then

$$\left\| f_i \middle/ L_p^{(m_i)}(0,1) \right\| = \left\{ \int\limits_0^1 \left[\left(\frac{d^{m_i}}{dx_i^{m_i}} f(x_i) \right)^2 \right]^{\frac{p}{2}} dx_i \right\}^{\frac{1}{p}},\ \ i = 1, 2, \dots, n. \tag{34}$$

So we have

$$|< \ell_N(x_1, x_2), f(x_1, x_2) >| = |< \ell_{N_2}(x_2), < \ell_{N_1}(x_1), f(x_1, x_2) >>| \le$$

$$\le \left\| \ell_{N_2}(x_2) \middle/ L_p^{(m_2)*}(0,1) \right\| \cdot \left\| < \ell_{N_1}(x_1), f(x_1, x_2) > \middle/ L_p^{(m_2)}(0,1) \right\|. \tag{35}$$

We compute the following norm:

$$\left\| < \ell_{N_1}(x_1), f(x_1,x_2) > \Big/ L_p^{(m_2)}(0,1) \right\| = \left\{ \int_0^1 \left[\left| \frac{d^{m_2}}{dx_2^{m_2}} < \ell_{N_1}(x_1), f(x_1,x_2) > \right|^2 \right]^{\frac{p}{2}} dx_2 \right\}^{\frac{1}{p}} =$$

$$= \left\{ \int_0^1 \left[\left| < \ell_{N_1}(x_1), \frac{d^{m_2}}{dx_2^{m_2}} f(x_1,x_2) \right|^2 \right]^{\frac{p}{2}} dx_2 \right\}^{\frac{1}{p}} \leq$$

$$\leq \left\{ \int_0^1 \left[\left(\left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot \left\| \frac{d^{m_2}}{dx_2^{m_2}} f(x_1,x_2) \Big/ L_p^{(m_1)}(0,1) \right\| \right)^2 \right]^{\frac{p}{2}} dx_2 \right\}^{\frac{1}{p}} =$$

$$= \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot \left\{ \int_0^1 \left\{ \int_0^1 \left[\left(\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x_1,x_2) \right)^2 \right]^{\frac{p}{2}} dx_1 \right\} dx_2 \right\}^{\frac{1}{p}} =$$

$$= d' \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot \left\| f(x) \Big/ L_p^{(m)}(K_2) \right\|, \tag{36}$$

where d' -constants.
Thus, from (35) and (36) we obtain
$|< \ell_N(x_1,x_2), f(x_1,x_2) >| \leq$

$$\leq d' \left\| \ell_{N_2}(x_2) \Big/ L_p^{(m_2)*}(0,1) \right\| \cdot \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot \left\| f(x) \Big/ L_p^{(m)}(K_2) \right\|. \tag{37}$$

From (37), using the definition of norm, we have

$$\left\| \ell_N(x) \Big/ L_p^{(m)^*}(K_2) \right\| \leq d' \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot \left\| \ell_{N_2}(x_2) \Big/ L_p^{(m_2)*}(0,1) \right\|. \tag{38}$$

Taking into account (31), on the basis of (38) we obtain

$$\left\| \ell_N \Big/ L_p^{(m)^*}(K_2) \right\| \leq d' \cdot d_1 \cdot d_2 \frac{1}{N_1^{m_1} N_2^{m_2}},$$

that is

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_2) \right\| \leq d_3 O\left(h_1^{m_1}\right) O\left(h_2^{m_2}\right), \tag{39}$$

where

$$d_3 = d' \cdot d_1 \cdot d_2.$$

Now suppose that the inequality (33) holds for $n = k$, then on the basis of the above calculations we obtain
$|< \ell_N(x), f(x) >| = |< \ell_N(x_1,x_2,...,x_k), f(x_1,x_2,...,x_k) >| =$
$= \left| < \ell_{N_k}(x_k), < \ell_{N_{k-1}}(x_{k-1}), ..., < \ell_{N_2}(x_2), < \ell_{N_1}(x_1), f(x_1,x_2,..,x_k) >> ... >> \right| \leq$
$\leq \left\| \ell_{N_k}^{(x_k)} \Big/ L_p^{(m_k)*}(0,1) \right\| \cdot \left\| \ell_{N_{k-1}}(x_{k-1}) \Big/ L_p^{(m_{k-1})*}(0,1) \right\| \cdot ...$
$... \cdot \left\| < \ell_{N_1}(x_1), f(x_1,x_2,...,x_k) > \Big/ L_p^{(m_1)}(0,1) \right\| \leq$

$$\leq d'' \left\| \ell_{N_k}(x_k) \Big/ L_p^{(m_k)*}(0,1) \right\| \cdot ... \cdot \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(x_1) \right\| \cdot \left\| f(x) \Big/ L_p^{(m)}(K_k) \right\|, \tag{40}$$

where d'' - constants.
From (40), using the definition for norm of the error functional, we obtain

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_k) \right\| \leq d'' \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot ... \cdot \left\| \ell_{N_k}(x_k) \Big/ L_p^{(m_k)*}(0,1) \right\|. \tag{41}$$

Then, using (31), the inequality (41) reduces to

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_k) \right\| \le d'' \cdot d \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_k^{m_k}},$$

where $d = \prod_{i=1}^{k} d_i$
or

$$\left\| \ell_N \Big/ L_p^{(m)^*}(K_k) \right\| \le d'' \cdot d \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_k^{m_k}\right).$$

Using the validity of lemma (2) at $n = k$, we prove that the assertion holds for $n = k+1$. Taking into account (40), at $n = k+1$ we estimate error of cubature formulas for the form (28)
$\left| < \ell_{N_{k+1}}(x_1, x_2, ..., x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > \right| =$
$= \left\| \ell_{N_1}(x_1), < \ell_{N_2}(x_2), < ... < \ell_{N_k}(x_k), < \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, ... x_{k+1}) >> .. >>> \right\| \le$
$\le \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)^*}(0,1) \right\| \cdot ... \cdot \left\| \ell_{N_k}(x_k) \Big/ L_p^{(m_k)^*}(0,1) \right\| \cdot$

$$\cdot \left\| < \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > \Big/ L_p^{(m_{k+1})^*}(0,1) \right\|. \tag{42}$$

Hence, as above, using the definition of norms of functionals, we get

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_{k+1}) \right\| \le d''' \left\| \ell_{N_1}(x_1) \Big/ L_p^{(m_1)*}(0,1) \right\| \cdot ...$$

$$... \cdot \left\| \ell_{N_k}(x_k) \Big/ L_p^{(m_k)*}(0,1) \right\| \cdot \left\| \ell_{N_{k+1}}(x_{k+1}) \Big/ L_p^{(m_{k+1})*}(0,1) \right\|. \tag{43}$$

From inequalities (31) and (43) we obtain

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_{k+1}) \right\| \le d''' \cdot d_{k+1} \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_{k+1}^{m_{k+1}}}. \tag{44}$$

or

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_{k+1}) \right\| \le d''' \cdot d_{k+1} \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_{k+1}^{m_{k+1}}\right),$$

where $d_{k+1} = \prod_{i=1}^{k+1} d_i$
Summarizing the results obtained, in conclude by noting that

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_n) \right\| \le d \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2} \ldots N_n^{m_n}}. \tag{45}$$

(45) or, taking into account (32), from (44) we have

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_n) \right\| \le d \cdot O\left(h_1^{m_1}\right) \ldots O\left(h_1^{m_n}\right), \quad d - constant.$$

Lemma 2 is proved.
With the help of this lemma it is easy to prove the following theorem.

Theorem 2. The weight cubature formula (28) with the error functional (29) at $N_1 = N_2 = ... = N_n$, $\prod_{i=1}^{n} N_i = N$ and $m_1 + m_2 + ... + m_n = m$ is optimal in order of convergence in the space $L_p^{(m)}(K_n)$, for the norms of error functional (29) of the cubature formula (28) have the equality

$$\left\| \ell_N \Big/ L_p^{(m)*}(K_n) \right\| = O\left(N^{-\frac{m}{n}}\right).$$

Proof. On the basis of Lemma 2 at $N_1 = N_2 = ... = N_n$ we have $N_i = \sqrt[n]{N}$, $i = 1,2,...,n$. Thus,

$$\prod_{i=1}^{n} N_i^{m_i} = N_1^{m_1+m_2+...+m_n} = N^{\frac{m}{n}}. \tag{46}$$

By substituting (46) into inequality (45), we obtain

$$\left\| \ell_N \middle/ L_p^{(m)*}(K_n) \right\| \le C \cdot N^{-\frac{m}{n}}. \tag{47}$$

From the N.S. Bakhvalov theorem [24] and the inequality (47) follows the proof of the theorem.

WEIGHT CUBATURE FORMULAS IN THE SPACE $\bar{L}_2^{(m)}(K_n)$

Multidimensional cubature formulas differ from the one-dimensional with two features
1) infinitely varied forms of multidimensional areas of integration;
2) rapidly grows number of integration nodes with increasing space dimension.
Problem 2) requires special attention to the construction of the most efficient formulas.

Here, we discuss the formula with taking into account this requirement. It is known, that such formulas are called by N.S. Bakhvalov as "practical formulas" [24].
We regard the weight cubature formula

$$\int_{\Omega} p(x) f(x)\, dx \approx \sum_{\lambda=1}^{N} C_\lambda f\left(x^{(\lambda)}\right). \tag{48}$$

in space $\bar{L}_2^{(m)}(K_n)$, where K_n - n - dimensional unit cube and $p(x) \in L_2(K_n)$ is weight function. In the cubature formula (48) comparable to generalized function

$$\ell_N(x) = p(x)\, \varepsilon_{K_n}(x) - \sum_{\lambda=1}^{N} C_\lambda \delta\left(x - x^{(\lambda)}\right). \tag{49}$$

and we call it a error functional.
Definition 5. The space $\bar{L}_2^{(m)}(K_n)$ is defined as the space of functions, given on the K_n and the norm of a function, which is determined by the following equation

$$\left\| f \middle/ \bar{L}_2^{(m)}(K_n) \right\| = \left\{ \int_{K_n} \left(\frac{\partial^m f(x)}{\partial x_1{}^{m_1} \partial x_2{}^{m_2} ... \partial x_n{}^{m_n}} \right)^2 dx \right\}^{\frac{1}{2}}, \tag{50}$$

where $m_1 + m_2 + ... + m_n = m$, $m_i > 0$, $\quad i = \overline{1,n}$
with the scalar product

$$(f, \varphi)_{\bar{L}_2^{(m)}(K_n)} = \int_{K_n} \left(\frac{\partial^m f(x)}{\partial x^m} \right) \left(\frac{\partial^m \varphi(x)}{\partial x^m} \right) dx,$$

where $\partial x^m = \partial x_1^{m_1} \partial x_2^{m_2} ... \partial x_n^{m_n}, \quad m = m_1 + m_2 + ... m_n, \quad dx = dx_1 dx_2 ... dx_n.$

As it is known [1], the norm of a function in the space $L_2^{(m)}(K_n)$ determined by the formula

$$\left\| f \middle/ L_2^{(m)}(K_n) \right\| = \left\{ \int_{K_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha f(x))^2 dx \right\}^{\frac{1}{2}}, \tag{51}$$

where $|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_n$, $\quad \alpha! = \alpha_1! \cdot \alpha_2! \cdot ...\alpha_n!$ and $D^{\alpha} f(x) = \frac{\partial^{|\alpha|} f(x_1,...,x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$.

Suppose that in (51) $n = 2$ and $m = 2$, then we obtain the following

$$\int\limits_{K_2} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\frac{\partial^m f(x)}{\partial x^m} \right)^2 dx = \int\limits_{K_2} \sum_{\alpha_1+\alpha_2=2} \frac{2!}{\alpha!\alpha_2!} \left(\frac{\partial^2 f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right)^2 dx =$$

$$= \int\limits_{K_2} \left[\left(\frac{\partial^2 f(x)}{\partial x_2^2} \right)^2 + \frac{2!}{1! \cdot 1!} \left(\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \right)^2 + \left(\frac{\partial^2 f(x)}{\partial x_1^2} \right)^2 dx \right]. \tag{52}$$

When $n = 2$ and $m = 2$ equality (50) takes the following form:

$$\left\| f \big/ \bar{L}_2^{(2)}(K_2) \right\|^2 = \int\limits_{K_2} \left(\frac{\partial^2 f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}} \right)^2 dx. \tag{53}$$

Obviously, in the right hand side of (53) is less computing, than in (52), and it follows that the norm of the function in space $\bar{L}_2^{(2)}(K_2)$ the number of computing operations will be much less, than in space $L_2^{(2)}(K_2)$, as in the norm (53), involved only the mixed derivatives. Now we prove the the following theorem, which is one of the main results of this work.

Theorem 3. If, for the error functional (49) of the weight cubature formula (48) in the space $barL_2^{(m)}(K_n)$ the following conditions are fulfilled

$$\ell_N(x) = \ell_{N_1}(x_1) \cdot \ell_{N_2}(x_2) \cdot ... \cdot \ell_{N_n}(x_n)$$

and

$$\left\| \ell_{N_i} \big/ \bar{L}_2^{(m_i)*}(0,1) \right\| \le d_i \frac{1}{N_i^{m_i}}. \quad d_i - \quad constants \tag{54}$$

that is

$$\left\| \ell_{N_i} \big/ \bar{L}_2^{(m_i)^*}(0,1) \right\| \le d_i O\left(h_i^{m_i}\right), \quad d_i - constants, (i = 1, \bar{n}), h_i = \frac{1}{N_i} \tag{55}$$

then

$$\left\| \ell_N \big/ \bar{L}_2^{(m)^*}(K_n) \right\| \le d \cdot \frac{1}{\prod\limits_{i=1}^{n} N_i^{m_i}}, \qquad d - constants, \tag{56}$$

or

$$\left\| \ell_N \big/ \bar{L}_2^{(m)^*}(K_n) \right\| \le d \cdot O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m2}\right) \cdot ... \cdot O\left(h_n^{m_n}\right), \tag{57}$$

where

$$\ell_{N_i}(x_i) = p_i(x_i)\, \varepsilon_{[0,1]}(x_i) - \sum_{\lambda_i=1}^{N_i} C_{\lambda_i} \delta\left(x_i - x_i^{(\lambda_i)}\right), \quad p(x) = \prod_{i=1}^{n} p_i(x_i),$$

$d = \prod\limits_{i=1}^{n} d_i$, $m = m_1 + m_2 + ... + m_n$ and m_i- is arbitrary $\left(i = \overline{1,n}\right)$, and $m_i \ge 1$.

Proof. We are conducting proof by mathematical induction.
Suppose $n = 2$, then
$x = (x_1, x_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $m = m_1 + m_2$, $dx = dx_1 dx_2$, $f(x) = f(x_1, x_2)$,

$p(x) = p_1(x_1) \cdot p_2(x_2)$ and $\ell_N(x) = \ell_{N_1}(x_1) \cdot \ell_{N_2}(x_2)$.
If presume in (50) $n = 1$, then

$$\left\| f_i \Big/ \bar{L}_2^{(m_i)}(0,1) \right\| = \left\{ \int_0^1 \left(\frac{\partial^{m_i} f(x_i)}{\partial x_i^{m_i}} \right)^2 dx_i \right\}^{\frac{1}{2}}, \left(i = \overline{1,n}\right).$$

Thus, we have

$$\begin{gathered} |< \ell_N(x_1,x_2), f(x_1 x_2) >| = |< \ell_{N_2}(x_2), < \ell_{N_1}(x_1), f(x_1,x_2) >>| \leq \\ \left\| \ell_{N_2}(x_2) \Big/ \bar{L}_2^{(m_2)^*}(0,1) \right\| \cdot \left\| < \ell_{N_1}(x_1), f(x_1,x_2) > \Big/ \bar{L}_2^{(m_2)^*}(0,1) \right\|. \end{gathered} \tag{58}$$

We compute the following norm:

$$\left\| < \ell_{N_1}(x_1), f(x_1.x_2) > \Big/ L_2^{(m_2)}(0,1) \right\| = \left\{ \int_0^1 \left| \frac{\partial^{m_2}}{\partial x_2^{m_2}} < \ell_{N_1}(x_1), f(x_1,x_2) > \right|^2 dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_0^1 \left| < \ell_{N_1}(x_1), \frac{\partial^{m_2}}{\partial x_2^{m_2}} f(x_1,x_2) > \right|^2 dx_2 \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ \int_0^1 \left[\left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_1)^*}(0,1) \right\| \cdot \left\| \frac{\partial^{m_2}}{\partial x_2^{m_2}} f(x_1,x_2) \Big/ \bar{L}_2^{(m_1)}(0,1) \right\| \right]^2 dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m)^*}(0,1) \right\| \cdot \left\{ \int_0^1 \left\{ \int_0^1 \left[\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} f(x_1,x_2) \right]^2 dx_1 \right\} dx_2 \right\}^{\frac{1}{2}} =$$

$$= \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_1)^*}(0,1) \right\| \cdot \left\| f(x) \Big/ \bar{L}_2^{(m)}(K_2) \right\|, \tag{59}$$

where $x = (x_1,x_2)$ and $m = m_1 + m_2$.
Thus, from (58) and (59) we obtain

$$|< \ell_N(x_1,x_2), f(x_1,x_2) >| \leq \left\| \ell_{N_2}(x_2) \Big/ \bar{L}_2^{(m_2)^*}(0,1) \right\| \cdot$$

$$\cdot \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m)^*}(0,1) \right\| \cdot \left\| f(x) \Big/ \bar{L}_2^{(m)}(K_2) \right\|. \tag{60}$$

Considering (50) from (60) we obtain

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)^*}(K_2) \right\| \leq \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_1)^*}(0,1) \right\| \cdot \left\| \ell_{N_2}(x_2) \Big/ \bar{L}_2^{(m_2)^*}(0,1) \right\|. \tag{61}$$

Taking into account (54) from (61) we have

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)^*}(K_2) \right\| \leq d_1 \cdot d_2 \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2}},$$

or

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)^*}(K_2) \right\| \leq d_3{}' O\left(h_1^{m_1}\right) \cdot O\left(h_2^{m_2}\right), \tag{62}$$

where $d_3{}' = d_1 \cdot d_2$.
When $n = k$ we obtain

$$|< \ell_N(x), f(x) >| = |< \ell_N(x_1, x_2, ..., x_k), f(x_1, x_2, ..., x_k) >| =$$

$$\left|< \ell_{N_k}(x_k), < \ell_{N_{k-1}}(x_{k-1}), ... < \ell_{N_2}(x_2), < \ell_{N_1}(x_1), f(x_1, x_2, ..., x_k) >> ... >>\right| \leq$$

$$\leq \left\| \ell_{N_k}(x_k) \Big/ \bar{L}_2^{(m_k)*}(0,1) \right\| \cdot \left\| \ell_{N_{k-1}}(x_{k-1}) \Big/ \bar{L}_2^{(m_{k-1})*}(0,1) \right\| ...$$

$$\cdot \left\| < \ell_{N_1}(x_1), f(x_1, x_2, ..., x_k) > \Big/ \bar{L}_2^{(m_1)}(0,1) \right\| \leq$$

$$\leq \left\| \ell_{N_k}(x_k) \Big/ \bar{L}_2^{(m_k)*}(0,1) \right\| ... \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_1)*}(x_1) \right\| \cdot \left\| f(x) \Big/ \bar{L}_2^{(m)}(K_k) \right\|. \tag{63}$$

From (63), taking into account (50) we have

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_k) \right\| \leq \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_1)*}(0,1) \right\| ... \left\| \ell_{N_k}(x_k) \Big/ \bar{L}_2^{(m_k)*}(0,1) \right\|. \tag{64}$$

Then in view of (54) from (64) we obtain

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_k) \right\| \leq d \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_k^{m_k}}, \quad where \quad d = \prod_{i=1}^{k} d_i, \tag{65}$$

or, taking into account (55), from (65) we will have

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_k) \right\| \leq d \cdot O\left(h_1^{m_1}\right) ... O\left(h_k^{m_k}\right).$$

Using validity of the assertion of theorem 1 for $n = k$, we prove that the assertion is performed when $n = k+1$.
Thus, suppose $n = k+1$, then taking into account (50), from (64) we have

$$|< \ell_{k+1}(x_1, x_2, ..., x_{k+1}), f(x_1, x_2, ..., x_{k+1}) >| =$$

$$= \left|< \ell_{N_1}(x_1), < \ell_{N_2}(x_2), <, ... < \ell_{N_k}(x_k), \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > ... >\right| \leq$$

$$\leq \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_1)*}(0,1) \right\| ... \left\| \ell_{N_k}(x_k) \Big/ \bar{L}_2^{(m_k)*}(0,1) \right\| \cdot$$

$$\cdot \left\| < \ell_{N_{k+1}}(x_{k+1}), f(x_1, x_2, ..., x_{k+1}) > \Big/ \bar{L}_2^{(m_{k+1})}(0,1) \right\|. \tag{66}$$

Using (50) and (64) from (66) we obtain

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_{k+1}) \right\| \leq \left\| \ell_{N_1}(x_1) \Big/ \bar{L}_2^{(m_k)*}(0,1) \right\| ...$$

$$\cdot \left\| \ell_{N_k}(x_k) \Big/ \bar{L}_2^{(m_k)*}(0,1) \right\| \cdot \left\| \ell_{N_{k+1}}(x_{k+1}) \Big/ \bar{L}_2^{(m_{k+1})*}(0,1) \right\|. \tag{67}$$

By using (54) from (67) we have

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_{k+1}) \right\| \leq d_{k+1} \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_{k+1}^{m_{k+1}}}, \tag{68}$$

or, considering (62) and (68) we obtain

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_{k+1}) \right\| \leq d_{k+1} \cdot O\left(h_1^{m_1}\right) ... O\left(h_{k+1}^{m_{k+1}}\right), \quad where \quad d_{k+1} = \prod_{i=1}^{k+1} d_i.$$

Thus obtained inequalities (56) and (57):

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_n) \right\| \leq d \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_n^{m_n}}, \quad d - constants, \tag{69}$$

or, considering (55) from (69) we obtain

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_n) \right\| \leq d \cdot O\left(h_1{}^{m_1}\right) ... O\left(h_n{}^{m_n}\right), \quad h_i = \frac{1}{N_i}, \quad i = \overline{1,n}, \tag{70}$$

where $d = \prod_{i=1}^{n} d_i$.
If, in (69) or (70) suppose $N = N_1 \cdot N_2 \cdot ... \cdot N_n$, $N_1 = N_2 = ... = N_n$ and $m_1 + m_2 + ... + m_n = m$ then we have

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_n) \right\| \leq d \cdot N^{-\frac{m}{n}}$$

or

$$\left\| \ell_N \Big/ \bar{L}_2^{(m)*}(K_n) \right\| \leq d \cdot O\left(h^m\right), \quad d - constant, \quad h = N^{-\frac{1}{n}}. \tag{71}$$

which was needed of proof. Theorem is proved.

Thus, we obtain an upper estimate for the norm of the error functional (49) for cubature formula (48) in the space $\bar{L}_2^{(m)*}(K_n)$.
A similar assessment was obtained previously for the norm of the error functional of cubature formula (48) on the quotient space of S.L. Sobolev $L_2^{(m)}(K_n)$ and as a result we have received the same order of convergence to zero as $N \to \infty$, although the norm of function was defined in different ways, this is confirmed by the inequality (27), (71).
For illustration, we present an example at $n = 2$. Suppose

$$(x_1, x_2) = e^{x_1} \left(\frac{1}{2} - x_2^2 \right)^{3/2}. \tag{72}$$

Obviously, that the derivatives
$\frac{\partial^{m-1} f(x_1,x_2)}{\partial x_1^{m-1}}$ and $\frac{\partial f(x_1,x_2)}{\partial x_2}$ are continuous on K_2, but $\frac{\partial^2 f(x_1,x_2)}{\partial x_2}$ has a feature on K_2. Therefore, from the condition $m = m_1 + m_2$ it is clear that $m_1 = m - 1$ and $m_2 = 1$, in that $m - 1 + 1 = m$.
Hence it follows that $f(x_1, x_2) \in \bar{L}_2^{(m)}(K_2)$ when $m_1 = m - 1$, $m_2 = 1$ and $f(x_1, x_2) \bar{\in} L_2^{(m)}(K_2)$.

CONCLUSION

In this paper we investigate weight cubature formula in function spaces of S.L. Sobolev $L_2^{(m)}, L_p^{(m)}, \bar{L}_2^{(m)}$ for the functions defined in the n - dimensional unit cube K_n and obtain an upper estimate for the norm of error functionals of weight cubature formulas. The basis of theorem N.S. Bahvalov it isproved that consideredviewed cubature formulas are optimal on order of convergence in these spaces.

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